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Compendium of useful formulas

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Abstract: Almost everything I know.

Contents

1. Miscellaneous math

1.1 A matrix identity

If $\{g_{ab}\}$ is a symmetric matrix, then

$$\det\begin{bmatrix} x & v_b \\ w_a & g_{ab}\end{bmatrix} = x\det\left\{g_{ab} - \frac{1}{x}w_a v_b\right\} = \alpha\det\{g_{ab}\}, \tag{1.1}$$

$$\begin{bmatrix} x & v_b \\ w_a & g_{ab}\end{bmatrix}^{-1} = \frac{1}{\alpha}\begin{bmatrix} 1 & -v^c \\ -w^b & \alpha g^{bc} + w^b v^c\end{bmatrix}, \tag{1.2}$$

$$\left\{g_{ab} - \frac{1}{x}w_a v_b\right\}^{-1} = \left\{g^{bc} + \frac{1}{\alpha}w^b v^c\right\} \tag{1.3}$$

where g^{bc} is the inverse of g_{ab}, $v^c = v_b g^{bc}$, $w^b = g^{bc}w_c$, and $\alpha = x - v_b g^{bc} w_c$.

1.2 Campbell-Baker-Hausdorff

If $[A, B]$ commutes with A and B, then

$$[A, e^B] = [A, B]e^B \tag{1.4}$$

$$e^A B e^{-A} = B + [A, B] \tag{1.5}$$

$$e^A e^B = e^{[A,B]} e^B e^A \tag{1.6}$$

$$e^{A+B} = e^{-[A,B]/2} e^A e^B. \tag{1.7}$$

1.3 Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0\end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}. \tag{1.8}$$

1.4 Level repulsion

In the space of $n \times n$ Hermitian matrices, the matrices with two degenerate eigenvalues lie on a (real) co-dimension 3 submanifold. Therefore, a 1-parameter family of matrices $H(\lambda)$ will not generically intersect the submanifold: as λ is varied, the eigenvalues will move around, but will not cross each other. However, if there is a symmetry, i.e. another Hermitian matrix S that commutes with $H(\lambda)$ for all λ, then the eigenvalues of $H(\lambda)$ belonging to different eigenspaces of S are independent, and will generically cross.

1.5 The Legendre transform

Let $f(x)$ be a function on a vector space that is bounded below and goes to plus infinity faster than linearly in every direction. The Laplace transform of $e^{-f(x)}$ is $e^{g(k)}$, where $g(k)$ is a function on the dual vector space:

$$e^{g(k)} = \int dx\, e^{k_i x^i - f(x)}. \tag{1.9}$$

It is easy to show that $g(k)$ is convex.

The Legendre transform of $f(x)$ is the saddle-point approximation to $g(k)$. Assume first that f is strictly convex and continuously differentiable. Then for every k the function $f(x) - k_i x^i$ has a unique (local and global) minimum, where

$$\frac{\partial f}{\partial x^i} = k_i. \tag{1.10}$$

In other words, we have a bijection between x and k. The saddle point approximation to the integral (1.9) is the following function of k.

$$g(k) = k_i x^i - f(x)\big|_{\frac{\partial f}{\partial x^i}=k_i}. \tag{1.11}$$

This will also be strictly convex and continuously differentiable. The inverse map is given by $x^i = \partial g/\partial k_i$, and we have

$$f(x) = k_i x^i - g(k)\big|_{\frac{\partial f}{\partial x^i}=k_i} = k_i x^i - g(k)\big|_{\frac{\partial g}{\partial k_i}=x^i}. \tag{1.12}$$

In other words, the Legendre transform is its own inverse.

If f is not convex, then for some values of k, $f(x) - k_i x^i$ will have multiple local minima (and saddle points). However, it is the global minimum that dominates the integral (1.9). The x value of this global minimum will jump discontinuously as a function of k. This shows that the Legendre transform $g(k)$ will have a discontinuous first derivative. In fact, the global minimum won't change at all if we replace $f(x)$ with its convex hull (as in the Maxwell construction). Therefore, two functions with the same convex hull will have the same Legendre transform.

If $f(x)$ has a discontinuous first derivative (at a convex point), then the global minimum of $f(x) - k_i x^i$ will be at the same value of x over a range of values of k. Over that range $g(k)$ will be a linear function, and therefore won't be strictly convex.

In summary, on convex functions, having a range over which the function is linear (i.e. not being strictly convex) makes its Legendre transform have a discontinuous first derivative, and vice versa; the Legendre transform continues to be its own inverse. The double Legendre transform of a non-convex function returns its convex hull.

If f depends on an auxiliary parameter y, then g will also depend on y. Their derivatives with respect to y obey a simple relation:

$$\frac{\partial g}{\partial y} = -\frac{\partial f}{\partial y}. \tag{1.13}$$

1.6 Gaussian integrals

$$\int dx\, e^{-ax^2/2} = (2\pi)^{1/2} a^{-1/2} \tag{1.14}$$

$$\int dx\, x^2 e^{-ax^2/2} = (2\pi)^{1/2} a^{-3/2} \tag{1.15}$$

$$\int dx\, x^4 e^{-ax^2/2} = 3(2\pi)^{1/2} a^{-5/2} \tag{1.16}$$

$$\int dx\, x^6 e^{-ax^2/2} = 15(2\pi)^{1/2} a^{-7/2} \tag{1.17}$$

$$\int dx\, x^{2n} e^{-ax^2/2} = (2n-1)!!\,(2\pi)^{1/2} a^{-(n+1/2)} = \Gamma\left(n+\frac{1}{2}\right)\left(\frac{2}{a}\right)^{n+1/2} \tag{1.18}$$

$$\int d^m x\, e^{-A_{ij}x^i x^j} = (2\pi)^{m/2}(\det A)^{-1/2}\,, \qquad \text{(where } A \text{ is symmetric)} \tag{1.19}$$

1.7 Saddle-point approximation

Here we treat only the simplest case of a single real integral of a real function. Let f, g be real analytic functions, with f be bounded below with a single global minimum at x_0. Set

$$f_0 = f(x_0)\,, \quad f_2 = f''(x_0)\,, \quad g_0 = g(x_0)\,, \quad g_1 = g'(x_0)\,, \quad \text{etc.} \tag{1.20}$$

(of course, $f'(x_0) = 0$). We will also assume $f_2, g_0 \neq 0$. Taking ϵ to be a small (positive) parameter, we derive the following approximation, working to the necessary order to get to order ϵ in the final answer:

$$\begin{aligned}
&\int dx\, g(x) e^{-f(x)/\epsilon} \\
&= \int dx\, g(x) \\
&\qquad \times \exp\left(-\frac{1}{\epsilon}\left(f_0 + \frac{1}{2}f_2(x-x_0)^2 + \frac{1}{6}f_3(x-x_0)^3 + \frac{1}{24}f_4(x-x_0)^4 + O(x-x_0)^5\right)\right) \\
&= e^{-f_0/\epsilon}\epsilon^{1/2}\int dy\left(g_0 + \epsilon^{1/2}g_1 y + \frac{1}{2}\epsilon g_2 y^2 + O(\epsilon^{3/2})\right) \\
&\qquad\qquad \times \exp\left(-\frac{1}{2}f_2 y^2 - \frac{1}{6}\epsilon^{1/2}f_3 y^3 - \frac{1}{24}\epsilon f_4 y^4 + O(\epsilon^{3/2})\right) \\
&= e^{-f_0/\epsilon}\epsilon^{1/2}g_0\int dy\left(1 + \epsilon^{1/2}\frac{g_1}{g_0}y + \frac{1}{2}\epsilon\frac{g_2}{g_0}y^2 + O(\epsilon^{3/2})\right) \\
&\qquad\qquad \times \left(1 - \frac{1}{6}\epsilon^{1/2}f_3 y^3 - \frac{1}{24}\epsilon f_4 y^4 + \frac{1}{72}\epsilon f_3^2 y^6 + O(\epsilon^{3/2})\right) e^{-f_2 y^2/2} \\
&= e^{-f_0/\epsilon}\epsilon^{1/2}g_0\int dy\left(1 + \epsilon\left(\frac{g_2}{2g_0}y^2 - \frac{f_3 g_1}{6g_0}y^4 - \frac{f_4}{24}y^4 + \frac{f_3^2}{72}y^6\right) + O(\epsilon^2)\right) e^{-f_2 y^2/2} \\
&= e^{-f_0/\epsilon}\epsilon^{1/2}g_0\left(\frac{2\pi}{f_2}\right)^{1/2}\left(1 + \epsilon\left(\frac{g_2}{2g_0 f_2} - \frac{f_3 g_1}{2g_0 f_2^2} - \frac{f_4}{8f_2^2} + \frac{5f_3^2}{24f_2^3}\right) + O(\epsilon^2)\right),
\end{aligned} \tag{1.21}$$

In the second equality we changed the integration variable to $y = \epsilon^{-1/2}(x - x_0)$. In the fourth equality we dropped odd powers of y, since those will integrate to 0. The result is conveniently written as a decreasing series in ϵ for its logarithm:

$$\ln \int dx\, g(x) e^{-f(x)/\epsilon} = -\frac{f_0}{\epsilon} + \frac{1}{2} \ln\left(\frac{2\pi\epsilon g_0^2}{f_2}\right) + \epsilon\left(\frac{g_2}{2g_0 f_2} - \frac{f_3 g_1}{2g_0 f_2^2} - \frac{f_4}{8f_2^2} + \frac{5f_3^2}{24f_2^3}\right) + O(\epsilon^2)\,. \tag{1.22}$$

In the case $g_0 = 0$ (but $g_2 - f_3 g_1/f_2 \neq 0$), one gets instead

$$\int dx\, g(x) e^{-f(x)/\epsilon} = e^{-f_0/\epsilon}\epsilon^{3/2}\left(\frac{\pi}{2f_2^3}\right)^{1/2}\left(g_2 - \frac{f_3 g_1}{f_2} + O(\epsilon)\right) \tag{1.23}$$

$$\ln \int dx\, g(x) e^{-f(x)/\epsilon} = -\frac{f_0}{\epsilon} + \frac{1}{2}\ln\left(\frac{\pi\epsilon^3}{2f_2^3}\right) + \ln\left(g_2 - \frac{f_3 g_1}{f_2}\right) + O(\epsilon)\,. \tag{1.24}$$

If there is another local minimum, or if the integration region is bounded (but includes x_0 in its interior), then this series is asymptotic, and there are non-perturbative corrections to (1.21) that go like $e^{-f(x_1)/\epsilon}$ (or to (1.22) like $e^{(f_0 - f(x_1))/\epsilon}$), where x_1 is the local minimum (other than x_0) or integration endpoint with the lowest value of f.

1.8 Cauchy principal value

The Cauchy principal value of an integral where the integrand $f(x)$ is singular at b is

$$\mathrm{P}\int_a^c dx\, f(x) := \lim_{\epsilon\to 0}\left[\int_a^{b-\epsilon} dx\, f(x) + \int_{b+\epsilon}^c dx\, f(x)\right]. \tag{1.25}$$

For a function $f(z)$ on the complex plane with a singularity at b, the Cauchy principal value of the integral over a contour C containing b is

$$\mathrm{P}\int_C dz\, f(z) := \lim_{\epsilon\to 0}\int_{C\setminus D_\epsilon} dz f(z)\,, \tag{1.26}$$

where D_ϵ is the disk of radius ϵ about b. In particular, if f is analytic and contains a pole at b, then the principal value is the average of the integral with the contour deformed to go around the pole both ways.

If we define $[\frac{1}{x}]$ to be the distribution whose integral against a test function f is

$$\int dx \left[\frac{1}{x}\right] f(x) := \mathrm{P}\int dx\, \frac{f(x)}{x}\,, \tag{1.27}$$

then we have

$$\frac{1}{x + i\epsilon} = \left[\frac{1}{x}\right] - i\pi\delta(x)\,. \tag{1.28}$$

1.9 A few Fourier transforms

According to the definition

$$\tilde{f}(k) := \int dx\, e^{-ikx} f(x)\,, \qquad f(x) = \frac{1}{2\pi}\int dk\, e^{ikx}\tilde{f}(k) \tag{1.29}$$

of the Fourier transform, we have:

$$\begin{array}{c|c} f(x) & \tilde{f}(k) \\ \hline g(x)^* & \tilde{g}(-k)^* \\ g(-x) & \tilde{g}(-k) \\ g(-x)^* & \tilde{g}(k)^* \\ g'(x) & ik\tilde{g}(k) \\ -ixg(x) & \tilde{g}'(k) \\ g(x-a) & e^{-ika}\tilde{g}(k) \\ e^{iax}g(x) & \tilde{g}(k-a) \\ 1 & 2\pi\delta(k) \\ e^{ik_0x} & 2\pi\delta(k-k_0) \\ \delta(x) & 1 \\ \delta(x-x_0) & e^{-ikx_0} \\ \theta(x) & -\frac{i}{k-i\epsilon} \\ \theta(-x) & \frac{i}{k+i\epsilon} \\ \frac{1}{x+i\epsilon} & -2\pi i\theta(k) \\ \frac{1}{x-i\epsilon} & 2\pi i\theta(-k) \\ [\frac{1}{x}] & \pi i(\theta(-k)-\theta(k)) \\ \theta(x)-\theta(-x) & -2i[\frac{1}{k}] \end{array} \tag{1.30}$$

2. Differential geometry

2.1 Diffeomorphisms vs. coordinate tranformations

Physicists often use the terms "diffeomorphism" and "change of coordinates" interchangeably. Here I want to make the relationship precise: given a coordinate system on a manifold, its diffeomorphisms are in one-to-one correspondence with other coordinate systems *that have the same transition functions on the patch overlaps.*

Let M be a manifold. A "coordinate system" or "atlas of charts" is a set of open subsets ("patches") $\{U_\alpha \subset M\}$ covering M (i.e. $\cup_\alpha U_\alpha = M$), together with a one-to-one function ("coordinates") from each patch to $\mathbf{R}^D$ (or $\mathbf{C}^n$ for a complex manifold)

$$x_\alpha : U_\alpha \to \mathbf{R}^D. \tag{2.1}$$

Then for every overlap $U_\alpha \cap U_\beta$, the transition function

$$f_{\alpha\beta} = x_\alpha x_\beta^{-1} : x_\beta(U_\alpha \cap U_\beta) \to \mathbf{R}^D, \tag{2.2}$$

is required to be continuous, differentiable, holomorphic, etc. depending on what kind of manifold you want.

