Combinatorial Proofs of Congruences

Ira M. Gessel

Department of Mathematics
Brandeis University

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In 1872 Julius Petersen published a proof of Fermat's theorem $a^p \equiv a \pmod{p}$, where p is a prime:
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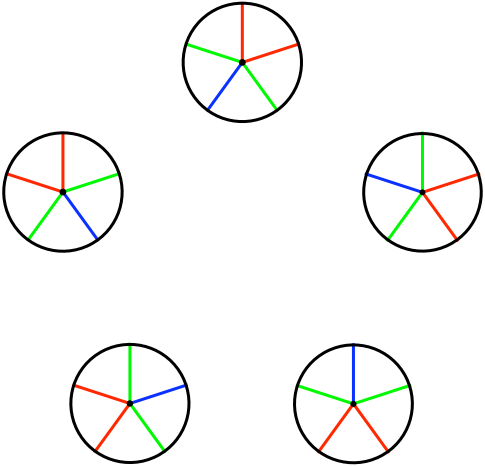

An equivalence class of colorings has size 1 if and only if every spoke has the same color, so there are a of these equivalence classes. Every other equivalence class contains p different colorings, so

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How do we know that every equivalence class has size 1 or p?

The equivalence classes are orbits under the action of a cyclic group of order p, and we know that the size of any orbit divides the order of the group.

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Another useful variation: If a group of order n acts on a set S then $|S|$ is congruent modulo n to the number of elements in orbits of size n.

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But the results we get from Burnside's lemma aren't in general as nice as those we get from counting fixed points.

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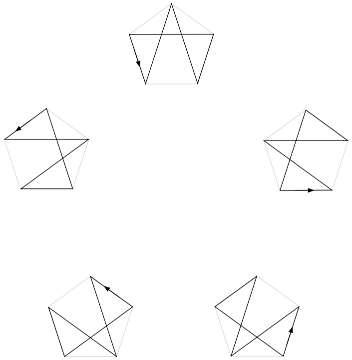

There are $p-1$ fixed cycles so $(p-1)! \equiv p-1 \pmod p$.

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If we take a wheel with p^k spokes and color each spoke in one of a colors, there are a^{p^k} colorings. There are $a^{p^{k-1}}$ colorings that are in orbits of size less than p^k, so

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More generally, we can show that if we take a wheel with n spokes, for any n, then the number of colorings in orbits of size n is $\sum_{d|n} \mu(d) a^{n/d}$, so

$$\sum_{d|n} \mu(d) a^{n/d} \equiv 0 \pmod{n}$$

(Gauss).

Another example of a combinatorial proof of a congruence is Lucas's theorem:

If $a = a_0 + a_1 p + \cdots + a_k p^k$ and $b = b_0 + b_1 p + \cdots + b_k p^k$, where $0 \leq a_i, b_i < p$ then

$$\binom{a}{b} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_k}{b_k} \pmod{p}.$$

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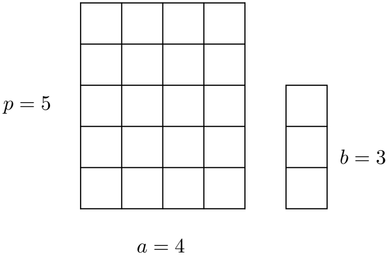
To prove this we take $ap + b$ boxes arranged in a $p \times a$ rectangle with an additional $b < p$ boxes.



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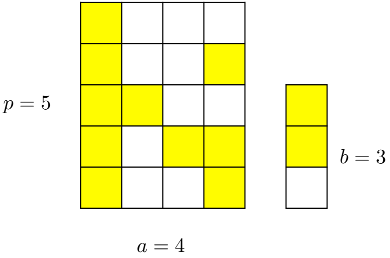

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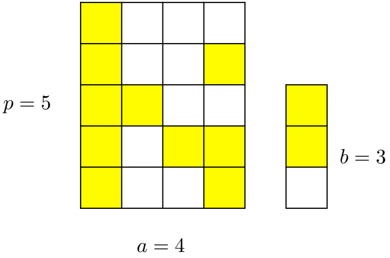

Now we rotate each of the a columns of p boxes independently. Each arrangement will be in an orbit of size divisible by p except for those arrangements that consist only of full and empty columns. Since b and d are less than p, we must choose d boxes from the b additional boxes, and then choose c whole columns from the a columns, which can be done in $\binom{a}{c}\binom{b}{d}$ ways.

