Chung-Feller Theorems

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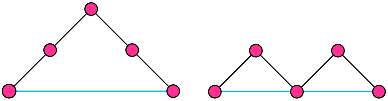
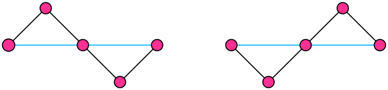
For example, with $n = 2$, of the 6 paths consisting of 2 up and down steps, 2 paths have 4 up steps above the x-axis



and 2 paths have 2 steps above the x-axis.



In particular, the case $k = n$ counts Dyck paths. This result is usually called the Chung-Feller theorem.

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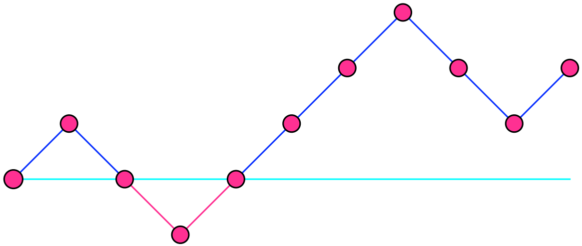

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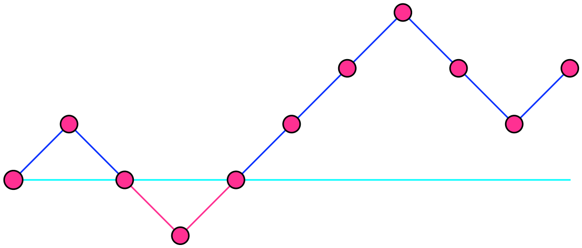

Note that here we don't care where the path ends. If we want to keep track of the number of steps above the x-axis and the endpoint of the path, then there isn't such a simple formula.

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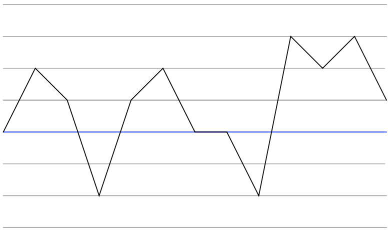
Are there Chung-Feller theorems for these formulas? (First given by David Callan) What about the Narayana numbers?

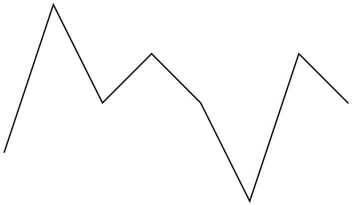
$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{n+1}\binom{n+1}{k}\binom{n-1}{k-1} = \cdots$$

A General Chung-Feller Theorem

We consider paths with arbitrary integer steps that start at the origin and end at height 1.



A conjugate or cyclic permutation of a path is a path obtained by cutting the original path and moving the first part to the end.



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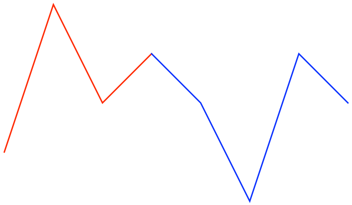

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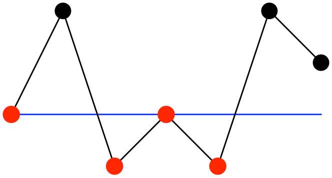
Note that if a path of length m ends at height 1, then it has m distinct conjugates.

A nonpositive point of a path is a point on or below the x-axis.



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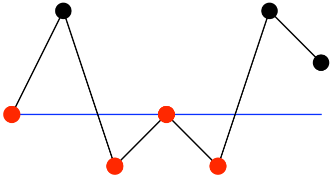

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Theorem. Let P be a path with m steps, ending at height 1. Then among the m conjugates of P, the number of nonpositive points takes on the values $1, 2, \ldots, m$.

We can apply the theorem to explain the formula $\frac{1}{2n+1}\binom{2n+1}{n}$ for Catalan numbers.

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We consider the $\binom{2n+1}{n}$ paths with $n+1$ up steps $(1,1)$ and n down steps $(1,-1)$. Each path ends at height 1. The set of paths is closed under conjugation and each conjugacy class contains $2n+1$ paths. Therefore by the theorem, in each conjugacy class the number of nonpositive points takes on the values 1, 2, ..., $2n+1$. Thus the number of nonpositive points is equidistributed among the values 1, 2, ..., $2n+1$, so for each $i = 1, \dots, 2n+1$, there are $\frac{1}{2n+1}\binom{2n+1}{n}$ paths with i nonpositive points.

