An Introduction to Symmetric Functions

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In other words, if $i_1, \dots, i_m$ are distinct positive integers and $\alpha_1, \dots, \alpha_m$ are arbitrary nonnegative integers then the coefficient of $x_{i_1}^{\alpha_1} \cdots x_{i_m}^{\alpha_m}$ in a symmetric function is the same as the coefficient of $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

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- $\sum_{i \leq j} x_i x_j$

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- $x_1^2 + x_2^2 + \ldots$
- $\sum_{i \leq j} x_i x_j$

But not $\sum_{i \leq j} x_i x_j^2$

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- Symmetric functions are useful in counting plane partitions.
- Symmetric functions are closely related to representations of symmetric and general linear groups
- Symmetric functions are useful in counting unlabeled graphs (Pólya theory).

The ring of symmetric functions

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There are several important bases for Λ^n, all indexed by partitions.

Monomial symmetric functions

If a symmetric function has a term $x_1^2 x_2 x_3$ with coefficient 1, then it must contain all terms of the form $x_i^2 x_j x_k$, with i, j, and k distinct, with coefficient 1. If we add up all of these terms, we get the monomial symmetric function

$$m_{(2,1,1)} = \sum x_i^2 x_j x_k$$

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We could write it more formally as

$$\sum_{i \neq j,\, i \neq k,\, j < k} x_i^2 x_j x_k .$$

More generally, for any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, m_λ is the sum of all distinct monomials of the form

$$x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}.$$

It's easy to see that $\{m_\lambda\}_{\lambda \vdash n}$ is a basis for Λ^n.

Multiplicative bases

There are three important multiplicative bases for Λ^n.

Suppose that for each n, u_n is a symmetric function homogeneous of degree n. Then for any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we may define u_λ to be $u_{\lambda_1} \cdots u_{\lambda_k}$.

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If $u_1, u_2, \ldots$ are algebraically independent, then $\{u_\lambda\}_{\lambda \vdash n}$ will be a basis for Λ^n.

We define the nth elementary symmetric function e_n by

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The nth power sum symmetric function is

$$p_n = \sum_{i=1}^{\infty} x_i^n,$$

so $p_n = m_{(n)}$.

Theorem. Each of $\{h_\lambda\}_{\lambda \vdash n}$, $\{e_\lambda\}_{\lambda \vdash n}$, and $\{p_\lambda\}_{\lambda \vdash n}$ is a basis for Λ^n.

Some generating functions

We have

$$\sum_{n=0}^{\infty} e_n t^n = \prod_{i=1}^{\infty} (1 + x_i t)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} h_n t^n &= \prod_{i=1}^{\infty} (1 + x_i t + x_i^2 t^2 + \cdots) \\ &= \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}. \end{aligned}$$

(Note that t is extraneous, since if we set $t = 1$ we can get it back by replacing each x_i with $x_i t$.)

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It follows that

$$\sum_{n=0}^{\infty} h_n t^n = \left(\sum_{n=0}^{\infty} (-1)^n e_n t^n \right)^{-1}.$$

Also

$$\log \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} = \sum_{i=1}^{\infty} \log \frac{1}{1 - x_i t}$$

$$= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} x_i^n \frac{t^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{p_n}{n} t^n.$$

Therefore

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Therefore

$$\sum_{n=0}^{\infty} h_n t^n = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n} t^n \right).$$

If we expand the right side and equate coefficients of t^n

We get

$$h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$$

Here if $\lambda = (1^{m_1} 2^{m_2} \cdots)$ then $z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots$.

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It is not hard to show that if λ is a partition of n then $n!/z_\lambda$ is the number of permutations in the symmetric group $\mathfrak{S}_n$ of cycle type λ and that z_λ is the number of permutations in $\mathfrak{S}_n$ that commute with a given permutation of cycle type λ.

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For example, for $n = 3$ we have $z_{(3)} = 3$, $z_{(2,1)} = 2$, and $z_{(1,1,1)} = 6$, so

$$\begin{aligned} h_3 &= \frac{p_{(1,1,1)}}{6} + \frac{p_{(2,1)}}{2} + \frac{p_{(3)}}{3} \\ &= \frac{p_1^3}{6} + \frac{p_2 p_1}{2} + \frac{p_3}{3}. \end{aligned}$$

The Cauchy kernel

The infinite product

$$\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}$$

is sometimes called the Cauchy kernel. It is symmetric in both $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$.

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First we note that the coefficient $N_{\lambda,\mu}$ of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots y_1^{\mu_1} y_2^{\mu_2} \cdots$ in this product is the same as the coefficient of $x_1^{\mu_1} x_2^{\mu_2} \cdots y_1^{\lambda_1} y_2^{\lambda_2} \cdots$.

