PROBLEMS RELATED TO ARTIN GROUPS

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Artin groups span a wide range of groups from braid groups to free groups to free abelian groups, as well as many more exotic groups. In recent years, Artin groups and their subgroups have proved to be a rich source of examples and counterexamples of interesting phenomena in geometry and group theory.

Artin groups, like Coxeter groups, are defined by presentations of a particular form. A *Coxeter graph* is a finite, simplicial graph Γ with vertex set S and edges labeled by integers $m \geq 2$. We denote by $m(s,t)$ the label on the edge connecting vertices s and t. By convention, $m(s,t) = \infty$ if s,t are not connected by an edge.

The *Artin group* associated to a Coxeter graph Γ is the group A given by the presentation

$$A = \langle S \mid \underbrace{sts\ldots}_{m(s,t)} = \underbrace{tst\ldots}_{m(s,t)} \text{ if } s,t \text{ are connected by an edge}\rangle.$$

We call Γ the *defining graph* for A. Adding additional relations $s^2 = 1$ for all $s \in S$ gives rise to a *Coxeter group* W,

$$W = \langle S \mid s^2 = 1, (st)^{m(s,t)} = 1 \text{ if } s,t \in S \text{ are connected by an edge}\rangle$$

Let $\rho : A \to W$ denote the canonical projection sending each $s \in S$ to the generator of the same name.

If the Coxeter group W associated to Γ is finite (resp. Euclidean), we say that A is *finite type* (resp. *Euclidean type*). Note, however, that Artin groups themselves are never finite groups. (Indeed, finite type Artin groups are torsion-free.) A *right-angled* Coxeter or Artin group is one in which all edges of the Coxeter graph are labeled 2.

If T is any subset of the vertex set S, then the subgroup A_T generated by T is naturally isomorphic to the Artin group associated to the full subgraph of Γ spanned by T [62]. Similarly for the subgroup W_T of the Coxeter group W. These subgroups are called *special subgroups* of A or W, respectively. The *dimension* of an Artin group is the maximum cardinality of a subset T such that A_T is finite type.

1. Geometric properties

There are several simplicial complexes associated to an Artin group A. The *Deligne complex* $\mathcal{D}_A$ is the flag complex associated to the partially ordered set of cosets,

$$\{aA_T \mid a \in A, \, T \subseteq S, \, A_T \text{ is finite type}\}.$$

Here we allow $T = \emptyset$ in which case $aA_T = \{a\}$. Note that the dimension of the simplicial complex $\mathcal{D}_A$ is equal to the dimension of the Artin group A as defined above. The group

A acts by left multiplication on this poset and hence acts simplicially on $\mathcal{D}_A$. The action is not proper, however, since the stabilizer of a vertex aA_T, $T \neq \emptyset$, is the infinite group aA_Ta^{-1}. (We remark that the analogous complex for W is the well-known Davis complex, but there the stabilizers wW_Tw^{-1} are finite groups, so the action is proper.)

Another complex associated to A is the Slavetti complex, first introduced by Salvetti in [72]. The canonical projection $\rho : A \to W$ has a set theoretic section σ defined by representing an element $w \in W$ by a minimal length positive word in S and setting $\sigma(w)$ to be the image of this word in A. It follows from fundamental facts about Coxeter groups that any two such minimal words define the same element of A. For a special subgroup W_T, we denote by $\hat{W}_T$ the set $\sigma(W_T) \subset A_T$. The *Salvetti complex* $\mathcal{S}_A$ is the flag complex associated to the partially ordered set,

$$\{a\hat{W}_T \mid a \in A,\, T \subseteq S,\, W_T \text{ is finite}\}.$$

As before, A acts simplicially on this complex, but in this case, the action is free. If $T \subset S$ generates a maximal finite subgroup W_T, then the star of $\hat{W}_T$ in $\mathcal{S}_A$ is isomorphic to the Coxeter cell for W_T. It follows that a fundamental domain for the action of A on $\mathcal{S}_A$ consists of one Coxeter cell for each such W_T.