Given such a coordinate system $\{(U_\alpha, x_\alpha)\}$, the diffeomorphisms of M onto itself are in one-to-one correspondence with coordinate systems $\{(U'_\alpha, x'_\alpha)\}$ that have the same images in $\mathbf{R}^D$ and the same transition functions:

$$x'_\alpha(U'_\alpha) = x_\alpha(U_\alpha), \qquad f'_{\alpha\beta} = f_{\alpha\beta} \qquad \forall \alpha, \beta. \tag{2.3}$$

2.2 Connection and curvatures

Antisymmetrization notation:

$$u_{[ab]} := \frac{1}{2}(u_{ab} - u_{ba}) \tag{2.4}$$

Covariant derivatives:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) \tag{2.5}$$

$$\Gamma^a_{ab} = \partial_b \ln \sqrt{|g|} \tag{2.6}$$

$$\nabla_a v^b = \partial_a v^b + \Gamma^b_{ac} v^c \tag{2.7}$$

$$\nabla_a v_b = \partial_a v_b - \Gamma^c_{ab} v_c \tag{2.8}$$

$$\nabla_a v^a = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} v^a) \tag{2.9}$$

$$[v, w]^a = v^b \nabla_b w^a - w^b \nabla_b v^a = v^b \partial_b w^a - w^b \partial_b v^a \tag{2.10}$$

Geodesic equation:

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0 \tag{2.11}$$

Riemann tensor:

$$R_{abc}{}^d = -2\partial_{[a} \Gamma^d_{b]c} + 2\Gamma^e_{c[a} \Gamma^d_{b]e} \tag{2.12}$$

$$R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc} \tag{2.13}$$

$$R_{[abc]}{}^d = 0 \tag{2.14}$$

$$\nabla_{[a} R_{bc]d}{}^e = 0 \tag{2.15}$$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v_c = R_{abc}{}^d v_d \tag{2.16}$$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v^d = -R_{abc}{}^d v^c \tag{2.17}$$

Ricci tensor and scalar:

$$R_{ac} = R_{abc}{}^b \tag{2.18}$$

$$R = R_a{}^a \tag{2.19}$$

Einstein tensor:

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \tag{2.20}$$

$$\nabla^a G_{ab} = 0 \tag{2.21}$$

Weyl tensor:

$$C_{abcd} = R_{abcd} - \frac{2}{D-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{2}{(D-1)(D-2)}Rg_{a[c}g_{d]b} \tag{2.22}$$

$$C_{abcd} = C_{cdab} = -C_{bacd} = -C_{abdc} \tag{2.23}$$

$$C_{[abc]}{}^{d} = 0 \tag{2.24}$$

$$C_{abc}{}^{a} = C_{abc}{}^{b} = 0 \tag{2.25}$$

$D = 2$:

$$K = \frac{1}{2}R \quad \text{(Gaussian curvature)} \tag{2.26}$$

$$R_{ab} = \frac{1}{2}g_{ab} \tag{2.27}$$

$$G_{ab} = 0 \tag{2.28}$$

$$R_{abcd} = Rg_{a[c}g_{b]d} \tag{2.29}$$

$$\tag{2.30}$$

$D = 3$:

$$R_{abcd} = 2(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - Rg_{a[c}g_{d]b} \tag{2.31}$$

$$C_{abcd} = 0 \tag{2.32}$$

$$C^{a}{}_{d} = \epsilon^{abc}\nabla_b\left(R_{cd} - \frac{1}{4}Rg_{cd}\right) \quad \text{(Cotton tensor)} \tag{2.33}$$

$$C^{a}{}_{a} = 0 \tag{2.34}$$

$$C_{ab} = C_{ba} \tag{2.35}$$

$$\nabla_a C^{a}{}_{b} = 0 \tag{2.36}$$

2.3 Geodesic deviation equation

Given a 1-parameter family of curves $\gamma_s(t)$, where t is a parameter along each curve and s parametrizes the choice of curve, define tangent vectors $T := \partial/\partial t$, i.e. $T^a := \partial x^a/\partial t$ (velocity) and $X := \partial/\partial s$, i.e. $X^a := \partial x^a/\partial s$ (deviation vector). The rate of change of the deviation vector with t, or the relative velocity of points on curves γ_s and γ_{s+ds}, is described by the vector field

$$v^a := T^b\nabla_b X^a = X^b\nabla_b T^a\,, \tag{2.37}$$

where the equality is due to the fact that $[X, T] = 0$. The relative acceleration is given by

$$a^a := T^b\nabla_b v^a = X^c\nabla_c(T^b\nabla_b T^a) - R_{bcd}{}^{a}X^cT^bT^d\,. \tag{2.38}$$

If the curves are geodesics then $T^b\nabla_b T^a = 0$ and

$$a^a = -R_{bcd}{}^{a}X^cT^bT^d \qquad \text{(geodesic deviation equation)}\,. \tag{2.39}$$

Note that in this case $a^aT_a = 0$. In fact, by an s-dependent redefinition of t, we can set $X^aT_a = 0$ everywhere, so $v^aT_a = 0$ as well.

2.4 Forms

Let v be a p-form and w a q-form.[1]

$$v = \frac{1}{p!} v_{a_1 \dots a_p} dx^{a_1} \wedge \cdots \wedge dx^{a_p} \tag{2.43}$$

$$(v \wedge w)_{a_1 \dots a_p b_1 \dots b_q} = \frac{(p+q)!}{p!q!} v_{[a_1 \dots a_p} w_{b_1 \dots b_q]} \tag{2.44}$$

$$v \wedge w = (-1)^{pq} w \wedge v \tag{2.45}$$

$$d = dx^a \wedge \partial_a \tag{2.46}$$

$$(dv)_{a_1 \dots a_{p+1}} = (p+1)\partial_{[a_1} v_{a_2 \dots a_{p+1}]} \tag{2.47}$$

$$\int_{\mathcal{M}} dv = \int_{\partial \mathcal{M}} v \qquad (\mathcal{M} \text{ is } p+1 \text{ dimensional}) \tag{2.48}$$

$$d(v \wedge w) = dv \wedge w + (-1)^p v \wedge dw \tag{2.49}$$

$$(*v)_{a_1 \dots a_{D-p}} = \frac{1}{p!} \epsilon_{a_1 \dots a_{D-p}}{}^{b_1 \dots b_p} v_{b_1 \dots b_p} \tag{2.50}$$

$$(*v) \wedge v = * \left(\frac{1}{p!} v_{a_1 \dots a_p} v^{a_1 \dots a_p} \right) = *|v|^2 \tag{2.51}$$

$$\int *\phi = \int d^D x \sqrt{|g|} \phi \quad (\phi \text{ is a scalar}) \tag{2.52}$$

Euclidean signature:

$$\epsilon_{1 \dots D} = \sqrt{|g|} \tag{2.53}$$

$$\epsilon^{1 \dots D} = \frac{1}{\sqrt{|g|}} \tag{2.54}$$

$$* * v = (-1)^{p(D-p)} v \tag{2.55}$$

$$d^\dagger v = (-1)^{Dp} * d * v = -\frac{1}{(p-1)!} \nabla_{a_1} v^{a_1}{}_{a_2 \dots a_p} dx^{a_2} \wedge \cdots \wedge dx^{a_p} \tag{2.56}$$

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[1] The definition of the Hodge star depends on who you ask. Above we gave Polchinski's definition. However, another common one is Wald's,

$$(*_{\mathrm{W}} v)_{a_1 \dots a_{D-p}} = \frac{1}{p!} v_{b_1 \dots b_p} \epsilon^{b_1 \dots b_p}{}_{a_1 \dots a_{D-p}} , \tag{2.40}$$

which is related to Polchinski's by $*_{\mathrm{W}} v = (-1)^{p(D-p)} * v$. I believe that all of the equations in this subsection continue to hold under $* \to *_{\mathrm{W}}$, except for (2.51), which becomes

$$v \wedge (*_{\mathrm{W}} v) = *_{\mathrm{W}} |v|^2 , \tag{2.41}$$

and (2.56),(2.60), which become

$$d^\dagger v = (-1)^{Dp+D+s} *_{\mathrm{W}} d *_{\mathrm{W}} v = -\frac{1}{(p-1)!} \nabla_{a_1} v^{a_1}{}_{a_2 \dots a_p} dx^{a_2} \wedge \cdots \wedge dx^{a_p} \tag{2.42}$$

($s = 1$ for Euclidean signature, $s = 0$ for Lorentzian signature).

Lorentzian signature:

$$\epsilon_{0\ldots(D-1)} = \sqrt{|g|} \tag{2.57}$$

$$\epsilon^{0\ldots(D-1)} = -\frac{1}{\sqrt{|g|}} \tag{2.58}$$

$$**v = (-1)^{p(D-p)+1}v \tag{2.59}$$

$$d^\dagger v = (-1)^{Dp+1} * d * v = -\frac{1}{(p-1)!}\nabla_{a_1}v^{a_1}{}_{a_2\ldots a_p}dx^{a_2}\wedge\cdots\wedge dx^{a_p} \tag{2.60}$$

2.5 Harmonic forms

The natural inner product on forms of the same degree is

$$\langle v, w\rangle := \frac{1}{p!}\int d^Dx\sqrt{|g|}g^{a_1b_1}\cdots g^{a_pb_p}v_{a_1\cdots a_p}w_{b_1\cdots b_p} = \int *v\wedge w\,, \tag{2.61}$$

where v and w are both p-forms. If the manifold is closed, or boundary conditions are imposed so that boundary terms vanish, then

$$\langle dv, w\rangle = \langle v, d^\dagger w\rangle \tag{2.62}$$

and

$$\langle v, dw\rangle = \langle d^\dagger v, w\rangle\,. \tag{2.63}$$

v is *harmonic* if

$$dd^\dagger v + d^\dagger dv = 0\,. \tag{2.64}$$

For the rest of this subsection we consider a closed Riemannian manifold. A harmonic form v is then both closed, $dv = 0$, and co-closed, $d^\dagger v = 0$:

$$dd^\dagger v + d^\dagger dv = 0 \;\Rightarrow\; \langle v, dd^\dagger v + d^\dagger v\rangle = 0 \tag{2.65}$$

$$\Rightarrow\; \langle d^\dagger v, d^\dagger v\rangle + \langle dv, dv\rangle = 0 \tag{2.66}$$

$$\Rightarrow\; \langle d^\dagger v, d^\dagger v\rangle = \langle dv, dv\rangle = 0 \tag{2.67}$$

$$\Rightarrow\; d^\dagger v = 0\,,\; dv = 0 \tag{2.68}$$

where in the last two steps we used the fact that the inner product is positive definite. The *Hodge decomposition theorem* states that any form can be decomposed uniquely into the sum of an exact form dw, a co-exact form $d^\dagger u$, and a harmonic form x:

$$v = dw + d^\dagger u + x\,. \tag{2.69}$$

The uniqueness can be seen as follows. First, a form that is both exact and co-exact necessarily vanishes, since if $dw = d^\dagger u$ then

$$\langle dw, dw\rangle = \langle dw, d^\dagger u\rangle = \langle w, d^{\dagger 2}u\rangle = 0 \tag{2.70}$$

so $dw = 0$. Second, a form that is both exact and harmonic also vanishes, since if dw is harmonic, then $d^\dagger dw = 0$, which implies $0 = \langle w, d^\dagger dw\rangle = \langle dw, dw\rangle$, hence $dw = 0$. Finally, a form that is both co-exact and harmonic similarly vanishes.

