The Konvalinka-Amdeberhan conjecture and plethystic inverses

Ira M. Gessel

Department of Mathematics
Brandeis University

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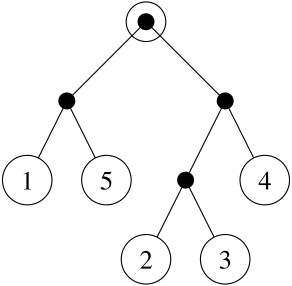
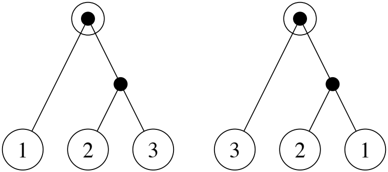
Tanglegrams

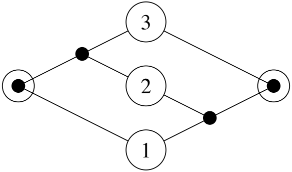
A binary tree is an unordered binary tree with labeled leaves and unlabeled internal vertices:



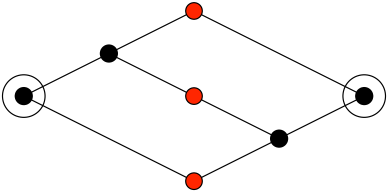
An ordered pair of trees sharing the same set of leaves is called a tanglegram. (The term comes from biology.)



which we can also draw as



Sara Billey, Matjaž Konvalinka, and Frederick A. Matsen IV wanted to count unlabeled tanglegrams



which may defined formally as orbits of tanglegrams under the action of the symmetric group permutating the labels on the leaves.

Burnside's Lemma

To count orbits, we use Burnside's Lemma: If a group G acts on a set S then the number of orbits is

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{fix}(g),$$

where $\mathrm{fix}(g)$ is the number of elements of S fixed by G. It's not hard to show that $\mathrm{fix}(g)$ depends only on the conjugacy class of g.

In the case of the symmetric group $\mathfrak{S}_n$, the conjugacy classes correspond to cycle types, which are indexed by partitions of n. If $\lambda = (1^{m_1} 2^{m_2} \cdots)$ is a partition of n then the number of elements of $\mathfrak{S}_n$ of cycle type λ is $n!/z_\lambda$, where $z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots$.

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If we define $\text{fix}(\lambda)$ be $\text{fix}(g)$ for any $g \in \mathfrak{S}_n$ of cycle type λ, then we may write Burnside's sum for $\mathfrak{S}_n$ as

$$\frac{1}{n!} \sum_{\lambda \vdash n} \text{fix}(\lambda) \frac{n!}{z_\lambda} = \sum_{\lambda \vdash n} \frac{\text{fix}(\lambda)}{z_\lambda}$$

Now let r_λ be the number of binary trees fixed by a permutation of cycle type λ. Then the number of unlabeled binary trees on n vertices is

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But an ordered pair (T_1, T_2) of binary trees is fixed by a permutation π if and only if T_1 and T_2 are both fixed by π. So the number of ordered pairs of binary trees fixed by a permutation of cycle type λ is r_λ^2.

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So the number of unlabeled tanglegrams with n leaves is

$$\sum_{\lambda \vdash n} \frac{r_\lambda^2}{z_\lambda}.$$

Tangled chains

Billey, Konvalinka, and Matsen define a tangled chain of length k to be a k-tuple of binary trees sharing the same set of leaves.

By the same reasoning, the number of unlabeled tangled chains of length k with n leaves is

$$\sum_{\lambda \vdash n} \frac{r_\lambda^k}{z_\lambda}.$$

A formula for r_λ

Billey, Konvalinka, and Matsen found a remarkable formula for r_λ:

r_λ is zero if λ is not a binary partition (a partition in which every part is a power of 2), and if λ is a binary partition, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$, then

$$r_\lambda = \prod_{i=2}^{k} \bigl(2(\lambda_i + \cdots + \lambda_k) - 1\bigr).$$

For example, $r_{(4,2,1)} = [2 \cdot (2+1) - 1](2 \cdot 1 - 1) = 5 \cdot 1 = 5$.