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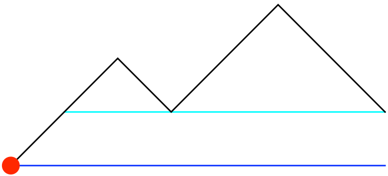
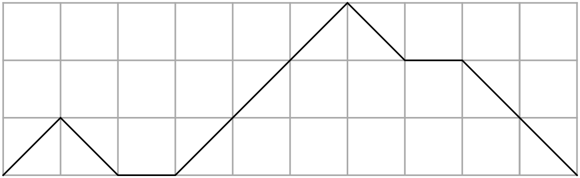
In particular, the paths with only 1 nonpositive point (the starting point) consist of an up step followed by a Dyck path.



Motzkin Paths

The Motzkin number M_n count Motzkin paths of length n: paths that never go below the x-axis with three kinds of steps: up steps $(1,1)$, flat steps $(1,0)$ and down steps $(1,-1)$:



The Motzkin numbers are given by the formula

$$M_n = \frac{1}{n+1} \sum_{k \le n/2} \binom{n+1}{k, k+1, n-2k}.$$

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Our theorem gives a Chung-Feller interpretation of this formula. The multinomial coefficient $\binom{n+1}{k,k+1,n-2k}$ counts paths with k down steps, $k+1$ up steps, and $n-2k$ flat steps. Since these paths end at height 1, we can apply the theorem to get that for i from 1 to $n+1$, the number of these paths with i nonpositive points is $\frac{1}{n+1}\binom{n+1}{k,k+1,n-2k}$. Summing over k, we get that for i from 1 to $n+1$, the Motzkin number M_n counts paths of length $n+1$ ending at height 1 with up, down, and flat steps, with exactly i nonpositive points. In particular, for $i=0$ these paths consist of an up step followed by a Motzkin path of length n.

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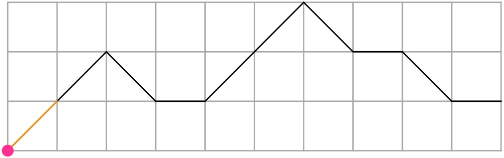
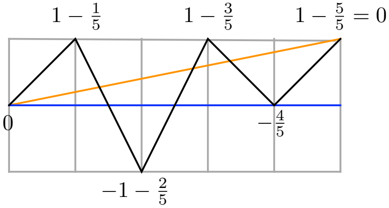


Sketch of the proof of the theorem. We take a path ending at height 1 and draw the line from the starting point to the endpoint. We'll call the distance of a point of the path above this line the elevation of the point.

Note that the elevations of the points are all different since their fractional parts are different (except that the starting and ending points both have elevation 0).



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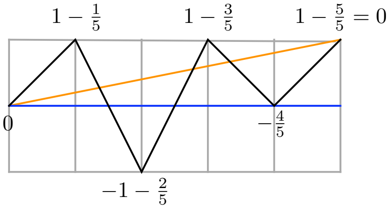
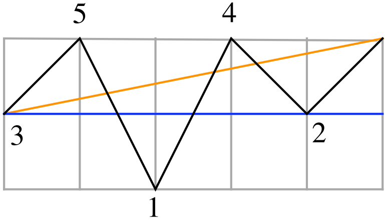
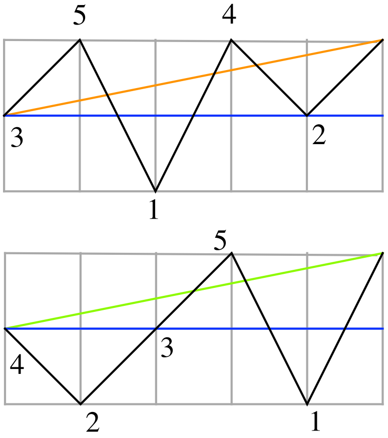


Now let's number the points (except the last point) in increasing order of elevation.



Note that the points are numbered from bottom to top, with points at the same height numbered from right to left. Because of this fact, the points with numbers less than or equal to the number of the starting point lie on or below the axis, but points with greater numbers lie above the *x*-axis.

But since the numbers are defined by increasing elevation, the numbering is compatible with conjugation:



Therefore the conjugate that starts with point *i* has exactly *i* nonpositive points.

To interpret the Catalan number formulas $\frac{1}{n+1}\binom{2n}{n}$ and $\frac{1}{n}\binom{2n}{n-1}$ we need a slight generalization of our theorem. For each path we pick some set of points of the path and call them special points. We require that the starting point be special but the endpoint is not, and that special points are preserved under conjugation.