A closed form w can be written as an exact form plus a harmonic form. This follows from the Hodge decomposition theorem (2.69). If $dv = 0$ then $dd^\dagger u = 0$, which implies

$$0 = \langle u, dd^\dagger u\rangle = \langle d^\dagger u, d^\dagger u\rangle \tag{2.71}$$

so $d^\dagger u = 0$, so

$$v = dw + x\,. \tag{2.72}$$

(2.72) implies that every cohomology class contains a harmonic representative. Furthermore, this representative is unique, since, as explained above, a form that is both harmonic and exact must vanish. The harmonic representative is distinguished by the fact that it is co-closed, $d^\dagger v = 0$. It also minimizes the norm $\langle v, v\rangle$ within the class. This can be seen as follows. The functional $\langle v, v\rangle$ is convex, so an extremum is necessarily a minimum. At the extremum we have, for an arbitrary $p-1$ form w and to first order in ϵ,

$$0 = \langle v + \epsilon dw, v + \epsilon dw\rangle - \langle v, v\rangle = \epsilon(\langle v, dw\rangle + \langle dw, v\rangle) = \epsilon(\langle d^\dagger v, w\rangle + \langle w, d^\dagger v\rangle) \tag{2.73}$$

Setting $w = d^\dagger v$, we see that $d^\dagger v = 0$, so v is harmonic.

2.6 Vielbein formalism

Let μ, ν be indices in the vector representation of the appropriate Lorentz group (raised and lowered with $\eta_{\mu\nu}$), and a, b be the usual tangent and co-tangent space indices. The vielbein 1-forms e_μ are defined by

$$(e_\mu)_a (e^\mu)_b = g_{ab} \tag{2.74}$$

which is equivalent to

$$(e_\mu)^a (e_\nu)_a = \eta_{\mu\nu}. \tag{2.75}$$

The connection 1-forms $\omega_\nu{}^\mu$ are valued in the Lie algebra of the Lorentz group (i.e. $\omega_{\mu\nu} = -\omega_{\nu\mu}$), and defined by

$$de_\sigma = e_\mu \wedge \omega_\sigma{}^\mu \tag{2.76}$$

or in components

$$(\omega_{\mu\nu})_a = (e_\mu)^b \nabla_a (e_\nu)_b. \tag{2.77}$$

(Wald writes this $\omega_{a\mu\nu}$.) The Riemann 2-forms $R_\mu{}^\nu$ are also valued in the Lie algebra of the Lorentz group, and are defined by

$$(R_\mu{}^\nu)_{ab} = (e_\mu)^c (e^\nu)_d R_{abc}{}^d. \tag{2.78}$$

They are given by

$$R_\mu{}^\nu = d\omega_\mu{}^\nu + \omega_\mu{}^\alpha \wedge \omega_\alpha{}^\nu. \tag{2.79}$$

In order to calculate the connection 1-forms from the vielbein, one may do it either by inspection from (2.76), or, more algorithmically, as follows. First compute

$$K_{\mu\sigma\nu} \equiv (e_\sigma)_a [e_\mu, e_\nu]^a \,, \tag{2.80}$$

which equals $\omega_{\mu\sigma\nu} - \omega_{\nu\sigma\mu}$, and from that compute

$$\omega_{\mu\sigma\nu} = \frac{1}{2}\left(K_{\mu\sigma\nu} + K_{\sigma\mu\nu} + K_{\sigma\nu\mu}\right) . \tag{2.81}$$

The scalars $\omega_{\mu\sigma\nu}$ are called Ricci rotation coefficients. The connection 1-forms can be computed from them:

$$(\omega_{\sigma\nu})_a = (e^\mu)_a \omega_{\mu\sigma\nu} \,. \tag{2.82}$$

(Note that it is the *first* index on the Ricci rotation coefficients that becomes the 1-form index on the connection.) The Riemann tensor can then be computed either using (2.78) or directly from the Ricci rotation coefficients:

$$R_{\rho\sigma\mu\nu} = (e_\rho)^a \partial_a \omega_{\sigma\mu\nu} - (e_\sigma)^a \partial_a \omega_{\rho\mu\nu} - \eta^{\alpha\beta}\left(\omega_{\rho\beta\mu}\omega_{\sigma\alpha\nu} - \omega_{\sigma\beta\mu}\omega_{\rho\alpha\nu} + \omega_{\rho\beta\sigma}\omega_{\alpha\mu\nu} - \omega_{\sigma\beta\rho}\omega_{\alpha\mu\nu}\right) . \tag{2.83}$$

The Christoffel symbols can also be computed from the connection 1-forms:

$$\Gamma^c_{ab} = \eta^{\mu\nu}(e_\mu)^c \left(\partial_a (e_\nu)_b - \eta^{\alpha\beta}(e_\alpha)_b (\omega_{\beta\nu})_a\right) . \tag{2.84}$$

2.7 Lie derivatives and Killing fields

Let v^a be a vector field. Then

$$\pounds_v w^a = [v,w]^a = v^b \partial_b w^a - w^b \partial_b v^a = v^b \nabla_b w^a - w^b \nabla_b v^a \tag{2.85}$$

$$\pounds_v w_a = v^b \partial_b w_a + w_b \partial_a v^b = v^b \nabla_b w_a + w_b \nabla_a v^b \tag{2.86}$$

$$\pounds_v g_{ab} = \nabla_a v_b + \nabla_b v_a \tag{2.87}$$

If in some coordinate system the components of v are constant, $\partial_a v^b = 0$, then in that coordinate system the Lie derivative coincides with the ordinary derivative:

$$\pounds_v = v^a \partial_a . \tag{2.88}$$

Under a small diffeomorphism

$$x^a = x'^a + \epsilon v^a , \tag{2.89}$$

a tensor T will transform into

$$T' = T + \epsilon \pounds_v T . \tag{2.90}$$

So this diffeomorphism is an isometry if

$$\pounds_v g_{ab} = 0 \tag{2.91}$$

(Killing's equation).

2.8 Weyl transformations

Let

$$\tilde{g}_{ab} = e^{2\omega} g_{ab}. \tag{2.92}$$

Then

$$\sqrt{|\tilde{g}|} = e^{D\omega}\sqrt{|g|} \tag{2.93}$$

$$\tilde{g}^{ab} = e^{-2\omega} g^{ab} \tag{2.94}$$

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \partial_b\omega\delta^a_c + \partial_c\omega\delta^a_b - \partial^a\omega g_{bc} \tag{2.95}$$

$$\tilde{R}_{abc}{}^d = R_{abc}{}^d + 2\delta^d_{[a}\left(\nabla_{b]} + \partial_{b]}\omega\right)\partial_c\omega - 2g_{c[a}\left(\nabla_{b]} + \partial_{b]}\omega\right)\partial^d\omega - 2g_{c[a}\delta^d_{b]}\partial_e\omega\partial^e\omega \tag{2.96}$$

$$\tilde{R}_{ac} = R_{ac} - g_{ac}\nabla^2\omega + (D-2)\Big(-\nabla_a\partial_c\omega + \partial_a\omega\partial_c\omega - g_{ac}\partial_d\omega\partial^d\omega\Big) \tag{2.97}$$

$$\tilde{R} = e^{-2\omega}\left(R - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial_a\omega\partial^a\omega\right) \tag{2.98}$$

$$\tilde{C}_{abc}{}^d = C_{abc}{}^d \tag{2.99}$$

$$\tilde{C}_{ab} = e^{-\omega} C_{ab} \qquad (D=3) \tag{2.100}$$

In $D = 2$, all metrics are locally conformally flat. In $D = 3$, the metric is locally conformally flat if and only if $C_{ab} = 0$. In $D \geq 4$, the metric is locally conformally flat if and only if $C_{abcd} = 0$.

2.9 Metric perturbations

Let us perturb the metric g_{ab} by

$$\delta g_{ab} = h_{ab}. \tag{2.101}$$

Then, to first order in h_{ab} (and always using g_{ab} to raise and lower indices),

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}h \tag{2.102}$$

$$\delta g^{ab} = -h^{ab} \tag{2.103}$$

$$\delta\Gamma^a_{bc} = \nabla_{(b}h_{c)}{}^a - \frac{1}{2}\nabla^a h_{bc} \tag{2.104}$$

$$\delta R_{abc}{}^d = -2\nabla_{[a}\delta\Gamma^d_{b]c} \tag{2.105}$$

$$\delta R_{ab} = \nabla_c\nabla_{(a}h_{b)}{}^c - \frac{1}{2}\nabla_a\partial_b h - \frac{1}{2}\nabla^2 h_{ab} = \Delta_{\mathrm{L}}h_{ab} - \nabla_{(a}v_{b)}, \tag{2.106}$$

$$\Delta_{\mathrm{L}}h_{ab} \equiv -\frac{1}{2}\nabla^2 h_{ab} - R_a{}^c{}_b{}^d h_{cd} + R_{(a}{}^c h_{b)c}\,, \qquad v_a \equiv \frac{1}{2}\partial_a h - \nabla_b h^b{}_a \tag{2.107}$$

$$\delta R = \nabla_a\nabla_b h^{ab} - \nabla^2 h - h^{ab}R_{ab} \tag{2.108}$$

2.10 Warped products

Consider the warped product geometry

$$ds^2 = ds^2_{(1)} + e^{2\tau(x_{(1)})}ds^2_{(2)} \tag{2.109}$$

$$= g_{mn}(x^m)dx^m dx^n + e^{2\tau(x^m)}g_{\mu\nu}(x^\mu)dx^\mu dx^\nu, \tag{2.110}$$

where $x_{(1)} = x^m$ and $x_{(2)} = x^\mu$. Let k be the dimensionality of the space $ds^2_{(2)}$. The Christoffel symbols are:

$$\Gamma^\mu_{\nu\rho} = \Gamma^{(2)\mu}_{\nu\rho} \tag{2.111}$$
$$\Gamma^\mu_{\nu p} = \partial_p \tau \delta^\mu_\nu \tag{2.112}$$
$$\Gamma^m_{\nu\rho} = -g^{mn} \partial_n \tau e^{2\tau} g_{\nu\rho} \tag{2.113}$$
$$\Gamma^m_{n\rho} = \Gamma^\mu_{np} = 0 \tag{2.114}$$
$$\Gamma^m_{np} = \Gamma^{(1)m}_{np} \tag{2.115}$$

The Riemann tensor is:

$$R_{\mu\nu\rho}{}^\sigma = R^{(2)}_{\mu\nu\rho}{}^\sigma + e^{2\tau} g^{mn} \partial_m \tau \partial_n \tau \left(g_{\nu\rho} \delta^\sigma_\mu - g_{\mu\rho} \delta^\sigma_\nu \right) \tag{2.116}$$
$$R_{\mu\nu\rho}{}^s = R_{\mu\nu p}{}^\sigma = R_{m\nu\rho}{}^\sigma = R_{\mu n\rho}{}^\sigma = 0 \tag{2.117}$$
$$R_{mn\rho}{}^\sigma = R_{\mu\nu p}{}^s = 0 \tag{2.118}$$
$$R_{m\nu p}{}^\sigma = -\left(\nabla^{(1)}_m \partial_p \tau + \partial_m \tau \partial_p \tau \right) \delta^\sigma_\nu \tag{2.119}$$
$$R_{\mu np}{}^\sigma = \left(\nabla^{(1)}_n \partial_p \tau + \partial_n \tau \partial_p \tau \right) \delta^\sigma_\mu \tag{2.120}$$
$$R_{m\nu\rho}{}^s = g^{sa} \left(\nabla^{(1)}_m \partial_a \tau + \partial_m \tau \partial_a \tau \right) e^{2\tau} g_{\nu\rho} \tag{2.121}$$
$$R_{\mu n\rho}{}^s = -g^{sa} \left(\nabla^{(1)}_n \partial_a \tau + \partial_n \tau \partial_a \tau \right) e^{2\tau} g_{\mu\rho} \tag{2.122}$$
$$R_{mnp}{}^\sigma = R_{\mu np}{}^s = R_{m\nu p}{}^s = R_{mn\rho}{}^s = 0 \tag{2.123}$$
$$R_{mnp}{}^s = R^{(1)}_{mnp}{}^s \tag{2.124}$$

The Ricci tensor is:

$$R_{\mu\rho} = R^{(2)}_{\mu\rho} - \left(\nabla^2_{(1)} \tau + k g^{mn} \partial_m \tau \partial_n \tau \right) e^{2\tau} g_{\mu\rho} \tag{2.125}$$
$$R_{\mu p} = 0 \tag{2.126}$$
$$R_{mp} = R^{(1)}_{mp} - k \left(\nabla^{(1)}_m \partial_p \tau + \partial_m \tau \partial_p \tau \right) \tag{2.127}$$

The Ricci scalar is:

$$R = R_{(1)} + e^{-2\tau} R_{(2)} - 2k \nabla^2_{(1)} \tau - k(k+1) g^{mn} \partial_m \tau \partial_n \tau \tag{2.128}$$

2.11 Flat fibrations

Consider the following metric ansatz:

$$ds^2 = g_{mn}(x) dx^m dx^n + h_{\mu\nu}(x) dy^\mu dy^\nu . \tag{2.129}$$

We denote quantities derived from the base metric $d\hat{s}^2 = g_{mn} dx^m dx^n$ by hats. The Christoffel symbols are (here and below we write only non-zero components):

$$\Gamma^l_{mn} = \hat{\Gamma}^l_{mn} \tag{2.130}$$

$$\Gamma^m_{\mu\nu} = -\frac{1}{2} g^{mn} \partial_n h_{\mu\nu} \tag{2.131}$$

$$\Gamma^\mu_{m\nu} = \Gamma^\mu_{\nu m} = \frac{1}{2} h^{\mu\lambda} \partial_m h_{\lambda\nu}. \tag{2.132}$$

