AN INVESTIGATION OF SKYLINE POLYOMINOES

ANTON GERASCHENKO

Abstract. In this paper, we investigate the number of skyline polyominoes for a given perimeter. A formula is derived along with a combinatorial interpretation of the formula.

1. Introduction

Definition 1. A *skyline path* from $(0,0)$ to $(m,0)$ is a path with steps $(0,1)$, $(1,0)$, and $(0,-1)$ (also called up, horizontal, and down steps, respectively) which does not contian the substrings $(0,1),(0,-1)$ or $(0,-1),(0,1)$ and which remains strictly above the x-axis.

A *skyline polyomino* is a polyomino bounded above by a skyline path and bounded below by the x axis.

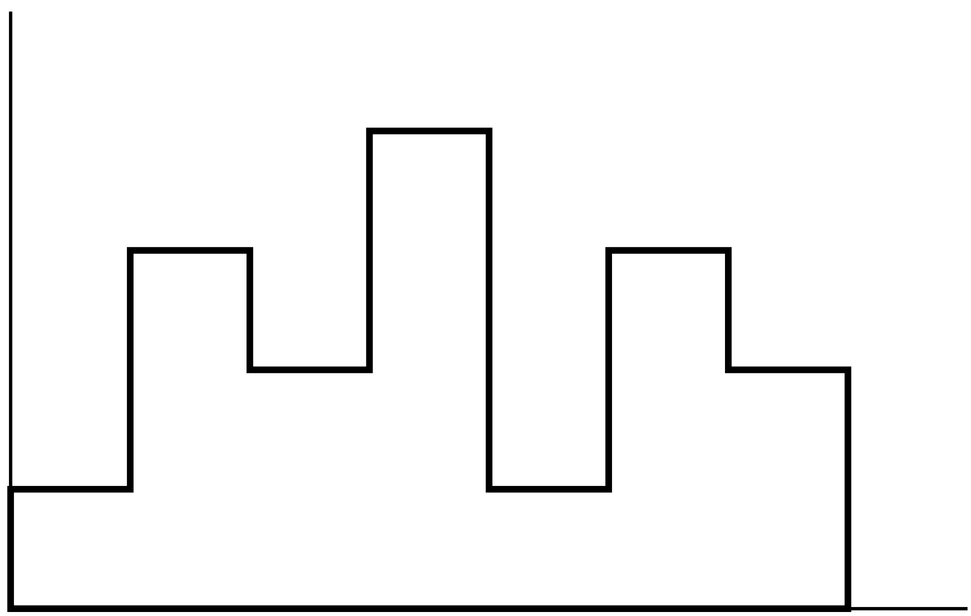

Figure 1

Let $S_{m,n}$ be the number of skyline paths with m $(0,1)$ steps and n $(1,0)$ steps. Note that $S_{m,n}$ counts the number of skyline paths from $(0,0)$ to $(m,0)$ of length $m+2n$ as well as the number of skyline polyominoes with $2m$ horizontal edges and $2n$ vertical edges. We will consider the generating function $f(x,y)=\sum_{m,n=0}^{\infty} S_{m,n}x^m y^n$.

2. Counting skyline paths

Note that a skyline path must begin with an up step, end with a down step, and remain at height at least one in between. In between the initial up step and the final down step, the path may fall into one of two cases. The path either stays at height one until it returns to the x-axis in which case the path takes only horizontal steps, of which there may be any positive number. If the path exceeds height one, then it must decompose in the following way: after the initial up step, there is a possibly empty horizontal path followed by a skyline path, followed by some number of paths which decompose into a non-empty horizonal path followed by a skyline path (possibly zero of these paths), followed by a possibly empty horizontal path. This results in a functional equation for our generating function f.

$$\begin{aligned} f &= y\left(\frac{x}{1-x}+\frac{1}{1-x}\cdot f\cdot\frac{1}{1-f\dfrac{x}{1-x}}\cdot\frac{1}{1-x}\right) \\ &= y\left(\frac{x}{1-x}+\frac{1}{(1-x)^2}\cdot\frac{f}{1-f\dfrac{x}{1-x}}\right) \end{aligned} \tag{1}$$