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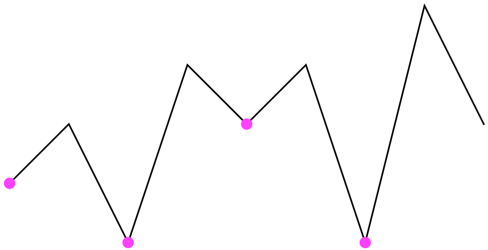

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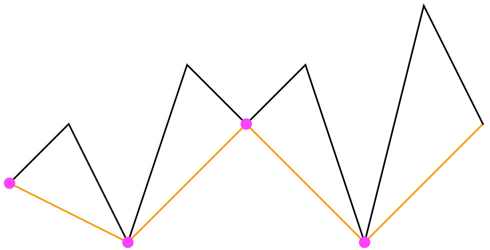

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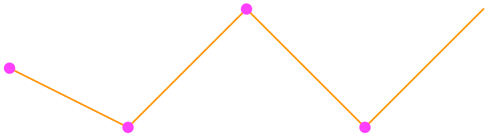

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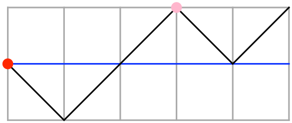

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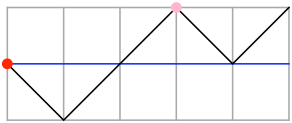

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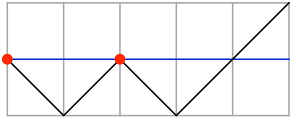

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We get Dyck paths from these paths with $i = 1$ (for $n > 0$) by deleting the first two steps (which must be down and up) and adding a down step at the end:

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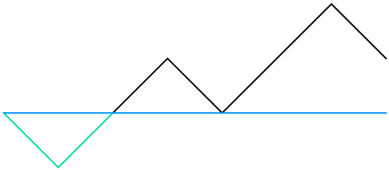

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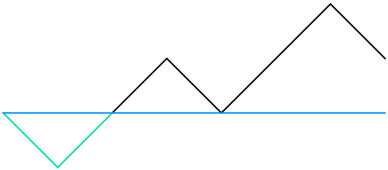

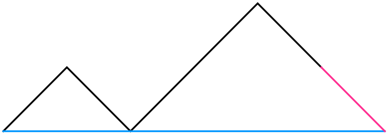

We can restate the theorem by deleting the first (down) step of each path:

Theorem. Among the $\binom{2n}{n-1}$ paths with $n+1$ up steps and $n-1$ down steps, the number with i down steps ending on or below the x-axis, for $i = 0, 1, \ldots n-1$, is $\frac{1}{n}\binom{2n}{n-1}$.

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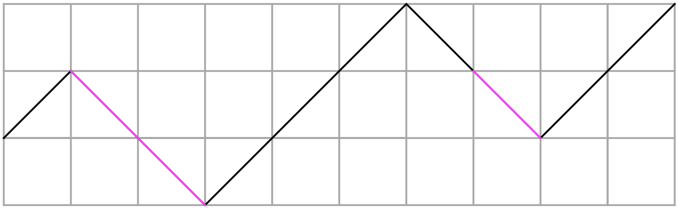

The original Chung-Feller Theorem

We obtain the original Chung-Feller Theorem, for the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$ in the same way. We count paths with $n+1$ up steps and n down steps, and we take the special points to be the starting points of up steps. We find that among these $\binom{2n}{n}$ points, the number with i up steps starting on or below the x-axis is $\frac{1}{n+1}\binom{2n}{n}$. Removing the first step, which is an up step, gives the original Chung-Feller Theorem.

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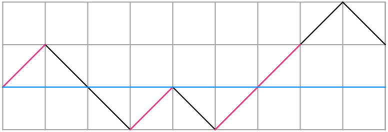

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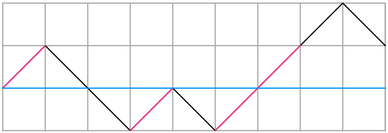
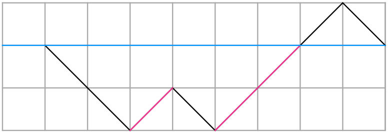


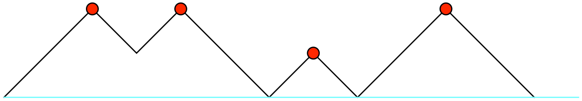


Narayana Numbers

The Narayana numbers may be defined by the formula

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

It is the number of Dyck paths of length $2n$ with k peaks.