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The total number of of binary trees with n leaves is

$$r_{(1^n)} = 1 \cdot 3 \cdots (2n-3).$$

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But the formula is still somewhat mysterious.

The Konvalinka-Amdeberhan conjecture

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Let q be a prime. We say that a partition λ is q-ary if every part of λ is a power of q. Define $r_{\lambda,q}$ by

$$r_{\lambda,q} = \begin{cases} 0, & \text{if } \lambda \text{ is not } q\text{-ary} \\ \prod_{j=2}^{l(\lambda)} (q\lambda_j + q\lambda_{j+1} + \cdots + q\lambda_{l(\lambda)} - 1) & \text{if } \lambda \text{ is } q\text{-ary} \end{cases}$$

(Here $l(\lambda)$ is the number of parts of λ.)

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The Konvalinka-Amdeberhan Conjecture: For every positive integer k,

$$\sum_{\lambda \vdash n} \frac{r_{\lambda,q}^k}{z_\lambda}$$

is an integer.

Symmetric functions

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The symmetric functions that are homogeneous of degree n form a vector space Λ^n whose dimension is the number of partitions of n.

There are several important bases for Λ^n, indexed by partitions of n, but we only need three of them.

First, the monomial symmetric functions: If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ then m_λ is the sum of all distinct monomials of the form

$$x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}.$$

Next, the power sum symmetric functions are defined by

$$p_n = \sum_{i=1}^{\infty} x_i^n$$

and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$.

Finally, the complete symmetric functions

$$h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}.$$

and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$.

Integral symmetric functions

A symmetric function is called integral if its coefficients are integers. (This is equivalent to its coefficients being integers in the monomial basis, or any of the other common bases except for the power sum basis.)

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If f is an integral symmetric function expressed in terms of the p_λ, then setting each p_n to 1 gives an integer, since setting each p_n to 1 is equivalent to setting $x_1 = 1$, $x_i = 0$ for $i > 1$.

The Kronecker product

Next we define the operation of Kronecker product on symmetric functions. It is defined by

$$p_\lambda * p_\mu = z_\lambda \delta_{\lambda,\mu} p_\lambda$$

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Theorem. If f and g are integral symmetric functions then so is $f * g$.

Konvalinka-Amdeberhan symmetric functions

We can generalize the Konvalinka-Amdeberhan conjecture to prime powers. Let m be a power of the prime q and define the symmetric function

$$u_m(n,\alpha) = \sum_{\lambda \vdash_q n} \frac{p_\lambda}{z_\lambda} \alpha \prod_{j=2}^{l(\lambda)} (m\lambda_j + m\lambda_{j+1} + \cdots + m\lambda_{l(\lambda)} + \alpha),$$

Here $\lambda \vdash_q n$ means that n is a q-ary partition.

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Main Theorem. For any integer α, $u_m(n,\alpha)$ is an integral symmetric function.

The Konvalinka-Amdeberhan conjecture follows from this theorem, since the Konvalinka-Amdeberhan number $\sum_{\lambda \vdash n} r^k_{\lambda,q}/z_\lambda$ is obtained by setting each p_λ to 1 in the Kronecker power $[-u_q(n,-1)]^{*k}$.

Let

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where $u_m(0, \alpha) = 1$.

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So the only partitions λ that contribute are $\lambda = (1^n)$.

Then for $\lambda = (1^n)$ we have $p_\lambda / z_\lambda = x^n / n!$ and

$$\alpha \prod_{j=2}^{l(\lambda)} (m\lambda_j + m\lambda_{j+1} + \cdots + m\lambda_{l(\lambda)} + \alpha) = \alpha(m+\alpha)(2m+\alpha)\cdots((n-1)m+\alpha)$$

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Then $F_m(\alpha)$ becomes

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So it seems that $F_m(\alpha)$ is a kind of symmetric function binomial expansion. In fact, we will see that $F_m(\alpha) = F_m(1)^\alpha$.