The interest in these complexes originates with the study of hyperplane complements. The Coxeter group W acts as a (complexified real) reflection group on $\mathbb{C}^n$. The action preserves an open cone Ω (the "Tits cone") on which W acts properly [77]. The set of regular points in Ω (points with trivial isotropy) is the complement of the reflecting hyperplanes. Denote this hyperplane complement by $\mathcal{H}_W$. In the case of a finite Coxeter group, Deligne [46] proved that the orbit space $\mathcal{H}_W/W$ is a $K(A,1)$-space for the corresponding Artin group A. More generally, van der Lek [62] proved that for all W, $\mathcal{H}_W/W$ has fundamental group isomorphic to A.

Theorem 1. [32] [33] [72] *The complexes $\mathcal{D}_A$ and $\mathcal{S}_A$ are both homotopy equivalent to the universal cover of $\mathcal{H}_W$. In particular, the following are equivalent.*

(1) *$\mathcal{D}_A$ is contractible.*
(2) *$\mathcal{H}_W/W$ is a $K(A,1)$-space.*
(3) *$\mathcal{S}_A/A$ is a (finite) $K(A,1)$-space.*

Problem 1. The $K(\pi,1)$ Conjecture: *Prove that the conditions above hold for all Artin groups A.*

As noted above, this was proved by Deligne [46] for finite type Artin groups. For infinte type Artin groups, the most successful approach to this problem so far has been to find CAT(0) metrics on $\mathcal{D}_A$.

Problem 2. *Show that $\mathcal{D}_A$ supports a CAT(0) metric.*

In [32], Charney and Davis describe CAT(0) metrics on the Deligne complex of two types of Artin groups, the 2-dimensional Artin groups and the Artin groups of FC type. An Artin group is said to be of *FC type* if every complete subgraph of the Coxeter graph generates a finite type special subgroup. For example, right-angled Artin groups are always of FC type. For the FC type groups, the CAT(0) metric introduced in [32] is a cubical metric. For the 2-dimensional groups, the metric is the analogue of the CAT(0) metric defined by Moussong on the Davis complex for the corresponding Coxeter group [63]. The authors conjecture that the Moussong metric on $\mathcal{D}_A$ is CAT(0) for all A. Some progress on this conjecture may be found in [27]. Other approaches to the $K(\pi, 1)$ Conjecture in the case of Artin groups of Euclidean type may be found in [24], [34], and [67].

When the $K(\pi, 1)$ Conjecture holds for A, the Salvetti complex can be used to obtain information about the cohomology of A. For right-angled Artin groups the cohomology ring is easily determined [33] and for finite and some Euclidean type Artin groups, the cohomology has been extensively studied by Salvetti and others. See for example, [72], [73], [41] and [24]. In the case of right-angled Artin groups, the connectivity of A at infinity and its cohomology of with group-ring coefficients was determined by Brady-Meier [18], and Jensen-Meier [57]. One might hope to do the same for other cases in which one has a good $K(\pi, 1)$-space.

Problem 3. *Compute the cohomology of A with coefficients in $\mathbb{Z}[A]$. Which Artin groups are duality groups?*

Since the action of A on $\mathcal{D}_A$ is not proper, the existence of a CAT(0) metric on $\mathcal{D}_A$ does not imply that A is a CAT(0) group. Indeed, aside from the right-angled Artin groups, very few Artin groups are known to be CAT(0). This is even the case for finite type Artin groups, though Bestvina [9] has shown that these groups act geometrically on a space with CAT(0)-like properties. In addition, some low dimensional Artin groups have been shown to be CAT(0) by Brady [19] and Bell [7].

Problem 4. *Which Artin groups are CAT(0) groups?*

Niblo and Reeves [66] have shown that every Coxeter group acts properly (but not necessarily cocompactly) on a CAT(0) cube complex. With the exception of the right-angled Artin groups, it is not known whether Artin groups have such actions.