The Riemann tensor is:

$$R_{mnpl} = \hat{R}_{mnpl} \tag{2.133}$$

$$R_{mn\rho\lambda} = R_{\rho\lambda mn} = \frac{1}{4} h^{\mu\nu} \left(\partial_m h_{\lambda\nu} \partial_n h_{\mu\rho} - \partial_n h_{\lambda\nu} \partial_m h_{\mu\rho} \right) \tag{2.134}$$

$$R_{\mu n\rho l} = -R_{n\mu\rho l} = -R_{\mu n l\rho} = R_{n\mu l\rho} = -\frac{1}{2} \hat{\nabla}_n \partial_l h_{\mu\rho} + \frac{1}{4} h^{\nu\lambda} \partial_n h_{\lambda\rho} \partial_l h_{\mu\nu} \tag{2.135}$$

$$R_{\lambda\mu\nu\rho} = \frac{1}{4} g^{mn} \left(\partial_n h_{\mu\nu} \partial_m h_{\rho\lambda} - \partial_n h_{\nu\lambda} \partial_m h_{\mu\rho} \right) \tag{2.136}$$

The Ricci tensor is:

$$R_{mn} = \hat{R}_{mn} - \frac{1}{2} h^{\mu\rho} \hat{\nabla}_m \partial_n h_{\mu\rho} - \frac{1}{4} \partial_m h^{\mu\rho} \partial_n h_{\mu\rho} \tag{2.137}$$

$$R_{\mu\nu} = -\frac{1}{2} \hat{\nabla}^2 h_{\mu\nu} + \frac{1}{2} g^{mn} h^{\rho\lambda} \partial_n h_{\lambda\mu} \partial_m h_{\rho\nu} - \frac{1}{2} g^{mn} \partial_m h_{\mu\nu} \partial_n \ln \sqrt{h}. \tag{2.138}$$

The Ricci scalar is:

$$R = \hat{R} - h^{\mu\rho} \hat{\nabla}^2 h_{\mu\rho} - \frac{3}{4} g^{mn} \partial_m h^{\mu\rho} \partial_n h_{\mu\rho} - g^{mn} \partial_m \ln \sqrt{h} \partial_n \ln \sqrt{h}. \tag{2.139}$$

2.12 Gauss-Bonnet theorems

Here we discuss only closed manifolds. In odd dimensions, the Euler number always vanishes. In zero dimensions, it is the number of connected components (i.e. points). In two dimensions it is given by:

$$\chi = \frac{1}{4\pi} \int d^2x \sqrt{g} R. \tag{2.140}$$

In four dimensions it is given by

$$\chi = \frac{1}{32\pi^2} \int d^4x \sqrt{g} \left(R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd} \right). \tag{2.141}$$

For an even-dimensional hypersurface in flat space,

$$\chi = \frac{2}{\Omega_D} \int d^D x \sqrt{\gamma} \det\{K_i{}^j\}, \tag{2.142}$$

where Ω_D is the volume of the unit D-sphere (see 2.19), and γ_{ij} and K_{ij} are the induced metric and extrinsic curvature respectively (see 2.13).

2.13 Hypersurfaces

Let's begin with the most covariant description of a hypersurface in a manifold, in which the hypersurface S and the manifold M have independent coordinate systems: M has D coordinates x^μ while S has $D-1$ coordinates y^i. The embedding of S in M is described by the functions $x^\mu(y)$, whose first derivatives give a $D \times (D-1)$ matrix:

$$M_i^\mu = \frac{\partial x^\mu}{\partial y^i}. \tag{2.143}$$

This matrix can be used to pull a co-vector back from M onto S,

$$v_i = M_i^\mu v_\mu, \tag{2.144}$$

or to push a vector forward from S into M,

$$v^\mu = M_i^\mu v^i. \tag{2.145}$$

A vector v^μ in M is said to be tangent to S if there exists a vector v^i in the tangent space to S such that 2.145 is satisfied. The notion of being tangent to S does not exist for co-vectors (in the absence of a metric), which instead have the notion of being normal to S: a co-vector v_μ in M is said to be normal to S if it pulls back to a vanishing co-vector in S. In fact, up to rescaling, there is a unique normal co-vector. For example, if $f(x)$ is a scalar field on M that vanishes on S, $f(x(y)) = 0$ for all y, then its gradient $v_\mu = \partial_\mu f$ is normal to S. Given a foliation by hypersurfaces and a normal co-vector field v_μ, it is easy to show that

$$v \wedge dv = 0\,, \tag{2.146}$$

where $v = v_\mu dx^\mu$ is the corresponding one-form. Frobenius's theorem asserts that (2.146) is also a sufficient condition for the existence of a foliation by hypersurfaces normal to v_μ.

None of the above depends on the metric. If M is endowed with a metric $g_{\mu\nu}$, then several structures follow from it. First of all, we can pull it back to a metric on S:

$$\gamma_{ij} = M_i^\mu M_j^\nu g_{\mu\nu} \tag{2.147}$$

We'll always use $g_{\mu\nu}$ to raise and lower μ, ν indices, and γ_{ij} to raise and lower i, j indices. Now that we don't have to distinguish so carefully between vectors and co-vectors, we can make statements like: The pull-back of the push-forward of a vector v^i is itself. The push-forward of the pull-back of a vector v^μ lies tangent to S; this defines an orthogonal projection operator:

$$P^\mu{}_\nu = M_i^\mu M_\nu^i. \tag{2.148}$$

Unless the hypersurface is null, we can define a unit normal co-vector n_μ by

$$M_i^\mu n_\mu = 0\,, \qquad n_\mu n^\mu = \sigma \tag{2.149}$$

where $\sigma = \pm 1$, which makes n_μ unique up to a sign. It's clear that

$$P^\mu{}_\nu = \delta^\mu_\nu - \sigma n^\mu n_\nu, \tag{2.150}$$

since both 2.148 and 2.150 are 0 acting on n^μ and 1 on vectors tangent to S.

The induced metric γ_{ij} defines the intrinsic geometry of S, and one can use it to calculate covariant derivatives and curvature tensors. The curvature of the embedding of S in M is captured by the (covariant) derivative of the unit normal along S,

$$K_{i\nu} = M^\rho_i \nabla_\rho n_\nu. \tag{2.151}$$

Of course, this tensor is a bit awkward, having one index in M and the other in S. However, since the ν index is tangent to S ($n^\nu K_{i\nu} = 0$), we can pull it back to S without losing any information:

$$K_{ij} = M^\nu_j M^\rho_i \nabla_\rho n_\nu = -n_\nu M^\rho_i \nabla_\rho M^\nu_j. \tag{2.152}$$

Alternatively, we can push the i index forward to M:

$$K_{\mu\nu} = P^\rho{}_\mu \nabla_\rho n_\nu. \tag{2.153}$$

One can show that both K_{ij} and $K_{\mu\nu}$ are symmetric. They are both known as the extrinsic curvature.

Recall that the normal vector n^μ was unique up to a sign. Under $n^\mu \to -n^\mu$, K_{ij} and $K_{\mu\nu}$ change sign.

The mean curvature is the trace of the extrinsic curvature:

$$K = g^{\mu\nu} K_{\mu\nu} = \gamma^{ij} K_{ij}\,. \tag{2.154}$$

If M is foliated by a family of hypersurfaces, then their normal vectors n^μ will define a vector field on M. In this case there are a few other useful formulas for the extrinsic curvature and its trace:

$$K_{\mu\nu} = \frac{1}{2}\pounds_n P_{\mu\nu} = \frac{1}{2} P^\alpha{}_\mu P^\beta{}_\nu \pounds_n g_{\alpha\beta} = \frac{1}{2}\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu - \sigma n^\alpha \nabla_\alpha (n_\mu n_\nu)\right), \tag{2.155}$$

$$K = \nabla_\mu n^\mu\,. \tag{2.156}$$

If the coordinates on M are $\{\lambda, y^i\}$, with S being a surface of constant λ, and we use the coordinates y^i on S, then we have

$$M^\mu_i: \qquad M^\lambda_i = 0\,, \qquad M^j_i = \delta^j_i\,, \tag{2.157}$$

$$\gamma_{ij} = g_{ij}\,, \tag{2.158}$$

$$n_\lambda = \sigma |g^{\lambda\lambda}|^{-1/2}\,, \qquad n_i = 0\,, \tag{2.159}$$

$$K_{ij} = -n_\lambda \Gamma^\lambda_{ij}\,. \tag{2.160}$$

(The coefficient σ in n_μ is put in to agree with (2.161) below.)

Gaussian normal coordinates is a special case of the coordinate system of the last paragraph that is adapted to metric, with the coordinates for M fixed (at least in a neighborhood of S) in terms of those for S. From each point on S, send out a geodesic with initial velocity n^μ. Sufficiently close to S, each point p of M will have a single such geodesic passing through it. Call the point on S from which that geodesic emanates $q(p)$. The point p can then be labelled by the coordinates y^i of $q(p)$ together with the affine parameter λ of the geodesic at p.

Obviously, on S (which is the $\lambda = 0$ surface) the unit normal vector in Gaussian normal coordinates is

$$n^\lambda = 1, \qquad n^i = 0. \tag{2.161}$$

Less obvious, but straightforward to show (see section 3.3 of Wald), is that the vector 2.161 is the unit normal vector to the surfaces of constant λ even away from S. Therefore in the neighborhood of S where the Gaussian normal coordinates are defined, the metric may be written

$$ds^2 = \sigma d\lambda^2 + \gamma_{ij}(y, \lambda) dy^i dy^j. \tag{2.162}$$

In these coordinates the extrinsic curvature is

$$K_{ij} = \frac{1}{2} \partial_\lambda \gamma_{ij}. \tag{2.163}$$

The formulas below give useful quantities for M in Gaussian normal coordinates in terms of those for the surfaces of constant λ. We write $\dot{K}_{ij} = \partial_\lambda K_{ij}$, and a prime indicates a quantity derived from γ_{ij} rather than $g_{\mu\nu}$, i.e. describing the intrinsic geometry of the surfaces of constant λ:

$$g^{\lambda\lambda} = \sigma \tag{2.164}$$
$$g^{i0} = 0 \tag{2.165}$$
$$g^{ij} = \gamma^{ij} \tag{2.166}$$
$$\sqrt{|g|} = \sqrt{|\gamma|} \tag{2.167}$$
$$\Gamma^i_{jk} = \Gamma'^i_{jk} \tag{2.168}$$
$$\Gamma^\lambda_{ij} = -\sigma K_{ij} \tag{2.169}$$
$$\Gamma^i_{\lambda j} = K^i{}_j \tag{2.170}$$
$$\Gamma^i_{\lambda\lambda} = \Gamma^\lambda_{i\lambda} = \Gamma^\lambda_{\lambda\lambda} = 0 \tag{2.171}$$

$$
\begin{aligned}
R_{ijkl} &= R'_{ijkl} - \sigma\left(K_{ik}K_{jl} - K_{jk}K_{il}\right) && (2.172)\\
R_{ijk\lambda} &= \nabla'_i K_{jk} - \nabla'_j K_{ik} && (2.173)\\
R_{i\lambda j\lambda} &= -\dot{K}_{ij} + K_i{}^m K_{mj} && (2.174)\\
R_{ij\lambda\lambda} = R_{i\lambda\lambda\lambda} = R_{\lambda\lambda\lambda} &= 0 && (2.175)\\
R_{ij} &= R'_{ij} - \sigma\left(\dot{K}_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\right) && (2.176)\\
R_{i\lambda} &= \nabla'_j K^j{}_i - \partial_i K && (2.177)\\
R_{\lambda\lambda} &= -\gamma^{ij}\dot{K}_{ij} + K^{ij}K_{ij} && (2.178)\\
R &= R' - \sigma\left(2\gamma^{ij}\dot{K}_{ij} + K^2 - 3K^{ij}K_{ij}\right) && (2.179)
\end{aligned}
$$

Combining (2.178) and (2.179) gives the following equation, valid in any coordinate system:

$$
R' = R + \sigma\left(K^2 - K^{ij}K_{ij} - 2n^\mu n^\nu R_{\mu\nu}\right) \tag{2.180}
$$

Under a small change in the metric $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, the tensors defined above will change. Here we write their changes in the Gaussian normal coordinates of the original metric $g_{\mu\nu}$:

$$
\begin{aligned}
\delta M_i^\mu &= 0 && (2.181)\\
\delta\gamma_{ij} &= h_{ij} && (2.182)\\
\delta n_\lambda &= \frac{1}{2}h_{\lambda\lambda}\,, \qquad \delta n_i = 0 && (2.183)\\
\delta n^\lambda &= -\frac{1}{2}\sigma h_{\lambda\lambda}\,, \qquad \delta n^i = -h^i{}_\lambda && (2.184)\\
\delta P^\lambda{}_\lambda &= \delta P^\lambda{}_i = \delta P^i{}_j = 0\,, \qquad \delta P^i{}_\lambda = h^i{}_\lambda && (2.185)\\
\delta K_{ij} &= -\nabla'_{(i}h_{j)\lambda} - \frac{1}{2}\sigma h_{\lambda\lambda}K_{ij} + \frac{1}{2}\dot{h}_{ij} && (2.186)
\end{aligned}
$$

(note that $h_{i\lambda}$ is a co-vector field from the point of view of S).