Plethysm

In order to describe the equation that $F_m(\alpha)$ satisfies, we need a kind of composition of symmetric functions called plethysm. Let f and g be symmetric functions. The plethysm of f and g is denoted $f[g]$ or $f \circ g$.

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First suppose that g can be expressed as a sum of monic terms, that is monomials $x_1^{\alpha_1} x_2^{\alpha_2} \ldots$ with coefficient 1.
In this case, if $g = t_1 + t_2 + \cdots$, where the t_i are monic terms, then

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We can give a different characterization of plethysm when f and g are expressed in terms of power sums. First, $p_j[g]$ is the result of replacing each p_i in g with p_{ij}. Then $f[g]$ is obtained by replacing each p_j in f with $p_j[g]$.

It is not hard to show that plethysm is associative and preserves integrality.

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If f is a symmetric function of the form

$$p_1 + \text{higher order terms}$$

then f has a unique plethystic inverse of the same form, which we write as $f^{[-1]}$, satisfying

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If f is integral then so is $f^{[-1]}$.

Now let's look at the cycle index for binary trees,

$$Z_R := \sum_{\lambda} r_\lambda \frac{p_\lambda}{z_\lambda}.$$

One can show (e.g., using the theory of combinatorial species) that Z_R satisfies the plethystic equation

$$Z_R = p_1 + h_2[Z_R].$$

Here h_2 is the complete symmetric function

$$h_2 = \sum_{i \leq j} x_i x_j = \tfrac{1}{2} p_1^2 + \tfrac{1}{2} p_2.$$

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This is the symmetric function refinement of the exponential generating function equation

$$B(x) = x + B(x)^2/2.$$

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We let $g = 1 - Z_R$ and we rearrange the equation

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$$g = (p_2[g] - 2p_1)^{1/2}.$$

(It's not obvious that this is the right thing to do!)

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We can use this formula to get an explicit formula for the expansion of g, or more generally, of $g^{-\alpha}$ in power sums.

Applying the binomial theorem gives

$$g^{-\alpha} = (p_2[g] - 2p_1)^{-\alpha/2} = \sum_{m_1=0}^{\infty} (-2)^{m_1} \binom{-\alpha/2}{m_1} p_1^{m_1} p_2[g]^{-\alpha/2-m_1}$$

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Now from $g = (p_2[g] - 2p_1)^{1/2}$ we get $p_2[g] = (p_4[g] - 2p_2)^{1/2}$

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Now from $g = (p_2[g] - 2p_1)^{1/2}$ we get $p_2[g] = (p_4[g] - 2p_2)^{1/2}$ so

$$\begin{aligned} g^{-\alpha} &= \sum_{m_1=0}^{\infty} (-2)^{m_1} \binom{-\alpha/2}{m_1} p_1^{m_1} (p_4[g] - 2p_2)^{-\alpha/4-m_1/2} \\ &= \sum_{m_1,m_2=0}^{\infty} (-2)^{m_1+m_2} \binom{-\alpha/2}{m_1} \binom{-\alpha/4 - m_1/2}{m_2} \\ &\qquad \times p_1^{m_1} p_2^{m_2} p_4[g]^{-\alpha/4-m_1/2-m_2}. \end{aligned}$$

Continuing in this way, we get the expansion of $g^{-\alpha}$ into powers of $p_1, p_2, p_4, p_8, \ldots$

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We can rearrange the binomial coefficients to get

$$g^{-\alpha} = F_2(\alpha) = \sum_{n=0}^{\infty} \sum_{\lambda \vdash_2 n} \frac{p_\lambda}{z_\lambda}\, \alpha \prod_{j=2}^{l(\lambda)} (2\lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{l(\lambda)} + \alpha),$$

and in particular,

$$Z_R = 1 - g = \sum_{n=1}^{\infty} \sum_{\lambda \vdash_2 n} \frac{p_\lambda}{z_\lambda} \prod_{j=2}^{l(\lambda)} (2\lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{l(\lambda)} - 1),$$