Problem 5. *Which Artin groups act properly on a CAT(0) cube complex?*

The geometry of the Deligne complex is also of interest for another reason. While Moussong [63] showed that many Coxeter groups are hyperbolic, it is easy to see that no Artin groups, other than the free groups, are hyperbolic. This is because any finite type special subgroup A_T, with $|T| \geq 2$, contains a $\mathbb{Z}^2$ subgroup. Moreover, freely indecomposable Artin groups are never strongly relatively hyperbolic with respect to any collection of proper subgroups as shown in [3] and [5]. However, in some cases, they are weakly relatively hyperbolic with respect to the set of finite type special subgroups. (Here, we use the term weakly relatively hyperbolic to mean that the coned off Cayley graph is hyperbolic, as in [51], while strongly relatively hyperbolic requires, in addition, that the bounded coset penetration property hold.) This question is studied by Charney and Crisp in [30] where the following theorem is proved.

Theorem 2. *A is weakly relatively hyperbolic with respect to its finite type special subgroups if and only if $\mathcal{D}_A$ is hyperbolic (with respect to some piecewise Euclidean equivariant metric).*

Problem 6. *Determine when $\mathcal{D}_A$ is hyperbolic.*

Some results on this question are given by Kapovich and Schupp [58] and Crisp [37] for 2-dimensional Artin groups and by Charney and Crisp in [30] for Artin groups of FC type. These results suggest the following conjecture.

Conjecture 7. *$\mathcal{D}_A$ is hyperbolic if and only if the corresponding Coxeter group W is hyperbolic.*

Another approach to studying relative hyperbolicity is by means of the asymptotic cone of a group. Drutu and Sapir [50] give a characterization of strong relative hyperbolicity in terms of a "tree-grading" on the asymptotic cone, and in [49] they undertake a more general study of groups with tree-graded asymptotic cone.

Problem 8. *Study the asymptotic cones of A. When are they tree-graded? Identify the pieces.*

In the case of a Coxeter group W, every special subgroup of W is quasi-convex (with respect to the standard generators S). If all of the edge labels in the Coxeter graph are even, then the same is true for the associated Artin group A since, in this case, there is a retraction of A onto any special subgroup defined by sending all of the other generators to the identity. In general, however, it is not known if special subgroups are quasi-convex. For finite type Artin groups, a more convenient generating set is given by the Garside structure (see discussion below). It follows from results of [26] that with respect to this generating set, every special subgroup is convex. Since FC-type Artin groups have nice normal forms constructed from their finite type subgroups [1], [2], a natural case to consider would be the following.

Problem 9. *Is every special subgroup of an Artin group of FC-type quasi-convex (with respect to some appropriate generating set)?*

2. Algebraic and algorithmic properties

Finite type Artin groups have nice algorithmic properties. In particular, they are biautomatic [25], [26]. There are two key features of finite type Artin groups which give rise to this biautomatic structure. First, the moniod, A^+, generated by the positive words in S forms a lattice with respect to left (or right) divisibility, that is, any two elements have a least common multiple and a greatest common divisor. Second, the full Artin group A can be obtained from A^+ be inverting a single element, Δ, the image under the section $\sigma : W \to A$ of the longest element of W. In the case of the braid group, these properties were discovered and used by Garside [53] to give a solution to the word problem. The analogue for other finite type Artin groups was done by Deligne [46] and Breiskorn and Saito [22]. Since then, the concept of a "Garside structure" has been generalized and applied to other groups [45], [42]. An alternate Garside structure for finite type Artin groups in which Δ is replaced by the image of the Coxeter element δ, also provides a useful tool [16], [21], [8].

Very little is known about algebraic or algorithmic properties of infinite type Artin groups. Right-angled Artin groups are known to be biautomatic by Hermiller-Meier [55] and VanWyk [76], and some automaticity results are known for Euclidean type [34], FC type [1], and 2-dimensional Artin groups [70], [36], [20].

For infinite type Artin groups in general, the positive monoid A^+ is still nicely behaved [22] and, by a theorem of Paris [68], still injects into the group A, but there is no analogue of the "longest element" Δ that allows one to pass from A^+ to A. It is possible that the Coxeter element δ could be used to find some analogue of a Garside structure in, say, the Euclidean case, but it seems likely that new techniques will be required to address the following questions in the general case.