Under a Weyl transformation,

$$
\tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}\,, \tag{2.187}
$$

the tensors mentioned above transform as follows:

$$
\begin{aligned}
\tilde{M}_i^\mu &= M_i^\mu && (2.188)\\
\tilde{\gamma}_{ij} &= e^{2\omega}\gamma_{ij} && (2.189)\\
\tilde{P}^\mu{}_\nu &= P^\mu{}_\nu && (2.190)\\
\tilde{n}_\mu &= e^\omega n_\mu && (2.191)\\
\tilde{n}^\mu &= e^{-\omega} n^\mu && (2.192)\\
\tilde{K}_{ij} &= e^\omega(K_{ij} + \gamma_{ij}n^\rho\partial_\rho\omega) && (2.193)\\
\tilde{K}_{\mu\nu} &= e^\omega(K_{\mu\nu} + P_{\mu\nu}n^\rho\partial_\rho\omega) && (2.194)\\
\tilde{K} &= e^{-\omega}\left(K + (D-1)n^\rho\partial_\rho\omega\right) && (2.195)
\end{aligned}
$$

2.14 General submanifolds

The extrinsic curvature generalizes easily to a codimension-k submanifold. We work in the most covariant description (the first one used in the previous subsection). As before, we have the matrix $M_i^\mu = \partial x^\mu / \partial y^i$, the induced metric $\gamma_{ij} = M_i^\mu M_j^\nu g_{\mu\nu}$, and the matrix $P^\mu{}_\nu = M_i^\mu M_\nu^i$ which projects vectors onto S. The normal vector is generalized into an orthonormal frame for the normal bundle, a set of k vectors n_a^μ satisfying

$$n_a^\mu n_{\mu b} = \eta_{ab} \,, \tag{2.196}$$

where η_{ab} is diagonal with eigenvalues ± 1. The frame indices a, b are raised and lowered with η_{ab}. There is an $SO(k)$ (or one of its Lorentzian generalizations) gauge (i.e. y-dependent) ambiguity in the choice of n_a^μ. We now have, instead of (2.150),

$$P^\mu{}_\nu = \delta_\nu^\mu - n_a^\mu n_\nu^a \,. \tag{2.197}$$

The extrinsic curvature now carries an index in the normal bundle:

$$K_{ij}^a = M_j^\nu M_i^\rho \nabla_\rho n_\nu^a = -n_\nu^a M_i^\rho \nabla_\rho M_j^\nu \,. \tag{2.198}$$

Note that this transforms covariantly under y-dependent $SO(k)$ transformations of the basis n_a^μ. Any or all of these indices can be put into embedding-coordinate indices; for example, we can write

$$K_{\mu\kappa}^\lambda = n_a^\lambda P^\nu{}_\kappa P^\rho{}_\mu \nabla_\rho n_\nu^a \,. \tag{2.199}$$

The mean curvature is the normal vector obtained by tracing on the tangent indices:

$$K^\lambda = g^{\mu\kappa} K_{\mu\kappa}^\lambda \,. \tag{2.200}$$

Under a Weyl transformation, $\tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}$, the extrinsic and mean curvatures transform as follows:

$$\tilde{K}_{\mu\kappa}^\lambda = K_{\mu\kappa}^\lambda + P_{\mu\kappa} Q^{\lambda\alpha} \partial_\alpha \omega \,, \tag{2.201}$$

$$\tilde{K}^\lambda = e^{-2\omega} \left(K^\lambda + (D-k) Q^{\lambda\alpha} \partial_\alpha \omega \right) , \tag{2.202}$$

where

$$Q^{\lambda\alpha} = g^{\lambda\alpha} - P^{\lambda\alpha} \,. \tag{2.203}$$

2.15 Kähler geometry

A point to keep in mind is that the complexified tangent space of a manifold is twice as big (in real dimension) as its real tangent space—you have the same number of basis vectors (D, the real dimension of the manifold), but they can have complex coefficients. So it is possible to impose a reality condition on a vector, $v^{\bar{\imath}} = (v^i)^*$. Of course, the holomorphic and anti-holomorphic parts of the tangent space each have complex dimension $D/2$.

An almost complex structure is a (real) tensor $J_a{}^b$ satisfying

$$J_a{}^b J_b{}^c = -\delta_a{}^c. \tag{2.204}$$

Complex coordinates z^i adapted to $J_a{}^b$ are ones in which it takes the form

$$J_i{}^j = i\delta_i{}^j, \qquad J_{\bar\imath}{}^{\bar\jmath} = -i\delta_{\bar\imath}{}^{\bar\jmath}, \qquad J_{\bar\imath}{}^j = J_i{}^{\bar\jmath} = 0. \tag{2.205}$$

For an almost complex structure to be a complex structure its Niejenhuis tensor must vanish (see Candelas). The complex structure is preserved by holomorphic changes of coordinates.

A metric g_{ab} is hermitian (with respect to a given complex structure) if it satisfies any of the following conditions, which are all equivalent:

$$g_{ij} = g_{\bar\imath\bar\jmath} = 0, \qquad g_{ab} = J_a{}^c J_b{}^d g_{bd}, \qquad J_{ab} = -J_{ba}. \tag{2.206}$$

Thus metric and complex structure together define a 2-form, the Kähler form J, whose components in complex coordinates are given by

$$J_{i\bar\jmath} = -J_{\bar\jmath i} = i g_{i\bar\jmath}, \qquad J_{ij} = J_{\bar\imath\bar\jmath} = 0. \tag{2.207}$$

The measure is

$$\sqrt{|\det\{g_{ab}\}|} = \det\{g_{i\bar\jmath}\}, \tag{2.208}$$

and the volume form is

$$\frac{1}{n!} J^n, \tag{2.209}$$

where $n = D/2$ is the complex dimension.

If the Kähler form is closed, $dJ = 0$, then the metric is called Kähler. By dividing dJ into (1,2) and (2,1) components, we find

$$\partial_k g_{i\bar\jmath} = \partial_i g_{k\bar\jmath}, \qquad \partial_{\bar k} g_{\bar\imath j} = \partial_{\bar\imath} g_{\bar k j}. \tag{2.210}$$

If follows that the mixed components of the Christoffel symbols vanish, and the pure ones are given by

$$\Gamma^i_{jk} = g^{i\bar\imath}\partial_j g_{k\bar\imath}, \qquad \Gamma^{\bar\imath}_{\bar\jmath\bar k} = g^{\bar\imath i}\partial_{\bar\jmath} g_{\bar k i}. \tag{2.211}$$

It then follows that the only non-zero components of the Riemann tensor are

$$-R_{i\bar\jmath\bar k}{}^{\bar l} = R_{\bar\jmath i\bar k}{}^{\bar l} = \partial_i \Gamma^{\bar l}_{\bar\jmath\bar k}, \qquad -R_{\bar\imath jk}{}^l = R_{j\bar\imath k}{}^l = \partial_{\bar\imath}\Gamma^l_{jk}, \tag{2.212}$$

and of the Ricci tensor are

$$R_{i\bar\jmath} = R_{\bar\jmath i} = -\partial_i\partial_{\bar\jmath} \ln\sqrt{|g|}. \tag{2.213}$$

(Be careful: I think that Candelas uses the opposite sign convention for the curvature tensors.)

It also follows from Kählerity that the metric can locally be expressed in terms of a Kähler potential K:

$$g_{i\bar\jmath} = \partial_i\partial_{\bar\jmath} K \tag{2.214}$$

or in other words

$$J = i\partial\bar{\partial}K. \tag{2.215}$$

The Kähler potential is a scalar under (holomorphic) diffeomorphisms, but it has an additional gauge freedom: the metric is invariant under Kähler transformations

$$K \to K + f(z) + \bar{f}(\bar{z}). \tag{2.216}$$

In fact, the Kähler potential may not be globally well-defined, and it may be necessary to do Kähler transformations in going from patch to patch.

Example: CP^n is defined by

$$(z^1, \ldots, z^{n+1}) \in C^{n+1} \setminus \{0\}, \qquad (z^1, \ldots, z^{n+1}) \sim \lambda(z^1, \ldots, z^{n+1}) \quad (\lambda \in C \setminus \{0\}). \tag{2.217}$$

On the patch where $z^m \neq 0$, we can define coordinates $\zeta^j_m = z^j/z^m$ (of course $\zeta^m_m = 1$ is not a coordinate). The Fubini-Study metric is defined by the Kähler potential

$$K_m = \ln\left(\sum_{j=1}^{n+1} |\zeta^j_m|^2\right). \tag{2.218}$$

A simple calculation shows that

$$\ln\sqrt{|g|} = -(n+1)K, \tag{2.219}$$

and therefore

$$R_{i\bar{\jmath}} = (n+1)g_{i\bar{\jmath}}. \tag{2.220}$$

2.16 Poincaré duality

Let M be a D-dimensional real compact orientable manifold without boundary.

For a given p ($0 \leq p \leq D$, $p \neq D/2$) consider the following four vector spaces: the pth and $(D-p)$th de Rham cohomologies $H^p(\mathbf{R})$ and $H^{D-p}(\mathbf{R})$, and the pth and $(D-p)$th homologies $H_p(\mathbf{R})$ and $H_{D-p}(\mathbf{R})$. If we arrange these four spaces at the vertices of a square, then they are connected along each edge by a natural bilinear form:

$$\begin{array}{ccccc} H^p(\mathbf{R}) & \to & \int_M \alpha\wedge\beta & \leftarrow & H^{D-p}(\mathbf{R}) \\ \downarrow & & & & \downarrow \\ \int_a \alpha & & & & \int_b \beta \\ \uparrow & & & & \uparrow \\ H_p(\mathbf{R}) & \to & \#(a,b) & \leftarrow & H_{D-p}(\mathbf{R}) \end{array} \tag{2.221}$$

Here α, β, a, and b are elements of $H^p(\mathbf{R})$, $H^{D-p}(\mathbf{R})$, $H_p(\mathbf{R})$, and $H_{D-p}(\mathbf{R})$ respectively, and $\#(a,b)$ is the intersection number of the cycles a and b. It is important that these bilinear forms are independent of the choice of representative in each homology or cohomology class.

Poincaré duality is an isomorphism between the diagonally opposite vector spaces of 2.221 under which all four bilinear forms are equal. Given a p-cycle $a \in H_p(\mathbf{R})$, the Poincaré dual is an element of $H^{D-p}(\mathbf{R})$ for which we can construct a representative form β as follows. In each patch choose coordinates x^i ($i = 1, \dots, p$), y^m ($m = 1, \dots, D-p$) such that the submanifold a is the $y = 0$ locus. Then

$$\beta = \delta^{D-p}(y) dy^1 \wedge \cdots \wedge dy^{D-p} \tag{2.222}$$

is a closed $(D-p)$-form. It is possible to show that this map from $H_p(\mathbf{R})$ to $H^{D-p}(\mathbf{R})$ is bijective. Since under Poincaré duality all four bilinear forms in 2.221 are equal, we really have just two distinct vector spaces connected by a single bilinear form. Furthermore, de Rham's theorems show that this form is non-degenerate, in other words it is represented by an invertible matrix. This makes each vector space canonically isomorphic to the dual vector space (in the usual sense) of the other. Their dimension is called the pth Betti number b_p $(= b_{D-p})$ of M.

Poincaré duality is actually stronger than this, as it is an isomorphism between the integral homology and cohomology groups. Consider the lattice of integral homology $H_p(\mathbf{Z}) \subset H_p(\mathbf{R})$. $H^p(\mathbf{Z}) \subset H^p(\mathbf{R})$ is defined as the dual lattice in the dual vector space:

$$H^p(\mathbf{Z}) = \left\{ \alpha \in H^p(\mathbf{Z}) : \forall a \in H_p(\mathbf{Z}), \int_a \alpha \in \mathbf{Z} \right\}. \tag{2.223}$$

Since the intersection number of two elements of $H_p(\mathbf{Z})$ is always an integer, the image of $H_p(\mathbf{Z})$ under Poincaré duality sits inside $H^p(\mathbf{Z})$. The non-trivial statement (see section 0.4 of Griffiths and Harris) is that Poincaré duality is in fact an isomorphism between $H_p(\mathbf{Z})$ and $H^p(\mathbf{Z})$.

Let us now deal with the case $p = D/2$. We now start with just two vector spaces, $H^p(\mathbf{R})$ and $H_p(\mathbf{R})$. These are both equipped with inner products: on $H_p(\mathbf{R})$ by the intersection number and on $H^p(\mathbf{R})$ by $\int_M \alpha \wedge \beta$. These inner products are symmetric or antisymmetric depending on whether p is even or odd. Between $H_p(\mathbf{R})$ and $H^p(\mathbf{R})$ we also have a bilinear form given by $\int_a \alpha$. Poincaré duality is an isomorphism between $H_p(\mathbf{R})$ and $H^p(\mathbf{R})$ under which the two inner products and the bilinear form all agree. So we effectively have a single vector space equipped with a (symmetric or antisymmetric) inner product. Furthermore, Poincaré duality asserts that, under that inner product, $H_p(\mathbf{Z}) \subset H_p(\mathbf{R})$ is a self-dual lattice.