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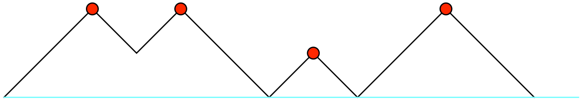

The Naryana number $N(n,k)$ can be written in several ways, each of which has a Chung-Feller Theorem:

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{n+1}\binom{n+1}{k}\binom{n-1}{k-1}$$
$$= \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1} = \frac{1}{n-k}\binom{n}{k-1}\binom{n-1}{k} = \dots$$

We will give the Chung-Feller formula only for the formula $N(n,k) = \frac{1}{n+1}\binom{n+1}{k}\binom{n-1}{k-1}$, which is a refinement of the original Chung-Feller theorem.

We consider paths with $n+1$ up steps and n down steps that start with an up step and have k peaks. It is easy to show that the number of such paths is $\binom{n+1}{k}\binom{n-1}{k-1}$. As in the classical Chung-Feller Theorem, we take the special points to be starting points of up steps. Then every conjugate starting with an up step will also have k peaks.

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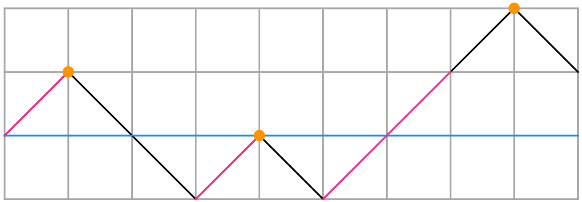
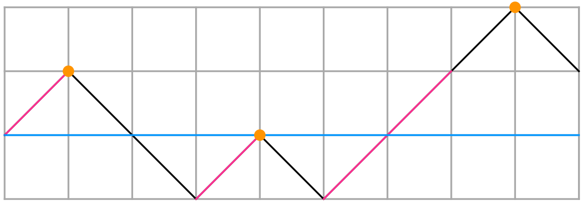
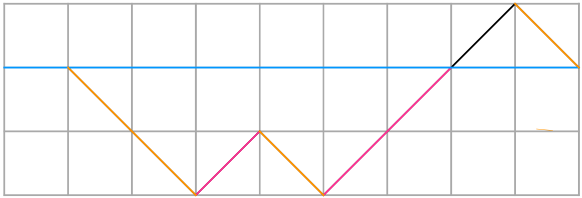
Proceeding as before, we find that the number of paths with $n+1$ up steps, n down steps, and k peaks, with i up steps starting on or below the x-axis, is $N(n,k) = \frac{1}{n+1}\binom{n+1}{k}\binom{n-1}{k-1}$ for $i = 1, \ldots, n+1$.





As before, we can remove the first (up) step, to get a path with n up steps and n down steps, but removing this step may (or may not) destroy the first peak. But an up step in a peak is always immediately followed by a descending run, so removing the first step always leaves a path with k descending runs:



Thus we have a refinement of the original Chung-Feller Theorem:

Theorem. The number of paths with n up steps, n down steps, k descending runs, and $2i$ steps below the x-axis is $N(n,k) = \frac{1}{n+1}\binom{n+1}{k}\binom{n-1}{k-1}$ for $i = 0, \ldots, n$.

Summing on k gives the original Chung-Feller Theorem.

Open Problem. Is there a q-analogue of the Chung-Feller Theorem?

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The q-binomial coefficient $\begin{bmatrix} m+n \\ m \end{bmatrix}$ counts paths with m up steps and n down steps weighted by $q^{\text{major index}}$, where the major index of a path is the sum of the positions of the valleys.

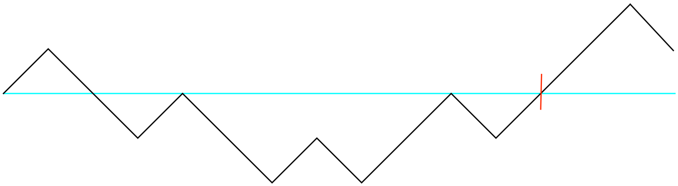
The q-Catalan number $\frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}$ counts Dyck paths by major index. Is there a Chung-Feller type combinatorial explanation?

The Second Chung-Feller Theorem

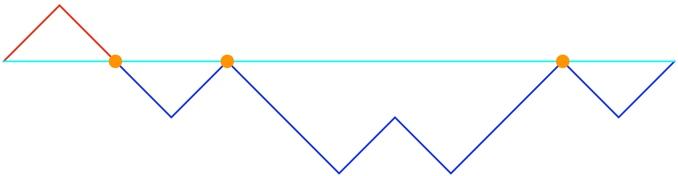
Our main tool for studying the second Chung-Feller Theorem is generating functions. We don't have a general theory, and all we can do is look at various cases which we can compute using generating functions, and see where there is a nice formula. My current student Apratim Roy is working on this problem.