Now let's expand the product:

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1 - x_i y_j} = \prod_{i=1}^{\infty} \sum_{k=0}^{\infty} x_i^k h_k[y]$$

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Now $N_{\lambda,\mu}$ is the coefficient of $x_1^{\lambda_1}x_2^{\lambda_2}\cdots y_1^{\mu_1}y_2^{\mu_2}\cdots$ in this product, which is the same as the coefficient of $y_1^{\mu_1}y_2^{\mu_2}\cdots$ in $h_\lambda[y]$.

MacMahon's law of symmetry

Since $N_{\lambda,\mu} = N_{\mu,\lambda}$, we have MacMahon's law of symmetry: The coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ in h_μ is equal to the coefficient of $x_1^{\mu_1} x_2^{\mu_2} \cdots$ in h_λ.

The scalar product

Now we define a scalar product on Λ by

$$\langle h_\lambda, f\rangle = \text{coefficient of } x_1^{\lambda_1} x_2^{\lambda_2} \cdots \text{ in } f$$

extended by linearity. By MacMahon's law of symmetry, $\langle h_\lambda, h_\mu\rangle = \langle h_\mu, h_\lambda\rangle$, so by linearity $\langle f, g\rangle = \langle g, f\rangle$ for all $f, g \in \Lambda$. Also

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$$

and

$$\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda,\mu}.$$

The characteristic map

Let ρ be a representation of the symmetric group $\mathfrak{S}_n$; i.e., an "action" of $\mathfrak{S}_n$ on a finite-dimensional vector space V (over $\mathbb{C}$).

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From ρ we can construct a function $\chi^\rho : \mathfrak{S}_n \to \mathbb{C}$, called the character of ρ, defined by

$$\chi^\rho(g) = \mathrm{trace}\, \rho(g).$$

Then the character of ρ determines ρ up to equivalence.

We define the characteristic of ρ to be the symmetric function

$$\mathrm{ch}\,\rho = \frac{1}{n!} \sum_{g \in \mathfrak{S}_n} \chi^\rho(g) p_{\mathrm{cyc}(g)},$$

where $\mathrm{cyc}(g)$ is the cycle type of g.

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where $\mathrm{cyc}(g)$ is the cycle type of g.

Since $\chi^\rho(g)$ depends only on the cycle type of g, if we define $\chi^\rho(\lambda)$, for λ a partition of n, by $\chi^\rho(\lambda) = \chi^\rho(g)$ for g with $\mathrm{cyc}(g) = \lambda$, then we can write this as

$$\begin{aligned}\operatorname{ch}\rho &= \frac{1}{n!}\sum_{\lambda\vdash n}\frac{n!}{z_\lambda}\chi^\rho(\lambda)p_\lambda \\ &= \sum_{\lambda\vdash n}\chi^\rho(\lambda)\frac{p_\lambda}{z_\lambda}\end{aligned}$$

Then $\mathrm{ch}\,\rho$ contains the same information as χ^ρ.

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Two very simple examples:

(1) The trivial representation. Here V is a one-dimensional vector space and for every $g \in \mathfrak{S}_n$, $\rho(g)$ is the identity transformation. Then $\chi^\rho(g) = 1$ for all $g \in \mathfrak{S}_n$ so

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(2) The regular representation. Here V is the vector space spanned by $\mathfrak{S}_n$ and $\mathfrak{S}_n$ acts by left multiplication. Then $\chi^\rho(g) = n!$ if g is the identity element of $\mathfrak{S}_n$ and $\chi^\rho(g) = 0$ otherwise. So

$$\operatorname{ch}\rho = p_1^n.$$

Group actions

Let G be a finite group and let S be a finite set. An action of G on S is map $\phi : G \times S \to S$, $(g, s) \mapsto g \cdot s$ satisfying

- $gh \cdot s = g \cdot (h \cdot s)$ for $g, h \in G$ and $s \in S$
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Given an action of G on S, we get a representation of G on the vector space spanned by S:

$$\rho(g)\left(\sum_{s \in S} c_s\, s\right) = \sum_{s \in S} c_s\, g \cdot s$$

Then the trace of $\rho(g)$ is the number of elements of S for which $g \cdot s = s$, which we denote by $\mathrm{fix}(g)$.

An important fact is Burnside's Lemma (also called the orbit-counting theorem): The number of orbits of G acting S is

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{fix}(g).$$

Now we take G to be the symmetric group $\mathfrak{S}_n$.

The characteristic of the corresponding representation is

$$\frac{1}{n!}\sum_{g\in\mathfrak{S}_n}\mathrm{fix}(g)\,p_{\mathrm{cyc}(g)}=\sum_{\lambda\vdash n}\mathrm{fix}(\lambda)\,\frac{p_\lambda}{z_\lambda}$$

It is called the cycle index of the action of $\mathfrak{S}_n$, denoted Z_ϕ.

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If we set all the p_i to 1 (or equivalently, set $x_1 = 1$, $x_i = 0$ for $i > 0$) then by Burnside's lemma we get the number of orbits. This is also equal to the scalar product $\langle Z_\phi, h_n\rangle$.