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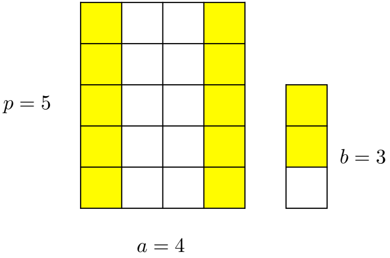

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In fact if $p \geq 5$ then $\binom{ap}{cp} \equiv \binom{a}{c} \pmod{p^3}$. The combinatorial approach reduces this to showing that $\binom{2p}{p} \equiv 2 \pmod{p^3}$. It's probably impossible to prove this combinatorially, but here is a simple proof due to Richard Stanley.

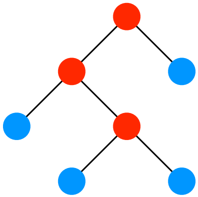
$$\binom{2p}{p} - 2 = \sum_{k=1}^{p-1} \binom{p}{k}^2 = \sum_{k=1}^{p-1} \left[\frac{p}{k}\binom{p-1}{k-1}\right]^2$$
$$= p^2 \sum_{k=1}^{p-1} \frac{1}{k^2}\binom{p-1}{k-1}^2$$

Since $\binom{p-1}{k-1} \equiv \binom{-1}{k-1} = (-1)^{k-1} \pmod{p}$, it's enough to show that $\sum_{k=1}^{p-1} 1/k^2$ is divisible by p. But

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \sum_{k=1}^{p-1} k^2 = \frac{1}{6}p(2p-1)(p-1) \equiv 0 \pmod{p}$$

if $p \neq 2$ or 3.

The Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts, among other things, binary trees with n internal vertices and $n+1$ leaves. For example, if $n = 3$ one such tree is



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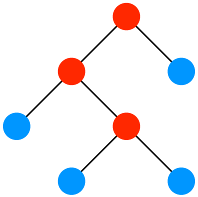
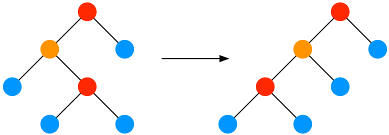


When is C_n odd?

A group of order 2^n acts on the binary trees counted by C_n: For each internal vertex we can switch the two subtrees rooted at its children:



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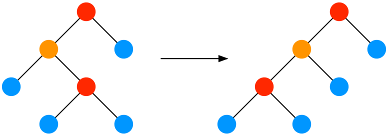
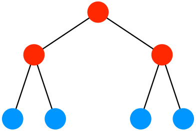


The size of every orbit will be a power of two, and the only orbits of size 1 are for trees in which every leave is at the same level:



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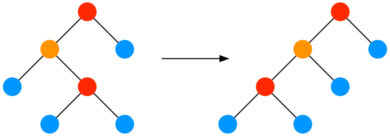

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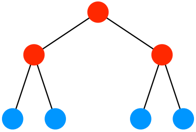

So there are 2^k leaves for some k, so $n = 2^k - 1$. Conversely, if $n = 2^k - 1$ then there is exactly one orbit of size 1, so C_n is odd.

Another class of applications of the combinatorial method is to sequences that counting “labeled objects" like permutations or graphs. For example, the derangement number d_n is the number of permutations of $[n] = \{1, 2, \ldots, n\}$ with no fixed points:

n	0	1	2	3	4	5	6	7	8
d_n	1	0	1	2	9	44	265	1854	14833

We think of a derangement as a set of cycles, each of length greater than 1:

(1 3 6) (2 5) (4 7)

The cyclic group C_n acts on the set of derangements of $[n + m]$ by cyclically permuting $1, 2, \ldots n$:

For $n = 3$ a generator of C_3 takes

(1 3 6) (2 5) (4 7) to (2 1 6) (3 5) (4 7)

If a derangement has elements of $[n]$ and of $[m] + n = \{n+1, n+2, \ldots, n+m\}$ in the same cycle, then it will be in an orbit of size n. Thus $d_{m+n} - d_m d_n$ is divisible by n, i.e.,

$$d_{m+n} \equiv d_m d_n \pmod{n}.$$

For a prime modulus p, we have $d_p \equiv p - 1 \pmod{p}$, so

$$d_{m+p} \equiv (p-1)d_m \equiv -d_m \pmod{p}.$$

The Bell number B_n is the number of partitions of an n-element set.

n	0	1	2	3	4	5	6	7	8	9
B_n	1	1	2	5	15	52	203	877	4140	21147

We will prove Touchard's congruence $B_{n+p} \equiv B_{n+1} + B_n \pmod{p}$, where p is a prime.

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So $B_{n+p} \equiv B_n + B_{n+1} \pmod{p}$.