To generalize this, we introduce a well-known symmetric function

$$L_m = \frac{1}{m}\sum_{d|m} \mu(d) p_d^{m/d},$$

Then L_m counts “primitive necklaces”, and in particular it is integral. (It also has a number of other applications.) In particular, if m is a power of a prime q then

$$L_m = \frac{1}{m}(p_1^m - p_q^{m/q}).$$

Lemma.

$$-L_m[1 - p_1] = p_1 + \text{higher order terms.}$$

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Proof. We have

$$\begin{aligned} -L_m[1-p_1] &= -\frac{1}{m}\sum_{d|m}\mu(d)p_d^{m/d}[1-p_1] \\ &= -\frac{1}{m}\sum_{d|m}\mu(d)(1-p_d)^{m/d}. \end{aligned}$$

The constant term is $-\frac{1}{m}\sum_{d|m}\mu(d) = 0$. The p_1 term comes from $d=1$:

$$-\frac{1}{m}(1-p_1)^m = -\frac{1}{m}(1-mp_1+\cdots) = -\frac{1}{m} + p_1 + \cdots .$$

Now let f_m be the plethystic inverse of $-L_m[1 - p_1]$, so $L_m[1 - f_m] = -p_1$. Then f_m is integral.

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Let $g_m = 1 - f_m$ so that $L_m[g_m] = p_1$. If m is a power of the prime q, then

$$L_m = \frac{1}{m}(p_1^m - p_q^{m/q}).$$

so

$$g_m^m - p_q[g_m]^{m/q} = -mp_1,$$

so

$$g_m = (p_q[g_m]^{m/q} - mp_1)^{1/m}$$

As before, we can expand by the binomial theorem and iterate to get the explicit formula $g_m^\alpha = F_m(\alpha)$ as defined before.

Summary

Let m be a an integer greater than 1, and let f_m be the plethystic inverse of $-L_m[1 - p_1]$, where

$$L_m = \frac{1}{m} \sum_{d|m} \mu(d) p_d^{m/d},$$

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Summary

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Then f_m is integral.

Now let m be a power of the prime q, and let $g_m = 1 - f_m$. Then for all α,

$$g_m^{-\alpha} = 1 + \sum_{n=1}^{\infty} \sum_{\lambda \vdash_q n} \frac{p_\lambda}{z_\lambda} \alpha \prod_{j=2}^{l(\lambda)} (m\lambda_j + m\lambda_{j+1} + \cdots + m\lambda_{l(\lambda)} + \alpha),$$

Conjectures and Guesses

I conjecture that g_m^{-1} is Schur positive (which implies that $g_m^{-\alpha}$ is Schur positive for $\alpha \in \mathbb{P}$) for all m (not just a prime power), and also that $1 - g_m^k$ is Schur positive for $k = 1, 2, \ldots, m-1$.

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One possible way to prove Schur positivity is to find a symmetric group representation whose characteristic is the symmetric function in question.

For $m = 2$ we have a permutation representation that works, but for $m > 2$ there does not seem to be such a permutation representation.

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We get

$$g_m^{-\alpha} \mapsto \frac{1}{(1 - mx)^{\alpha/m}} = \sum_{n=0}^{\infty} \alpha(m+\alpha)(2m+\alpha)\cdots((n-1)m+\alpha)\frac{x^n}{n!}$$

$$1 - g_m^k \mapsto \sum_{n=1}^{\infty} k(m-k)(2m-k)\cdots((n-1)m-k)\frac{x^n}{n!}$$

These formulas have combinatorial interpretations in terms of $(m+1)$-ary increasing trees (or equivalently, “generalized Stirling permutations” or multipermutations), so these might be correspond to bases for the representations we want, but it’s not clear how to construct representations from them.