Problem 10. *Suppose that A is of infinite type. Prove that A has solvable word and conjugacy problem.*

Problem 11. *Is every Artin group biautomatic?*

In the finite type case, using the normal forms for A given by one of the Garside structures, one can prove a variety of algebraic properties. For example, finite type Artin groups are torsion-free and have infinite cyclic center generated by a power of the Garside element. It is conjectured that the infinite type Artin groups are also torsion-free, but have no center. Brieskorn and Saito [22] showed that the positive monoid A^+ has no central elements, but as noted above, for infinite type Artin groups, there is no simple way to pass from the monoid to the group.

Conjecture 12. *Suppose A is of infinite type. Then A is torsion-free and has trivial center.*

The conjecture is proved in [34] for the case of Euclidean Artin groups of type $\tilde{A}_n$. Note that torsion-free is immediate for any Artin group known to satisfy the $K(\pi, 1)$ Conjecture since such groups have finite $K(\pi, 1)$ spaces. Another approach to proving Artin groups are torsion-free, as well as many other algebraic properties, is to show that they are right-orderable, that is, that there is a right-invariant total ordering on the group. This was proved for the braid groups by Dehornoy [43] (see also [44]) and for many of the other finite type Artin groups by Mulholland and Rolfsen [65]. The Euclidean Artin groups of type $\tilde{A}_n$ and $\tilde{C}_n$ inject into a braid group, hence these groups too are right-orderable.

Problem 13. *Is every finite (esp. Euclidean) type Artin group right-orderable?*

3. Right-angled Artin groups

Recall that a *right-angled Artin group* (or RAAG) is one in which all labels in the defining graph are 2's. In other words, in the presentation for the Artin group, all relations are commutator relations: $s_i s_j = s_j s_i$. Note that the dimension of a right-angled Artin groups is the maximal rank of an abelian subgroup in A, or equivalently, the rank of the largest complete subgraph of its defining graph. RAAGs range from free groups (the defining graph has no edges) to free abelian groups (the defining graph is a complete graph). On first glance the most elementary class of Artin groups, RAAGs turn out to have a surprising richness and flexibility that has led to some remarkable applications. For a more extensive discussion of these groups, see the survey article [28].

Many of the questions posed above have been answered for RAAGs. In particular, the Salvetti complex has a cubical structure which supports a CAT(0) metric [33]. It follows that RAAGs are CAT(0), biautomatic, and torsion-free. It also follows that they have finite cohomological dimension and their cohomology is easily computed from the Salvetii complex. The Deligne complex for a RAAG is hyperbolic if and only if the defining graph has no minimal circuits of length 4, or equivalently, if and only if the corresponding RA Coxeter group is hyperbolic [30]. Two RAAGs are isomorphic if and only if their defining graphs are isomorphic [47].

Letting A range over RAAGs with n generators, the automorphism groups $Aut(A)$ may be viewed as interpolating between $Aut(F_n)$, the automorphism group of a free group, and $GL_n(\mathbb{Z})$, the automorphism group of a free abelian group. Although Formanek and Procesi [52] showed $Aut(F_n)$ is not a linear group for $n \geq 3$, $Aut(F_n)$ has many properties in common with linear groups. It is natural to ask which of those properties hold for automorphism groups of all RAAGs. In the remainder of this section, we assume that A is a RAAG.

Problem 14. *For which RAAGs is $Aut(A)$ linear?*

Little is known about the groups $Aut(A)$, in general, with the exception of a finite generating set given by Servatius [75] and Laurence [60]. It is not known, for example, if these groups are finitely presented.

Problem 15. *Show that $Aut(A)$ is finitely presented. Find a presentation for $Aut(A)$ and $Out(A)$.*

The outer automorphism groups of RAAGs are studied by Charney, Crisp, and Vogtmann in [31] and [35]. There, it is shown that these groups are virtually torsion-free and have finite virtual cohomological dimension. In [23], the exact vcd of $Out(A)$ is determined in the case that the defining graph of A is a tree.

