Percolation on Triangulations, and a Bijective Path to Liouville Quantum Gravity

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ABSTRACT. We report on recent progress toward relating two notions of random surfaces introduced in theoretical physics. The first notion of random surface is *Liouville quantum gravity*, whose definition involves the Gaussian free field. The second notion is obtained by considering the scaling limit of random triangulations of the sphere. A key ingredient in proving an exact relation between these two notions is a bijective encoding of percolated triangulations by certain lattice paths.

This talk will attempt to achieve several goals:

1. Discuss the site-percolation model on random planar triangulations.
2. Provide an informal introduction to several probabilistic objects coming from theoretical physics: the *Gaussian free field*, *Schramm-Loewner evolutions*, and the *Brownian map*.
3. Present a bijective encoding of percolated triangulations, and explain its role in establishing exact relations between the above-mentioned random objects.

The results we will present are based on a collaboration between Nina Holden, Xin Sun, and me [1]. They build on a large body of work, among which a construction of Duplantier, Miller, and Sheffield [2] plays a particularly important role.

Random triangulations and site-percolation. A *planar triangulation* is a planar graph embedded in the sphere in such a way that each face is a triangle. A planar triangulation is represented in Figure 1. Triangulations are considered up to continuous deformation of the sphere, and are uniquely determined by the incidence relation between faces. Planar triangulations with n triangles are therefore obtained by taking a set of n triangles and "gluing" their edges in pairs, in such a way that the resulting surface has spherical topology.

Since the set of planar triangulations with n triangles is finite, one can consider the uniform probability measure on this set. Let us call *random triangulation of size n* a random planar triangulation with $2n$ triangles. A random triangulation can be thought of as a disorganized analogue of the triangular lattice. In fact, an important conjecture called *KPZ equation* relates the critical exponents of statistical mechanics models on the regular lattice to the same exponents for the disorganized lattice.

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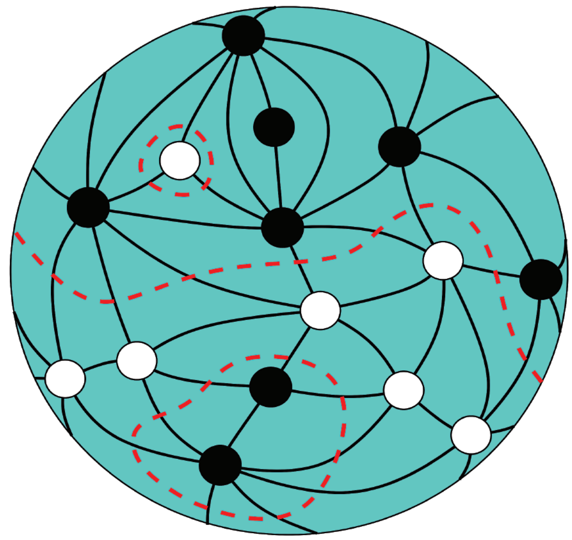

Figure 1. A planar triangulation endowed with a site-percolation configuration. The percolation interfaces are indicated by dashed lines.

The *site-percolation* model on a triangulation is a random assignment of color (either *black* or *white*) to each vertex of the triangulation. Figure 1 shows a percolation configuration. In the *critical setting* each vertex is colored black with probability 1/2 (independently of the other vertices). The clusters are the connected components of the unicolor subgraphs. The *percolation loops* are the set of closed curves on the sphere that separate the black clusters from the white clusters. Natural questions about the percolation model concern the size of the clusters and the distribution of the percolation interfaces.

A bijective tool. In [1] we establish a bijection which is key for our study of the percolation model on random triangulations. This bijection is represented in Figure 2 (see facing page). It relates (loopless) planar triangulations with $2n$ triangles endowed with a site-percolation configuration to lattice paths in $\mathbb{N}^2$ that start and end at $(0,0)$ and are made of $3n$ steps from the set $\{(0,1),(1,0),(-1,-1)\}$. This bijection is used to obtain the limiting distribution of several important observables of the percolation model (percolation loops, exploration tree, pivotal points measure).

Random surfaces and random curves. As discussed above, it is straightforward to define the notion of random

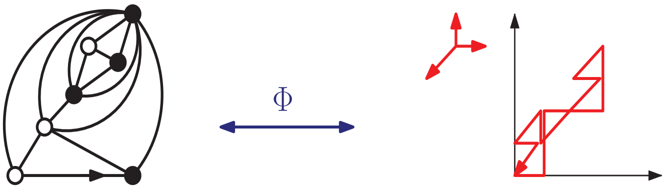

Figure 2. The bijection Φ between percolated triangulations and lattice walks.

triangulation of size n. This gives rise to an interesting notion of random metric space by considering the set of vertices of the triangulation endowed with the *graph distance* between them. In a major achievement, the *scaling limit* of this random metric space (when the number of triangles goes to infinity and their size goes to 0 at the appropriate rate) has been characterized in [5, 6]. It is a random metric space with spherical topology called the *Brownian map*, and it is a legitimate 2D analogue of the Brownian motion.

The *Gaussian free field* (GFF) is another 2D analogue of the Brownian motion, which is obtained by a completely different approach [8]. The GFF is a random distribution in (a domain of) the complex plane *PC*. The GFF can be used to define a random surface called *Liouville quantum gravity* (LQG) which was originally introduced by Polyakov [7] as a model for the random surface corresponding to the space-time evolution of a string. Heuristically, LQG_γ is a random surface which, when projected conformally on $\mathbb{C}$, gives rise to a density of area measure $e^{\gamma h}$, where h is the GFF and γ is a positive number.

The Brownian map and LQG have long been conjectured to be in some way related (although the proper definition of these objects and their possible relation is fairly recent). The proof of such a relation is under completion in a series of articles starting with [1] and culminating with [3]. Roughly speaking it is shown that, under a specific embedding of the random triangulations, the continuous limit of the vertex distribution has the law of the $\sqrt{8/3}$-LQG area measure, while the continuous limit of the percolation interfaces has the law of the *conformal loop ensemble* CLE_6 (an infinite collection of random loops closely related to Schramm–Loewner evolutions [9]).

References

[1] Bernardi O, Holden N, Xin S. Percolation on triangulations: a bijective path to Liouville Quantum Gravity. ArXiv e-prints 1807.01684.

[2] Duplantier B, Miller J, Sheffield S. Liouville quantum gravity as a mating of trees. ArXiv e-prints, September 2014.

[3] Holden N, Sun X. Quenched scaling limit of critical percolation on uniform triangulations. 2018. In preparation.

[4] Knizhnik V G, Polyakov A M, Zamolodchikov A B. Fractal structure of 2D quantum gravity. *Modern Phys. Lett. A*, 3(8):619–626, 1988.

[5] Le Gall J. Uniqueness and universality of the brownian map. *Ann. Probab.*, 41:2880–2960, 2013.

[6] Miermont G. The brownian map is the scaling limit of uniform random plane quadrangulations. *Acta. Math.*, 210:319–401, 2013.

[7] Polyakov A M. Quantum geometry of bosonic string. *Phys. Lett. B*, 3:207–210, 103.

[8] Sheffield S. Gaussian free fields for mathematicians. *Probab. Theory Relat. Fields*, 139:521–541, 2007.

[9] Werner W. Random planar curves and Schramm–Loewner evolutions. In *Lectures on probability theory and statistics*, volume 1840 of Lecture Notes in Mathematics, pages 107–195. Springer, Berlin, 2004.

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Credits

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