Let us first look at the original second Chung-Feller Theorem. We want to count paths of even length ending anywhere according to the number of steps above and below the x-axis.

We can factor such a path by cutting it at its last return to the x-axis.



Now we find the generating functions for the part of the path up to the last return and the part after the last return. Since both parts must have even length, let's weight each step above the x-axis by $x^{1/2}$ and each step below the x-axis by $y^{1/2}$.
For the first part, we can decompose the path into "positive and negative primes" by cutting the path at each return to the x-axis.



The generating for positive primes is $xc(x)$ and for negative primes is $yc(y)$, where

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is the generating function for the Catalan numbers.

Thus the generating function for all paths ending on the x-axis is

$$\frac{1}{1 - xc(x) - yc(y)}.$$

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The Chung-Feller theorem follows from the identity

$$\frac{1}{1 - xc(x) - yc(y)} = \frac{xc(x) - yc(y)}{x - y} = \sum_{n=0}^{\infty} C_n \sum_{i=0}^{n} x^i y^{n-i}.$$

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The total number of paths of length $2n$ that never return to the x-axis is $\binom{2n}{n}$, so since (for $n > 0$) half of these end above the x-axis and half below the x-axis, the generating function for these paths is

$$\frac{1}{2}\left(\frac{1}{\sqrt{1-4x}} + \frac{1}{\sqrt{1-4y}}\right).$$

So the generating function for all paths of even length is

$$\frac{1}{1 - xc(x) - yc(y)} \cdot \frac{1}{2}\left(\frac{1}{\sqrt{1-4x}} + \frac{1}{\sqrt{1-4y}}\right)$$

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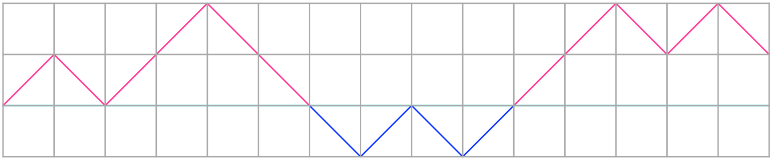
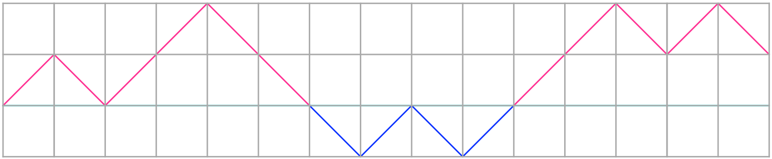
$$\frac{1}{1-xc(x)-yc(y)}\cdot\frac{1}{2}\left(\frac{1}{\sqrt{1-4x}}+\frac{1}{\sqrt{1-4y}}\right)$$
$$=\frac{1}{\sqrt{1-4x}\sqrt{1-4y}}=\sum_{m,n}\binom{2m}{m}\binom{2n}{n}x^m y^n.$$

(The second Chung-Feller theorem.)

Chung and Feller didn't consider paths of odd length. If we have a path of odd length then it will have either an odd number of steps above the x-axis and an even number below, or vice-versa. If it has an odd number of steps above the x-axis then it must end above the x-axis:



Chung and Feller didn’t consider paths of odd length. If we have a path of odd length then it will have either an odd number of steps above the *x*-axis and an even number below, or vice-versa. If it has an odd number of steps above the *x*-axis then it must end above the *x*-axis:



So without loss of generality, we may assume that the path ends above the *x*-axis.

The generating function for paths of odd length that stay strictly above the x-axis is $\frac{x^{1/2}}{\sqrt{1-4x}}$.
So (dividing by $x^{1/2}$) the generating function for paths of odd length that end above the x-axis is

$$\frac{1}{1 - xc(x) - yc(y)} \cdot \frac{1}{\sqrt{1-4x}}$$

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So (dividing by $x^{1/2}$) the generating function for paths of odd length that end above the x-axis is

$$\frac{1}{1 - xc(x) - yc(y)} \cdot \frac{1}{\sqrt{1-4x}} = \sum_{\substack{m \geq 1 \\ n \geq 0}} \frac{m}{2(m+n)} \binom{2m}{m} \binom{2n}{n} x^{m-1} y^n.$$

The coefficient of $x^{m-1}y^n$ here is the number of paths of length $2m + 2n - 1$ with $2m - 1$ steps above the x-axis and $2n$ steps below.

The numbers $\frac{m}{2(m+n)}\binom{2m}{m}\binom{2n}{n}$ are not well known, but they do come up in another lattice path problem: this is the number of paths with $m+n$ up steps and $m+n-1$ down steps whose last return to the x-axis is before the point $(2m, 0)$.