Problem 16. *Determine the vcd of $Out(A)$ for an arbitrary RAAG A.*

For 2-dimensional RAAGs with connected defining graph, $Out(A)$ satisfies the Tits alternative: every subgroup is either virtually solvable or contains a non-abelian free group [31].

Problem 17. *Does $Out(A)$ satisfy the Tits alternative for every RAAG A?*

Using techniques from [35], this problem can be reduced to showing that if Γ is a graph with multiple connected components $\Gamma_1, \dots \Gamma_k$, then the Tits alternative holds for $Out(A(\Gamma))$ providing it holds for each component group $Out(A(\Gamma_i))$. Thus we are led to the following more general question.

Problem 18. *If G is a free product of groups $G = G_1 * \cdots * G_k$ such that $Out(G_i)$ satisfies the Tits alternative, must $Out(G)$ also satisfies the Tits alternative?*

In [12] and [13], Bestvina, Feighn, and Handel prove the Tits alternative for $Out(F_n)$ by developing a theory of train tracks for free group automorphisms. A train track for an automorphism $\phi : F_n \to F_n$ is a geometric representation of ϕ as a homotopy equivalence of a graph satisfying some particular properties. Train tracks provide a means of studying the dynamics of an automorphism. For a RAAG automorphism $\phi : A \to A$, the natural analogue would be a homotopy equivalence of some carefully designed cubical complex with fundamental group A.

Problem 19. *Develop a theory of "train-tracks" for automorphisms of a RAAG.*

A key tool in the study of automorphism groups of free groups is Culler and Vogtmann's outer space, a contractible space with a proper $Out(F_n)$ action [40]. Outer space retracts onto a spine on which $Out(F_n)$ acts cocompactly. In [31], the authors construct an analogue of outer space for right-angled Artin groups whose defining graphs are connected and triangle-free. This space is contractible, has a proper $Out(A)$-action and retracts onto a spine of lower dimension. (However, in general, the action of $Out(A)$ on this spine is not cocompact.)

Problem 20. *Construct an analogue of outer space for higher dimensional RAAGs, that is, a finite-dimensional, contractible space X with a proper $Out(A)$ action.*

Problem 21. *Find a cocompact spine for this outer space.*

Culler and Vogtmann's outer space for a free group F may be described as the space of minimal, isometric actions of F on a tree. The outer space constructed in [31] for a 2-dimensional RAAG A is based on actions of free $\times$ free subgroups of A on a product of trees. Another, natural analogue of Culler-Vogtmann's space is the space of minimal, geometric actions of A on a CAT(0) space. We state the following question somewhat vaguely, as some condition (piecewise Euclidean? cubical?) is presumably needed on the spaces in question.

Problem 22. *Is the space of minimal geometric actions of A on some appropriate class of CAT(0) spaces contractible (with respect to the equivariant Gromov-Hausdorff topology)?*

For finitely generated free groups F, Bestvina and Feighn [11] contruct a cocompact bordification of outer space which they use to prove that $Out(F)$ is a virtual duality group.

Problem 23. *Is $Out(A)$ a virtual duality group?*

RAAGs have been shown to contain an amazing variety of interesting subgroups. For example, Bestvina and Brady [10] use subgroups of right-angled Artin groups to find examples of groups which distinguish between various types of finiteness properties. More recently, Haglund and Wise [54] have shown that every Coxeter group virtually injects into a right-angled Artin group.

A question that has received considerable attention lately is which RAAGs contain hyperbolic surface groups. In [48], Droms showed that any RAAG whose defining graph contains a minimal cycle of length at least 5 contains such hyperbolic surface group. Recently, Kim [59] and Crisp, Sageev and Sapir [38] have shown that many other RAAGs also contain such subgroups. However, the general question remains open. Crisp, Sageev, and Sapir pose the question as follows.