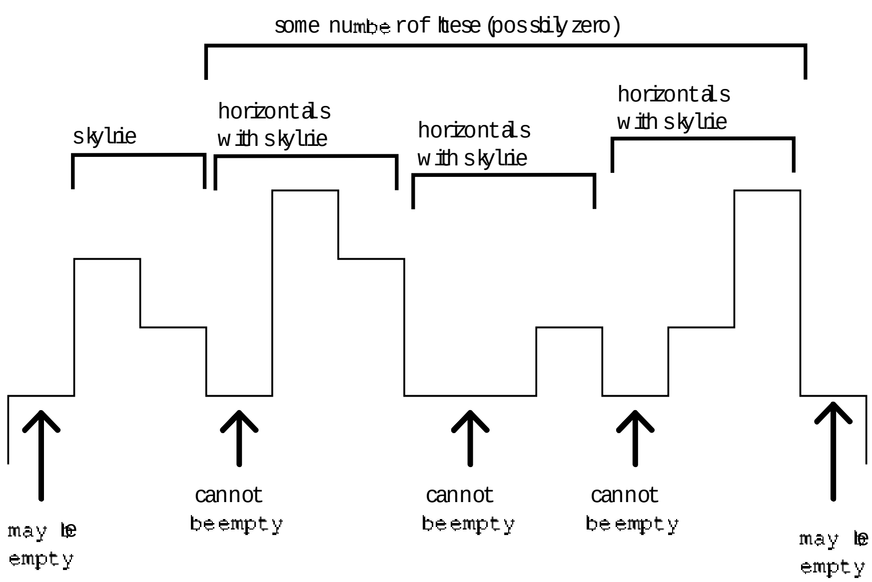

Figure 2. The second term in equation (1)

Observe that $f = y\cdot G(f)$ where $G(t)=\dfrac{x}{1-x}+\dfrac{1}{(1-x)^2}\cdot\dfrac{t}{1-t\dfrac{x}{1-x}}$, so Lagrange inversion may be applied: $[y^n]f^k=\frac{k}{n}[t^{n-k}]G(t)^n$ where $[y^n]f^k$ denotes the coefficient of y^n in the power series f^k. Taking the special case where $k=1$, we have

$$
\begin{aligned}
[y^n]f &= \frac{1}{n}[t^{n-1}]\left(\frac{x}{1-x} + \frac{1}{(1-x)^2}\cdot\frac{t}{1-t\dfrac{x}{1-x}}\right)^n \\
&= \frac{1}{n}[t^{n-1}]\sum_{i=0}^{n}\binom{n}{i}\frac{x^{n-i}}{(1-x)^{n-i}}\cdot\frac{1}{(1-x)^{2i}}\frac{t^i}{\left(1-t\dfrac{x}{1-x}\right)^i} \\
&= \frac{1}{n}[t^{n-1}]\sum_{i=0}^{n}\binom{n}{i}\frac{x^{n-i}}{(1-x)^{n+i}}t^i\sum_{j=0}^{\infty}\binom{j+i-1}{j}t^j\frac{x^j}{(1-x)^j} \\
&= \frac{1}{n}[t^{n-1}]\sum_{i=0}^{n}\sum_{j=0}^{\infty}\binom{n}{i}\binom{j+i-1}{j}t^{i+j}\frac{x^{n-i+j}}{(1-x)^{n+i+j}} \\
&= \sum_{i=0}^{n}\frac{1}{n}\binom{n}{i}\binom{n-2}{i-1}\frac{x^{2n-2i-1}}{(1-x)^{2n-1}}.
\end{aligned}
$$

From this, we may deduce a formula for the coefficients of f:

$$
\begin{aligned}
f(x,y) &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{1}{n}\binom{n}{i}\binom{n-2}{i-1}y^n\frac{x^{2n-2i-1}}{(1-x)^{2n-1}} \\
&= \sum_{m,n=0}^{\infty}\sum_{i=0}^{n}\frac{1}{n}\binom{n}{i}\binom{n-2}{i-1}y^n x^{2n-2i-1}\sum_{k=0}^{\infty}\binom{2n+k-2}{k}x^k \\
&= \sum_{m,n=0}^{\infty}\sum_{i=0}^{n}\frac{1}{n}\binom{n}{i}\binom{n-2}{i-1}\binom{m+2i-1}{2n-2}x^m y^n. \qquad (2)
\end{aligned}
$$

3. Interpretations

Observe from equation (2) that the coefficient in f of $x^m y^n$ is

$$\sum_{i=0}^{n} N(n-1,i)\binom{m+2i-1}{2n-2}$$

where $N(n-1,i)$ are the Narayana numbers, which count the number of Dyck paths of length $2(n-1)$ with i peaks. The Narayana numbers are usually written as $N(n,i) = \frac{1}{n}\binom{n}{i}\binom{n}{i-1}$, but it is easy to compute

$$
\begin{aligned}
N(n-1,i) &= \frac{1}{n-1}\binom{n-1}{i}\binom{n-1}{i-1} \\
&= \frac{1}{n-1}\cdot\frac{(n-1)!}{i!\,(n-i-1)!}\cdot\frac{(n-1)!}{(i-1)!\,(n-i)!} \\
&= \frac{1}{n}\cdot\frac{(n)!}{i!\,(n-i)!}\cdot\frac{(n-2)!}{(i-1)!\,(n-i-1)!} \\
&= \frac{1}{n}\binom{n}{i}\binom{n-2}{i-1}.
\end{aligned}
$$