2.17 Coordinates for S^2

There are three standard sets of coordinates on S^2, namely polar coordinates:

$$(\theta, \phi) : \quad 0 \le \theta \le \pi, \quad \phi \sim \phi + 2\pi; \tag{2.224}$$

embedding coordinates:

$$(x_1, x_2, x_3) \in \mathbf{R}^3 : \quad x_1^2 + x_2^2 + x_3^2 = 1; \tag{2.225}$$

and stereographic coordinates:

$$y \in \mathbf{C}. \tag{2.226}$$

They are related by:

$$x^1 + ix^2 = \sin\theta\, e^{i\phi} = \frac{2y}{1+|y|^2}, \qquad x^3 = \cos\theta = \frac{1-|y|^2}{1+|y|^2}, \tag{2.227}$$

$$|y| = \tan\frac{\theta}{2}, \qquad \arg y = \phi. \tag{2.228}$$

The unit round metric is

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2 = dx_1^2 + dx_2^2 + dx_3^2 = \frac{4dy\, d\bar{y}}{(1+|y|^2)^2}. \tag{2.229}$$

The metric is Kähler with respect to the y coordinate, with Kähler potential

$$K = 2\ln(1+|y|^2). \tag{2.230}$$

2.18 The Hopf fibration

The Hopf fibration is a fiber bundle of S^1 over S^2 that is topologically S^3. Define S^3 as the set $(z_1, z_2) \in \mathbf{C}^2$ satisfying

$$|z_1|^2 + |z_2|^2 = 1, \tag{2.231}$$

and S^2 as the set $(x_1, x_2, x_3) \in \mathbf{R}^3$ satisfying

$$x_1^2 + x_2^2 + x_3^2 = 1. \tag{2.232}$$

The Hopf fibration is defined by the following map from S^3 to S^2:

$$x_1 + ix_2 = 2z_1 z_2^*, \qquad x_3 = |z_1|^2 - |z_2|^2. \tag{2.233}$$

The inverse image of each point in S^2 is the S^1 defined by simultaneous phase rotations of z_1 and z_2 (note that z_1 and z_2 can never simultaneously vanish). If we parametrize S^3 by three angles as follows:

$$z_1 = \cos\frac{\theta}{2}\exp\frac{i}{2}(\psi+\phi), \qquad z_2 = \sin\frac{\theta}{2}\exp\frac{i}{2}(\psi-\phi), \tag{2.234}$$

where

$$0 \le \theta \le \pi, \qquad (\phi, \psi) \sim (\phi + 2\pi, \psi + 2\pi) \sim (\phi + 2\pi, \psi - 2\pi) \tag{2.235}$$

then θ and ϕ parametrize the base S^2 (in the usual way), and ψ parametrizes the fiber.

It is useful to define the following three one-forms on S^3, known as the "left-invariant one-forms" due to their role in $SU(2)$:

$$\sigma_x + i\sigma_y = -\frac{1}{2}e^{i\psi}(id\theta + \sin\theta d\phi), \qquad \sigma_z = \frac{1}{2}(\cos\theta d\phi + d\psi). \tag{2.236}$$

They satisfy

$$d\sigma_x = 2\sigma_y \wedge \sigma_z \tag{2.237}$$

and cyclic permutations. The unit round metric on S^3 is given by

$$d\Omega_3^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = \frac{1}{4}\left(d\theta^2 + d\phi^2 + 2\cos\theta d\phi d\psi + d\psi^2\right) \tag{2.238}$$

The volume form is given by

$$-\sigma_x \wedge \sigma_y \wedge \sigma_z = \frac{1}{8}\sin\theta d\theta \wedge d\phi \wedge d\psi. \tag{2.239}$$

The cross term in the metric 2.238 shows that the fibration is not trivial. In fact it has the form of the Kaluza-Klein ansatz,

$$d\Omega_3^2 = ds_2^2 + e^{2\sigma}\left(\frac{1}{2}d\psi + A_{\mathrm{KK}}\right)^2, \tag{2.240}$$

(We put $\frac{1}{2}d\psi$ because ψ has periodicity 4π rather than 2π as is conventional in the Kaluza-Klein ansatz.) Here

$$ds_2^2 = \sigma_x^2 + \sigma_y^2 = \frac{1}{4}d\Omega_2^2 \tag{2.241}$$

is the metric on the base, which is a round S^2. Then we have

$$e^{\sigma}\left(\frac{1}{2}d\psi + A_{\mathrm{KK}}\right) = \sigma_z, \tag{2.242}$$

so that the Kaluza-Klein scalar σ vanishes while the Kaluza-Klein gauge field,

$$A_{\mathrm{KK}} = \frac{1}{2}\cos\theta d\phi, \tag{2.243}$$

is the same as the gauge field on an S^2 surrounding a magnetic monopole in three spatial dimensions. Taub-NUT is a four-dimensional Ricci-flat metric that fills in the radial direction of the Kaluza-Klein monopole:

$$ds^2 = \frac{r+m}{r-m}dr^2 + 4(r^2 - m^2)(\sigma_x^2 + \sigma_y^2) + 16m^2\frac{r-m}{r+m}\sigma_z^2. \tag{2.244}$$

At long distances the base is asymptotically $\mathbf{R}^3$ while the fiber goes to a constant size.

The free $\mathbf{Z}_n$ action on S^3,

$$(z_1, z_2) \mapsto (e^{2\pi i/n}z_1, e^{2\pi i/n}z_2), \tag{2.245}$$

is rather convenient to describe in terms of the Hopf fibration, since it acts only on the fiber and not on the base:

$$(\theta, \phi, \psi) \mapsto (\theta, \phi, \psi + \frac{4\pi}{n}). \tag{2.246}$$

The metric obtained by modding out by this symmetry can still be described by the Kaluza-Klein ansatz, but we must adjust some of the coefficients since ψ now has periodicity $4\pi/n$:

$$e^{\sigma}\left(\frac{n}{2}d\psi + A_{\mathrm{KK}}\right) = \sigma_z \tag{2.247}$$

now yields

$$\sigma = -\ln n, \qquad A_{\mathrm{KK}} = n\cos\theta d\phi. \tag{2.248}$$

The sphere therefore now carries n units of magnetic flux. Corresponding four-dimensional multi-Taub-NUT solutions, with the individual monopoles either coincident or separated in three dimensions, are also known (see Eguchi-Gilkey-Hanson).

There is a more general Hopf fibration of S^{2n+1} over CP^n. Here we consider S^{2n+1} as non-zero points in $R^{2n+2} \cong C^{n+1}$, identified under multiplication by positive real numbers. These are mapped to points in CP^N simply by quotienting further by phases, which amounts to quotienting points in C^{n+1} by arbitrary non-zero complex numbers. It's clear that the fiber over every point in CP^n is a circle.

2.19 Unit sphere

The unit S^D has the following curvature tensors:

$$R_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}\,, \qquad R_{ab} = (D-1)g_{ab}\,, \qquad R = D(D-1)\,, \qquad G_{ab} = (D-1)(1-\frac{D}{2})g_{ab}\,. \tag{2.249}$$

Its total volume is

$$\Omega_D = \frac{2\pi^{(D+1)/2}}{\Gamma(\frac{D+1}{2})}\,. \tag{2.250}$$

For example,

D	0	1	2	3	4	5	6	7
Ω_D	2	2π	4π	$2\pi^2$	$8\pi^2/3$	π^3	$16\pi^3/15$	$\pi^4/3$

(2.251)

Its Euler character χ is 2 when D is even and 0 when D is odd.

The unit sphere embedded in $\mathbf{R}^{D+1}$ has extrinsic curvature $K_{ij} = \gamma_{ij}$ (in the notation of subsection 2.13).

3. Classical mechanics

3.1 Lagrangian and Hamiltonian formulations

To define Lagrangian mechanics we need a configuration space, which is a manifold, and a Lagrangian, which is a function $L(q,v)$ on its tangent bundle. (We are assuming that Lagrangian is first-order and not explicitly time-dependent.) A path is a map $q^i(t)$ from time (parametrized by the real line or a segment thereof) to the configuration space, and the action is a functional of the path:

$$S[q(t)] = \int dt\, L(q(t), \dot{q}(t)). \tag{3.1}$$

Extremizing this action with respect to the path gives the Euler-Lagrange equation, which is the equation of motion:

$$0 = \frac{\delta S}{\delta q^i} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial q^i}. \tag{3.2}$$

In Hamiltonian mechanics we start with a phase space, which is a symplectic manifold, and a Hamiltonian, which is a function $H(x)$ on it. The equation of motion is Hamilton's equation,

$$\dot{x}^a = \omega^{ab}\partial_b H, \tag{3.3}$$

where ω^{ab} is the inverse of the symplectic form ω_{ab}. It follows that the time dependence of any function on phase space is given by

$$\dot{f} = \partial_a f\, \dot{x}^a = \{f, H\}, \tag{3.4}$$

where the Poisson bracket $\{f, g\}$ is defined by

$$\{f, g\} = \partial_a f \omega^{ab} \partial_b g. \tag{3.5}$$

Given a Lagrangian system, it is straightforward to map it to a Hamiltonian system with the same dynamics. The phase space is taken to be the cotangent bundle of the configuration space, parametrized by coordinates q^i and momenta p_i, with symplectic structure defined by the Poisson brackets

$$\{q^i, q^j\} = \{p_i, p_j\} = 0, \qquad \{q^i, p_j\} = \delta^i_j. \tag{3.6}$$

The Hamiltonian is taken to be the Legendre transform of the Lagrangian with respect to the tangent vector v^i, at fixed coordinate q^i:

$$H(q, p) = p_i v^i - L(q, v)\Big|_{\frac{\partial L}{\partial v^i} = p_i}. \tag{3.7}$$

A history $q^i(t)$ is mapped into phase space using the map from the tangent space to the cotangent space implicit in the Legendre transform:

$$p_i(t) = \frac{\partial L}{\partial v^i}\bigg|_{v^i = \dot{q}^i(t)}. \tag{3.8}$$

The inverse of this map is

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{3.9}$$

which is one of Hamilton's equations. The other one is the translation of the Euler-Lagrange equation (3.2) into the Hamiltonian language, using (1.13) and (3.8):

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{3.10}$$

3.2 Symmetries

Symmetries are treated somewhat differently in the Lagrangian and Hamiltonian formulations. However, just as a Lagrangian system can be mapped into a Hamiltonian one, any symmetry of the former can be mapped into a symmetry of the latter.

For the rest of this subsection, we will deal only with continuous symmetries, but first we would like to make a comment that applies to discrete or continuous symmetries. We will need the following lemma: Suppose that a (differentiable) function S on some space is invariant under the action of a group G, and let I be the G-invariant subset of P. At any point $x \in I$, the gradient of S must be G-invariant. Therefore if x extremizes S within I then it extremizes it within the full space.

In Lagrangian mechanics, a symmetry is a group G acting on the space of paths which leaves the action S invariant. (We will make a further restriction below for continuous symmetries.) As a consequence of the above lemma, for G-invariant paths, it suffices to check that they extremize the action within the space of G-invariant paths. In other words, one can work consistently within the space of G-invariant paths, and any solution to the equations of motion derived there will solve the equations of motion of the full system. For example, for a static path of a time translation-invariant system, one just needs to check that the path extremizes the action within the space of static paths, in other words that the potential energy is extremized.

We now specialize to the case of continuous symmetries. Again, a symmetry of a Lagrangian system is a transformation on the space of paths under which the action is invariant. To be more specific, we require that the change in the Lagrangian at each point on the path be expressible as the time derivative of a locally defined quantity. It follows that paths obeying the equations of motion are mapped onto other such paths, i.e. that the equations of motion are invariant. We thus have

$$\delta L = \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i = \dot{K}, \tag{3.11}$$

where δq is the generator of the symmetry. When the equations of motion are satisfied, the following quantity ("charge") is conserved:

$$Q = \frac{\partial L}{\partial \dot{q}^i}\delta q^i - K. \tag{3.12}$$

This is Noether's theorem. If (as is usually the case) δq and K are functions only of q and $\dot{q}$, then obviously the charge Q is also.

In the Hamiltonian formulation, symmetries are special cases of canonical transformations, which are tranformations on the phase space which leave the symplectic form invariant. The generator δx^a of a canonical transformation must obey

$$0 = \pounds_{\delta x^a}\omega_{bc} = \delta x^a \partial_a \omega_{bc} + \omega_{ac}\partial_b \delta x^a + \omega_{ba}\partial_c \delta x^a = \partial_b(\omega_{ac}\delta x^a) - \partial_c(\omega_{ab}\delta x^a), \tag{3.13}$$

where in the last equality we used the fact that the symplectic form is closed. Since $\omega_{ab}\delta x^a$ is closed, it must be locally exact, in other words there must be a function f (possibly multiple-valued if the phase space is multiply connected) such that

$$\delta x^a = \omega^{ab}\partial_b f. \tag{3.14}$$

The function f is also sometimes called the generator of the canonical transformation δx^a. A symmetry is a canonical tranformation which preserves the Hamiltonian (and therefore the equations of motion), as well as the symplectic form:

$$\delta x^a \partial_a H = 0. \tag{3.15}$$

In other words, its generator commutes with the Hamiltonian,

$$\{f, H\} = 0, \tag{3.16}$$

and therefore by (3.4) is conserved. Since the Hamiltonian obviously commutes with itself, by Hamilton's equation time evolution acts by a symmetry.