Question 24. *Is there an algorithm to decide for a given graph* Γ *whether the associated Artin group contains the fundamental group of a closed hyperbolic surface?*

Another question about subgroups of right-angled Artin groups is posed by Hsu and Wise in [56]. They consider quasi-convex subgroups of A (or more generally, a class of subgroups they call quasi-full). They prove that if the defining graph of A is a tree, then every quasi-full subgroup of A is separable. They conjecture that the same holds for chordal graphs, that is, graphs in which no cycle of length ≥ 4 is a full subgraph.

Conjecture 25. *If the defining graph of* A *is chordal, then every quasi-full subgroup of* A *is separable.*

4. Isomorphisms and quasi-isometries

Another topic of interest is rigidity properties of Artin groups. It is known that two non-isomorphic Coxeter graphs can give rise to both isomorphic Coxeter groups and isomorphic Artin groups [17]. A great deal of work has been done on classifying Coxeter groups up to isomorphism by Bahls, Mihalik, Mühlherr, and others. For a good survey of this problem see [64]. As yet, however, little is known about the isomorphism classification of Artin groups.

Problem 26. *Determine which Coxeter graphs give rise to isomorphic Artin groups.*

It appears that Artin groups may be more rigid, in this respect, that Coxeter groups. For example, in [69], Paris shows that two finite type Artin groups are isomorphic if and only if their Coxeter graphs are the same. This is not the case for finite Coxeter groups since, for example, $D_{12} \cong D_6 \times \mathbb{Z}_2$ where D_n denotes the dihedral group of order n. On the other hand, there are no examples know for which the Artin groups are isomorphic but the Coxeter groups are not.

Problem 27. *If two graphs define isomorphic Artin groups, must the corresponding Coxeter groups also be isomorphic?*

In particular, in [17], the authors described an operation on certain Coxeter graphs called diagram twisting which produces a new Coxeter graph with isomorphic Coxeter and Artin groups. They conjectured that any two graphs which give rise to isomorphic Coxeter groups via an isomorphism taking reflections to reflections must be related by a series of diagram twists. This conjecture has been shown to be false by Ratcliffe and Tschantz [71]. In the case of Artin groups, the only known examples of non-isomorphic graphs with isomorphic Artin groups are those related by diagram twists.

Problem 28. *If two Coxeter graphs give rise to isomorphic Artin groups, can one graph necessarily be obtained from the other by a series of diagram twists?*

We can also consider a coarser classifications of Artin groups.

Problem 29. *Classify Artin groups up to quasi-isometry. What other finitely generated groups are quasi-isometric to an Artin group?*

The answer to these problems is likely to be quite complex. In [6], Behrstock and Neumann show that many Artin groups are quasi-isometric to each other, while in [14], Bestvina, Kleiner, and Sageev show that certain right-angled Artin groups, which they call atomic, are more rigid. An *atomic* Artin groups is one whose defining graph is connected, has no valence one vertices, no cycles of length less than 5, and no separating stars of vertices. Bestvina, Kleiner, and Sageev study quasi-isometries of the Salvetti complex for these groups. They show that although quasi-isometries of the Salvetti complex need not stay a bounded distance from an isometry, enough of the structure of flats is preserved to prove that two atomic RAAGs are quasi-isometric if and only if their defining graphs are isometric. They pose the following problem [14].

Problem 30. *Let A be an atomic, right-angled Artin group. If H is a finitely generated group quasi-isometric to A, is H commensurable to A?*

Using the relation between mapping class groups and braid groups, automorphism groups and abstract commensurators for some finite and Euclidean type Artin groups were computed in [29] and [61], and for some 2-dimensional Artin groups in [37]. In particular, in the 2-dimensional case, Crisp gives conditions under which the abstract commensurator is isomorphic to the automorphsim group.

Problem 31. *Compute the quasi-isometry group and abstract commensurator for other Artin groups.*

We conclude by remarking that a number of groups related to Artin groups have been introduced in recent years. These include Garside groups [45], [42], mock right-angled Artin groups [74], and singular braid groups [15] [4]. Some of the questions above could also be applied to these classes of groups.

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