The sum may be reindexed by setting $j = n-i$. Recalling the fact that $N(n-1,i) = N(n-1,n-i)$, we have that the coefficient of $x^m y^n$ is

$$\sum_{j=0}^{n} N(n-1,j)\binom{2n-1+(m-2j+1)-1}{m-2j+1}.$$

Note that the binomial coefficient that appears in the sum is $(2n-1)$ *multichoose* $(m-2j+1)$, which we will write $\left(\!\binom{2n-1}{m-2j+1}\!\right)$. By $\left(\!\binom{n}{k}\!\right)$, we mean the number of ways to choose k elements from an n element set, allowing repetitions. It is well known that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$. So we have that

$$S_{m,n} = [x^m y^n] f(x,y) = \sum_{j=0}^{n} N(n-1,j)\left(\!\!\binom{2n-1}{m-2j+1}\!\!\right). \tag{3}$$

This result may be interpreted directly. Given a Dyck path of length $2(n-1)$ with j peaks, we turn it into a prime Dyck path of length $2n$ with j peaks by adding an up step at the beginning and a down step at the end. Now we want to insert m horizontal edges at the vertices to obtain a skyline polyomino. Note that we must put a horizontal edge at every peak and at every valley. Since the number of valleys is one less than the number of peaks, this means that we have used up $(2j-1)$ of the horizontal edges, so we have $(m-2j+1)$ left. We may insert these at any vertex in the Dyck path with the exception of the first and last vertices. There are $(2n-1)$ of these, so there are $\left(\!\binom{2n-1}{m-2j+1}\!\right)$ ways to do this. It is clear that each skyline polyomino is obtained uniquely in this way. Summing over all Dyck paths and all j, we obtain equation (3).

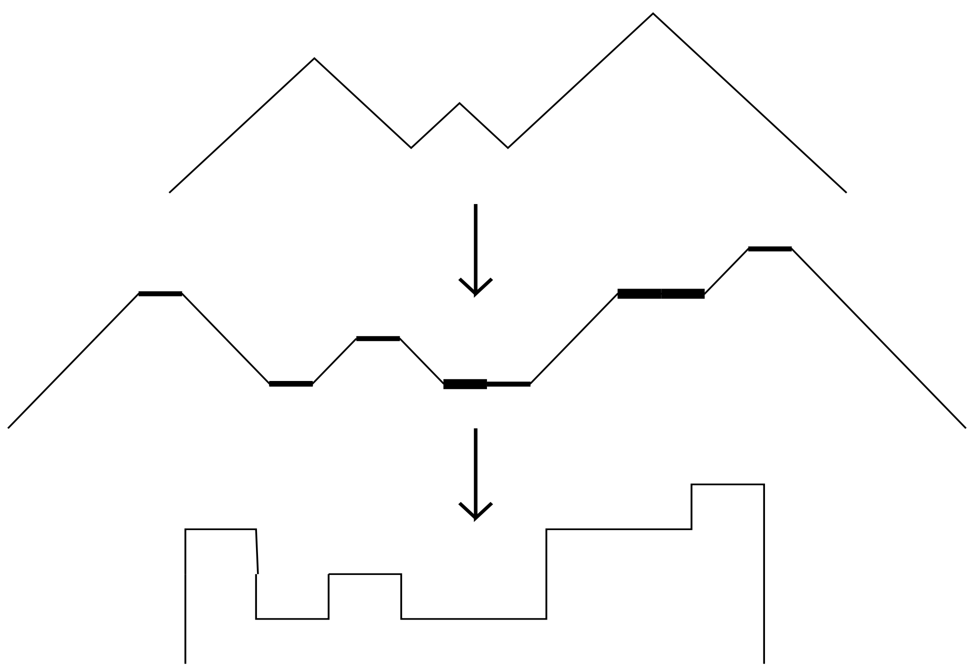

FIGURE 3. The obligatory horizontal edges are light bold. The optional horizontal edges are bold.

4. Additional Problems

Observe that we may use the quadratic formula to solve equation (1) for f explicitly, obtaining

$$f(x,y) = \frac{xy}{1-x-y-xy} \cdot \left(\frac{1 - \sqrt{1 - 4 \cdot \frac{x^2y}{(1-x-y-xy)^2}}}{2 \cdot \frac{x^2y}{(1-x-y-xy)^2}} \right)$$

$$= \frac{xy}{1-x-y-xy} \cdot c\left(\frac{x^2y}{(1-x-y-xy)^2}\right) \tag{4}$$

where $c(x) = \dfrac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers. This should have a fairly straightforward interpretation. It is likely that $\dfrac{x^2y}{(1-x-y-xy)^2}$ is a generating function for pairs of paths where the first path is increasing and the second is decreasing and the two are of the same height. By building a Dyck path using such increasing and decreasing paths in place of up steps and down steps and making some small modification, one gets a skyline path. Each skyline path should be obtained in this way uniquely. The author is not yet sure of exactly which paths $\dfrac{x^2y}{(1-x-y-xy)^2}$ is a generating function for.