Since f is conserved, it is consistent to restrict the dynamics to a subspace of the phase space with $f = f_0$ for some constant f_0. However, the symplectic form restricted to that subspace will be degenerate (as it must be, since the subspace is odd-dimensional): for any vector v^a lying in the subspace, $v^a\omega_{ab}\delta x^b = 0$. Hence, to obtain a sensible restricted phase space, it is necessary to quotient the surface $f = f_0$ by the action of the symmetry, i.e. by the vector field δx^a. Thus, one dimension is lost by the constraint $f = f_0$ and another by the quotient; the resulting restricted phase space has a well-defined symplectic form and Hamiltonian (since $\delta x^a \partial_a H = 0$). An application of this construction is to the global part of a gauge symmetry: as we will show below, the Noether charge associated to this symmetry is constrained to vanish, and the phase space should be quotiented by it to obtain the physical phase space.

As noted above, given a symmetry of a Lagrangian system such that δq and K depend only on q and $\dot{q}$, the same goes for corresponding conserved charge Q, which can therefore equally well be considered a function on the phase space of the corresponding Hamiltonian system. Furthermore, since it's conserved, it must commute with the Hamiltonian, and therefore must generate a symmetry in the Hamiltonian sense. In fact, it's straightforward to show that the transformation on the phase space that it generates is the original symmetry transformation, i.e. that $\partial Q/\partial p_i = \delta q^i$ and $\partial Q/\partial q^i = -\delta p_i$ (where δp is the image of δq and $\delta\dot{q}$ under the usual map $p_i = \partial L/\partial \dot{q}^i$). To prove the first identity, we use the fact that $\delta L - \dot{K}$ should vanish for a general path $\dot{q}$ (not necessarily one satisfying the equations of motion). Using $p_i = \partial L/\partial \dot{q}^i$, such a path can be mapped into a path $(q(t), p(t))$ in phase space, which will satisfy one of Hamilton's equations, $\dot{q}^i = \partial H/\partial p_i$, but not necessarily the other, $\dot{p}_i = -\partial H/\partial q^i$. We have

$$\delta L - \dot{K} = \frac{\partial L}{\partial q^j}\delta q^j + \frac{\partial L}{\partial \dot{q}^j}\delta\dot{q}^j - \dot{K} \tag{3.17}$$

$$= -\frac{\partial H}{\partial q^j}\delta q^j + \left(p_j\frac{\partial \delta q^j}{\partial q^i} - \frac{\partial K}{\partial q^i}\right)\dot{q}^i + \left(p_j\frac{\partial \delta q^j}{\partial p_i} - \frac{\partial K}{\partial p_i}\right)\dot{p}_i. \tag{3.18}$$

Since this must vanish for arbitrary $\dot{p}$ (independent of q and p), we find the relation

$$p_j \frac{\partial \delta q^j}{\partial p_i} - \frac{\partial K}{\partial p_i} = 0. \tag{3.19}$$

Using this relation, the equation

$$\frac{\partial Q}{\partial p_i} = \delta q^i \tag{3.20}$$

follows immediately from the definition of Q. To show that $\partial Q/\partial q^i = -\delta p_i$, we will need to use the equation of motion. This is because δp depends on $\delta \dot{q}$, which (if δq depends on $\dot{q}$) depends on $\ddot{q}$. So the equation of motion is necessary to write δp as a function of q and p, which $\partial Q/\partial p_i$ is. We won't calculate δp directly, but will instead use a trick, which is to notice that $\delta \dot{q}$ can be written as a function of q and p in two different ways. We have:

$$\begin{aligned} \delta \dot{q}^i &= \delta(\dot{q}^i) \\ &= \delta\left(\frac{\partial H}{\partial p_i}\right) \\ &= \frac{\partial^2 H}{\partial p_i \partial q^j}\delta q^j + \frac{\partial^2 H}{\partial p_i \partial p_j}\delta p_j. \end{aligned} \tag{3.21}$$

On the other hand we have

$$\begin{aligned} \delta \dot{q}^i &= \frac{d}{dt}\delta q^i \\ &= \left\{\frac{\partial Q}{\partial p_i}, H\right\} \\ &= -\left\{Q, \frac{\partial H}{\partial p_i}\right\} \\ &= -\frac{\partial Q}{\partial q^j}\frac{\partial^2 H}{\partial p_i \partial p_j} + \frac{\partial Q}{\partial p_j}\frac{\partial^2 H}{\partial p_i \partial q^j}. \end{aligned} \tag{3.22}$$

The first term of (3.21) and the second term of (3.22) are equal in view of the identity (3.20). Furthermore, the matrix $\partial^2 H/\partial p_i \partial p_j = \partial \dot{q}^i/\partial p_j$ is invertible in view of the assumed invertibility of the map $p_i = \partial L/\partial \dot{q}^i$. So we have, as promised,

$$\frac{\partial Q}{\partial q^i} = -\delta p_i. \tag{3.23}$$

3.3 Noether's theorem for higher-order actions

We will work in field theory notation. The treatment of Noether's theorem in most textbooks assumes a first-order Lagrangian. We will be more general, but will still assume that the action is the integral of a local Lagrangian $\mathcal{L}$. We define a (continuous) symmetry to be an infinitesimal variation $\epsilon \delta \phi^i$ of the fields ϕ^i, which is a local function of the fields and their derivatives, under which the change in the Lagrangian is a total divergence,

$$\delta \mathcal{L} = \epsilon \partial_a K^a, \tag{3.24}$$

where K^a is also a local function of the fields and their derivatives. (Here ϵ is a constant infinitesimal parameter.) This is slightly weaker than demanding that the action be invariant, but stronger than demanding that the equations of motion be unchanged.

Now consider letting ϵ be an arbitrary function of spacetime. Then we can expand the change in the Lagrangian in derivatives of ϵ:

$$\delta\mathcal{L} = \epsilon\partial_a K^a + \partial_a\epsilon J^a_{(1)} + \partial_a\partial_b\epsilon J^{ab}_{(2)} + \cdots \tag{3.25}$$

$$= -\epsilon\partial_a j^a + \text{total divergence}, \tag{3.26}$$

where

$$j^a = -K^a + J^a_{(1)} - \partial_b J^{ab}_{(2)} + \cdots . \tag{3.27}$$

When the fields satisfy the equations of motion, the change in the action must vanish for any ϵ that vanishes (along with enough of its derivatives) on the boundary. The equations of motion must therefore imply

$$\partial_a j^a = 0. \tag{3.28}$$

It is straightforward to derive explicit expressions for all the terms in the current (except K^a), order by order in derivatives of the fields. For example, up to second derivatives of the fields we have

$$J^a_{(1)} - \partial_b J^{ab}_{(2)} = \left(\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)} - \partial_b\frac{\partial\mathcal{L}}{\partial(\partial_a\partial_b\phi)} + \cdots\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_a\partial_b\phi)}\partial_b\delta\phi + \cdots \tag{3.29}$$

(sum over fields implied).

3.4 Gauge symmetries

A symmetry is called a gauge symmetry if its action on the system depends on a parameter which is a function of spacetime. This implies that histories (or paths) that are related by the symmetry are experimentally indistinguishable and should therefore be identified in the true space of histories. To see this, imagine that we are studying the system with an experimental apparatus which we imagine to be separated from it in space or time. If we act on the system but not the apparatus with a global symmetry, this is not a symmetry of the full system-plus-apparatus über-system; therefore the results reported by the apparatus may be different. On the other hand, acting only on the system with a gauge symmetry *is* a symmetry of the über-system, and the results of the apparatus will be unchanged.

Another way to see that histories related by a gauge symmetry must be identified is that to retain them as physically distinct would entail a massive loss of determinism. Given initial data specified at some initial time, and a history with those initial conditions satisfying the equations of motion, any other history related by a gauge transformation whose parameter vanishes at the initial time will have the same initial conditions and will also satisfy the equations of motion. This plethora of solutions for the same initial conditions is physically sensible only if we declare all of them to be physically the same. Thus we see that a gauge symmetry necessarily reveals a redundancy in our description of the physics.

A gauge symmetry necessarily includes a global part, in which the gauge parameter is constant. The global part will have an associated conserved Noether charge. Let us show that this charge in fact vanishes on the equations of motion. We work in the notation of a mechanical $(0+1$ dimensional) system, with positions q^i and Lagrangian $L(q,\dot{q})$ (we assume a first-order action). The gauge transformation will be parametrized by a function $\epsilon(t)$. We assume that the corresponding change in q^i is local in time, but can depend on both ϵ and its time derivative:

$$\delta q^i = \epsilon\delta_0 q^i + \dot{\epsilon}\delta_1 q^i. \tag{3.30}$$

Hence we have

$$\delta \dot{q}^i = \epsilon\delta_0 \dot{q}^i + \dot{\epsilon}\delta_0 q^i + \dot{\epsilon}\delta_1 \dot{q}^i + \ddot{\epsilon}\delta_1 q^i. \tag{3.31}$$

We now assume that under this transformation the Lagrangian changes by a total derivative:

$$\delta L = \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i = \dot{K}; \tag{3.32}$$

this guarantees that the equations of motion are unchanged. K will itself depend on ϵ and $\dot{\epsilon}$:

$$K = \epsilon K_0 + \dot{\epsilon} K_1. \tag{3.33}$$

Equating coefficients of ϵ, $\dot{\epsilon}$, and $\ddot{\epsilon}$ in 3.32 gives us

$$\frac{\partial L}{\partial q^i}\delta_0 q^i + \frac{\partial L}{\partial \dot{q}^i}\delta_0 \dot{q}^i = \dot{K}_0 \tag{3.34}$$

$$\frac{\partial L}{\partial q^i}\delta_1 q^i + \frac{\partial L}{\partial \dot{q}^i}\delta_0 q^i + \frac{\partial L}{\partial \dot{q}^i}\delta_1 \dot{q}^i = K_0 + \dot{K}_1 \tag{3.35}$$

$$\frac{\partial L}{\partial \dot{q}^i}\delta_1 q^i = K_1. \tag{3.36}$$

The first of these is Noether's theorem: defining the charge

$$Q = \frac{\partial L}{\partial \dot{q}^i}\delta_0 q^i - K_0, \tag{3.37}$$

it asserts

$$\frac{dQ}{dt} \cong 0, \tag{3.38}$$

where $\cong$ means "equal on the equations of motion". The last, 3.36, constrains the canonical momenta (without using the equations of motion); it asserts that the phase space is smaller than one would naively expect from the configuration space. The middle equation, 3.35, is equal to the time derivative of the last one plus

$$Q \cong 0. \tag{3.39}$$

Since Q can be calculated from the initial data, this equation asserts that the equations of motion not only dictate the time evolution but also constrain the initial data, making the physical phase space yet smaller.

Let us now generalize this story to multiple dimensions, with fields ϕ and Lagrangian density $\mathcal{L}(\phi, \partial_a \phi)$. Naively one might guess that now the Noether current j^a will vanish. However, we will see that the story is more complicated.

The field variations $\delta_0\phi$ and $\delta_1\phi$ are a scalar and a vector respectively:

$$\delta\phi = \epsilon\,\delta_0\phi + \partial_a\epsilon\,\delta_1\phi^a. \tag{3.40}$$

As before, we will assume that the change in the Lagrangian density is a total divergence:

$$\delta\mathcal{L} = \partial_a K^a, \qquad K^a = \epsilon K_0^a + \partial_b\epsilon K_1^{ab}. \tag{3.41}$$

The Noether current associated to the global part of the symmetry is

$$j^a = \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\delta_0\phi - K_0^a. \tag{3.42}$$

It is also useful to define the following tensor:

$$F^{ab} = \frac{\partial\mathcal{L}}{\partial(\partial_b\phi)}\delta_1\phi^a - K_1^{ba}. \tag{3.43}$$

The generalizations of equations 3.34, 3.35, 3.36 above are

$$\partial_a j^a \cong 0 \tag{3.44}$$
$$j^a + \partial_b F^{ab} \cong 0 \tag{3.45}$$
$$F^{(ab)} = 0. \tag{3.46}$$

The key difference is the last one, which is weaker than the naive generalization of 3.36, $F^{ab} = 0$. This occurs because it is the coefficient in $\delta\mathcal{L}$ of $\partial_a\partial_b\epsilon$, which is necessarily a symmetric tensor, so that only the symmetric part of F^{ab} is constrained.

3.5 Hamilton-Jacobi theory

Hamilton's equations of motion are first-order equations in phase space:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{3.47}$$

The Hamilton-Jacobi formalism is a trick for writing them as first-order equations on configuration space (the space of positions q^i) alone. Let $S(q,t)$ be a function satisfying the following differential equation:

$$\frac{\partial S}{\partial t} = -H|_{p_i = \frac{\partial S}{\partial q^i}}. \tag{3.48}$$

Then if we let the time evolution of the positions be governed by the first of Hamilton's equations 3.47, and at every time set the momenta equal to the gradient of S with respect to the positions,

$$p_i = \frac{\partial S}{\partial q^i}, \tag{3.49}$$

then they will automatically satisfy the second of Hamilton's equations.