There is a combinatorial interpretation to the coefficients of Z_ϕ:

The coefficient of $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ in Z_ϕ is the number of orbits of the Young subgroup $\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_m}$ of $\mathfrak{S}_n$, where $\mathfrak{S}_{\alpha_1}$ permutes $1, 2, \ldots, \alpha_1$; $\mathfrak{S}_{\alpha_2}$ permutates $\alpha_1 + 1, \ldots \alpha_1 + \alpha_2$, and so on.

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This result is a form of Pólya's theorem. If $\mathfrak{S}_n$ is acting on a set of "graphs" with vertex set $\{1, 2, \ldots, n\}$ then we can construct the orbits of $\mathfrak{S}_\alpha$ by coloring vertices $1, 2, \ldots, \alpha_1$ in color 1; vertices $\alpha_1 + 1, \ldots \alpha_1 + \alpha_2$ in color 2, and so on, and then "erasing" the labels, leaving only the colors.

Schur functions

Another important basis for symmetric functions is the Schur function basis $\{s_\lambda\}$. The Schur functions are the characteristics of the irreducible representations of $\mathfrak{S}_n$, and they are orthonormal with respect to the the scalar product:

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They are, up to sign, the unique orthonormal basis that can be expressed as integer linear combinations of the m_λ.

Plethysm

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First suppose that g can be expressed as a sum of monic terms, that is, monomials $x_1^{\alpha_1} x_2^{\alpha_2} \dots$ with coefficient 1. For example, m_λ is a sum of monic terms.

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If we have a sum of monomials with positive integer coefficients then we can also write it as a sum of monic terms:

$$2p_2 = x_1^2 + x_1^2 + x_2^2 + x_2^2 + \cdots$$

In this case, if $g = t_1 + t_2 + \cdots$, where the t_i are monic terms, then

$$f[g] = f(t_1, t_2, \dots).$$

For example

$$f[e_2] = f(x_1 x_2, x_1 x_3, x_2 x_3, \dots)$$
$$f[2p_2] = f(x_1^2, x_1^2, x_2^2, x_2^2, \dots)$$

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More specifically,

$$e_2[p_3] = \sum_{i<j} x_i^3 x_j^3$$
$$p_3[e_2] = \sum_{i<j} x_i^3 x_j^3$$

In this case, if $g = t_1 + t_2 + \cdots$, where the t_i are monic terms, then

$$f[g] = f(t_1, t_2, \dots).$$

For example

$$f[e_2] = f(x_1 x_2, x_1 x_3, x_2 x_3, \dots)$$
$$f[2p_2] = f(x_1^2, x_1^2, x_2^2, x_2^2, \dots)$$

More specifically,

$$e_2[p_3] = \sum_{i<j} x_i^3 x_j^3$$
$$p_3[e_2] = \sum_{i<j} x_i^3 x_j^3 = e_2[p_3]$$

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To do this we make several observations:

- For fixed g, the map $f \mapsto f[g]$ is an endomorphism of Λ.
- For any g, $p_n[g] = g[p_n]$
- $p_m[p_n] = p_{mn}$
- If c is a constant then $c[p_n] = c$.

These formulas allow us to define $f[g]$ for any symmetric functions f and g.

Examples of plethysm

First note that if c is a constant then

$$p_m[cp_n] = (cp_n)[p_m] = c[p_m]p_n[p_m] = cp_{mn}.$$

Then since $h_2 = (p_1^2 + p_2)/2$, we have

$$h_2[-p_1] = \frac{1}{2}(p_1[-p_1]^2 + p_2[-p_1]) = \frac{1}{2}((-p_1)^2 - p_2) = e_2.$$

More generally, we can show that $h_n[-p_1] = (-1)^n e_n$.
Also

$$\begin{aligned} h_2[1 + p_1] &= \frac{1}{2}(p_1[1 + p_1]^2 + p_2[1 + p_1]) \\ &= \frac{1}{2}((1 + p_1)^2 + (1 + p_2)) = 1 + p_1 + h_2 \end{aligned}$$

Another example: Since

$$\prod_{i=1}^{\infty}(1+x_i) = \sum_{n=0}^{\infty} e_n,$$

we have

$$\prod_{i<j}(1+x_i x_j) = \sum_{n=0}^{\infty} e_n[e_2].$$

Coefficient extraction

As we saw, the coefficient of x_1^n in a symmetric function f is the coefficient of x^n in $f(x, 0, 0, 0)$, and if f is expressed in terms of the p_i we get this by setting $p_i = x^i$ for all i.

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We can also get a simple generating function for the coefficient of $x_1 x_2 \cdots x_n$ in a symmetric function f.

Let $E(f)$ be obtained from the symmetric function f (expressed in the p_i) by setting $p_1 = x$ and $p_i = 0$ for all $i > 1$. Then

$$E(f) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

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Moreover, E is a homorphism that respects composition:

$$E(fg) = E(f)E(g)$$

and

$$E(f \circ g) = E(f) \circ E(g)$$

as long has g has no constant term.