An example of a function S that satisfies 3.48 (as well as the motivation for the name S) is provided by the action evaluated on trajectories that satisfy the equations of motion. Fix an initial time t_0 and position $q_0^i = q^i(t_0)$. Any q_1^i at any final time t_1 then uniquely determines a classical trajectory $q_c^i(t), p_{c,i}(t)$. The action evaluated along this classical trajectory, considered as a function of q_1^i and t_1,

$$S_{q_0,t_0}(q_1,t_1) = \int_{t_0}^{t_1} dt \left(p_{c,i}\dot{q}_c^i - H(q_c,p_c,t)\right), \tag{3.50}$$

can be shown to satisfy

$$\frac{\partial S_{q_0,t_0}}{\partial q_1^i} = p_{c,i}(t_1), \tag{3.51}$$

as well as 3.48.

In a saddle-point approximation to the path integral, the wave function at position q_1^i at time t_1 of a particle that started at position q_0^i at time t_0, is $\exp(iS_{q_0,t_0}(q_1,t_1)/\hbar)$ (times a factor depending on the second functional derivative of the action about the classical path). More generally, 3.48 is the $\hbar \to 0$ limit of the Schrödinger equation for the wave function $\psi(q,t) = e^{iS(q,t)/\hbar}$.

Consider now a situation where the Hamiltonian has no explicit time dependence. Then there exists a special class of solutions to 3.48 of the following form:

$$S(q,t) = S_0(q) - Et, \tag{3.52}$$

where E is a fixed constant and S_0 satisfies the following equation:

$$H\big|_{p_i = \frac{\partial S_0}{\partial q^i}} = E. \tag{3.53}$$

S_0 is called Hamilton's principal function. For such an S, all trajectories have the same energy E. This is the classical equivalent of an energy eigenstate.

Still assuming time-translation symmetry, we can also note an interesting property of the on-shell action $S_{q_0,t_0}(q_1,t_1)$. Fixing q_0, q_1, we write it $S(\Delta t)$ (where $\Delta t := t_1 - t_0$); from 3.48, we then have

$$\frac{dS(\Delta t)}{d\Delta t} = -E\,. \tag{3.54}$$

The Legendre transform of $S(\Delta t)$ is the area under the phase-space curve $(q(t), p(t))$ as a function of E:

$$E\Delta t + S(\Delta t) = \int_{t_0}^{t_1} dt(E+L) = \int_{q_0}^{q_1} dq^i p_i =: A(E) \tag{3.55}$$

It follows from properties of the Legendre transform that

$$\frac{dA(E)}{dE} = \Delta t\,, \qquad S(\Delta t) = A(E) - E\Delta t\,. \tag{3.56}$$

The first equation can easily be checked for the simple case $H = p^2/(2m) + V(q)$.

4. Gauge theories

4.1 Generalities

Consider a gauge group G. The connection (or gauge field) $A_\mu(x)$ is a Lie-algebra valued one-form that transforms under gauge transformations $U(x) \in G$ like

$$A_\mu \to U A_\mu U^{-1} + i(\partial_\mu U)U^{-1}. \tag{4.1}$$

The field strength $F_{\mu\nu}$ is a Lie-algebra valued 2-form:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \tag{4.2}$$

It transforms homogeneously under gauge transformations:

$$F_{\mu\nu} \to U F_{\mu\nu} U^{-1}. \tag{4.3}$$

When G is non-abelian, the Lagrangian for A_μ is conventionally

$$\mathcal{L}_{\rm YM} = -\frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu}, \tag{4.4}$$

where the trace is taken in the fundamental representation of G. When it is $U(1)$, the Lagrangian is conventionally

$$\mathcal{L}_{\rm em} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu}. \tag{4.5}$$

If G is not simple, its different simple parts can have different couplings g.

For a matter field ϕ transforming in the representation r of G,

$$\phi \to U_r \phi, \tag{4.6}$$

the covariant derivative

$$D^r_\mu = \partial_\mu + i A^r_\mu \tag{4.7}$$

transforms homogeneously:

$$D^r_\mu \phi \to U_r D^r_\mu \phi. \tag{4.8}$$

The field strength in this representation is simply

$$F^r_{\mu\nu} = -i[D^r_\mu, D^r_\nu]. \tag{4.9}$$

Another convention is obtained by the replacements $A_\mu \to gA_\mu$, $F_{\mu\nu} \to gF_{\mu\nu}$ in the above formulas:

$$
\begin{aligned}
A_\mu &\to UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} && (4.10)\\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] && (4.11)\\
F_{\mu\nu} &\to UF_{\mu\nu}U^{-1} && (4.12)\\
\mathcal{L}_{\mathrm{YM}} &= -\frac{1}{2}\operatorname{tr} F_{\mu\nu}F^{\mu\nu} && (4.13)\\
\mathcal{L}_{\mathrm{em}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} && (4.14)\\
\phi &\to U_r\phi && (4.15)\\
D^r_\mu &= \partial_\mu + igA^r_\mu && (4.16)\\
D^r_\mu\phi &\to U_r D^r_\mu\phi && (4.17)\\
F^r_{\mu\nu} &= -\frac{i}{g}[D^r_\mu, D^r_\nu]. && (4.18)
\end{aligned}
$$

4.2 Standard Model

Here is the field content before symmetry breaking ($i = 1, 2, 3$ labels generation):

field	Lorentz	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
g	$(\frac{1}{2}, \frac{1}{2})$	8	1	0
W	$(\frac{1}{2}, \frac{1}{2})$	1	3	0
B	$(\frac{1}{2}, \frac{1}{2})$	1	1	0
q_i	$(\frac{1}{2}, 0)$	3	2	$\frac{1}{6}$
u^c_i	$(\frac{1}{2}, 0)$	$\bar{3}$	1	$-\frac{2}{3}$
d^c_i	$(\frac{1}{2}, 0)$	$\bar{3}$	1	$\frac{1}{3}$
l_i	$(\frac{1}{2}, 0)$	1	2	$-\frac{1}{2}$
e^c_i	$(\frac{1}{2}, 0)$	1	1	1
H	$(0, 0)$	1	2	$-\frac{1}{2}$

(4.19)

The gauge bosons are g for $SU(3)_c$ with coupling g_3, W for $SU(2)_L$ with coupling g_2, and B for $U(1)_Y$ with coupling g_1. In the following, we write g_μ and W_μ as matrices in the fundamental representations of $SU(3)$ and $SU(2)$ respectively: $g_\mu = g^a_\mu T^a$ (where T^a are $\frac{1}{2}$ the 8 Gell-Mann matrices, satisfying $\operatorname{tr} T^a T^b = \frac{1}{2}\delta^{ab}$), $W_\mu = W^i_\mu t^i$ ($t^i = \frac{1}{2}\sigma_i$). We also use the second normalization convention of the previous subsection. The Lagrangian is:

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} \tag{4.20}$$

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\operatorname{tr} g_{\mu\nu}g^{\mu\nu} - \frac{1}{2}\operatorname{tr} W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{4.21}$$

$$\mathcal{L}_{\text{fermions}} = q_i^\dagger i\bar{\sigma}^\mu D_\mu q_i + u_i^{c\dagger} i\bar{\sigma}^\mu D_\mu u_i^c + d_i^{c\dagger} i\bar{\sigma}^\mu D_\mu d_i^c + l_i^\dagger i\bar{\sigma}^\mu D_\mu l_i + e_i^{c\dagger} i\bar{\sigma}^\mu D_\mu e_i^c \tag{4.22}$$

$$\mathcal{L}_{\text{Higgs}} = -(D_\mu H)^\dagger D^\mu H - \frac{\lambda}{2}(H^\dagger H - \frac{v^2}{2}) \tag{4.23}$$

$$\mathcal{L}_{\text{Yukawa}} = -H^\dagger q_i \lambda_{ij}^u u_j^c - H_A \epsilon^{AB} q_{Bi} \lambda_{ij}^d d_j^c - H_A \epsilon^{AB} l_{Bi} \lambda_{ij}^e e_j^c + \text{h.c.}, \tag{4.24}$$

where $\bar{\sigma}^\mu = (1, -\vec{\sigma})$, $\dagger$ conjugates spinors and transposes $SU(2)_L$ and $SU(3)_c$ indices, and $A, B = 1, 2$ are indices of the fundamental representation of $SU(2)_L$ with $\epsilon^{12} = 1$.

In the vacuum $H^\dagger H = v^2/2$, and by making an appropriate choice of $SU(2)_L \times U(1)_Y$ gauge we can always put H in the form

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} v + h(x) \\ 0 \end{bmatrix} \tag{4.25}$$

where $h(x)$ is a canonically normalized real scalar. This is called unitarity gauge. The unbroken $SU(2)_L \times U(1)_Y$ generator is

$$Q = t^3 + Y, \tag{4.26}$$

which is therefore the generator of the surviving $U(1)_{\text{em}}$ gauge group. In unitarity gauge the Higgs Lagrangian is (keeping only quadratic terms)

$$\begin{aligned}\mathcal{L}_{\text{Higgs}} &= -\frac{1}{2}\partial_\mu h \partial^\mu h - \frac{v^2}{8}\left((g_2 W_\mu^3 - g_1 B_\mu)^2 + (g_2 W_\mu^1)^2 + (g_2 W_\mu^2)^2\right) - \frac{\lambda v^2}{2}h^2 && (4.27)\\ &= -\frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m_h^2 h^2 - \frac{1}{2}m_Z^2 Z_\mu Z^\mu - m_W^2 W_\mu^+ W^{-\mu}, && (4.28)\end{aligned}$$

where the mass of the physical Higgs boson is

$$m_h = \sqrt{\lambda} v, \tag{4.29}$$

and we have defined the fields

$$Z_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}}(g_2 W_\mu^3 - g_1 B_\mu), \tag{4.30}$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm i W_\mu^2), \tag{4.31}$$

with masses

$$m_Z = \frac{1}{2}\sqrt{g_1^2 + g_2^2}\, v, \qquad m_W = \frac{1}{2} g_2 v. \tag{4.32}$$

The linear combination of W_μ^3 and B_μ that is orthogonal to Z_μ,

$$A_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}}(g_1 W_\mu^3 + g_2 B_\mu), \tag{4.33}$$

remains massless, and this is the gauge field for the unbroken $U(1)_{\text{em}}$, as we can see explicitly by expanding the covariant derivatives for a field in a general $SU(2)_L$ and $U(1)_Y$ representation (here t^i are the representation matrices for $SU(2)_L$ in a general representation):

$$\begin{align} D_\mu &= \partial_\mu + ig_2 W_\mu^i t^i + ig_1 Y B_\mu \tag{4.34} \\ &= \partial_\mu + i\frac{g_2}{\sqrt{2}}(W_\mu^+ t^+ + W_\mu^- t^-) + ieQA_\mu + \frac{i}{\sqrt{g_1^2 + g_2^2}}(g_2^2 t^3 - g_1 Y)Z_\mu, \tag{4.35} \end{align}$$

where $t^\pm = t^1 \pm it^2$,

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} \tag{4.36}$$

is the coupling of the $U(1)_{\text{em}}$ gauge field, and $Q = t^3 + Y$ as above. This goes for the vector fields $W_\mu^\pm$ as well, so they have electric charges $Q = \pm 1$. Defining the mixing angle θ_W by

$$\cos\theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}, \qquad \sin\theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \tag{4.37}$$

we have

$$\begin{align} Z_\mu = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu, &\qquad A_\mu = \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu \tag{4.38} \\ W_\mu^3 = \cos\theta_W Z_\mu + \sin\theta_W A_\mu, &\qquad B_\mu = -\sin\theta_W Z_\mu + \cos\theta_W A_\mu, \tag{4.39} \\ m_W = \cos\theta_W m_Z, \qquad g_1 = \frac{e}{cos\theta_W} &\qquad g_2 = \frac{e}{\sin\theta_W}. \tag{4.40} \end{align}$$

Inserting the Higgs vev into the Yukawa terms in the Lagrangian gives some of the fermions (Dirac) masses. Expand the $SU(2)_L$ doublet fermions out into components:

$$q_i = \begin{bmatrix} u_i \\ d_i \end{bmatrix}, \qquad l_i = \begin{bmatrix} \nu_i \\ e_i \end{bmatrix}. \tag{4.41}$$

The Yukawa Lagrangian becomes (again, keeping only quadratic terms):

$$\mathcal{L}_{\text{Yukawa}} = -\frac{v}{\sqrt{2}} u_i \lambda_{ij}^u u_j^c - \frac{v}{\sqrt{2}} d_i \lambda_{ij}^d d_j^c - \frac{v}{\sqrt{2}} e_i \lambda_{ij}^e e_j^c + \text{h.c.} \tag{4.42}$$

Note that the neutrinos remain massless. By global $U(3)$ rotations on the fields e_i and e_i^c, it is possible to make the matrix λ_{ij}^e diagonal, real, and positive. Similarly for λ_{ij}^u and λ_{ij}^d. However, the necessary $U(3)$ rotations on u_i and d_i will be different. Since these two fields are components of the same $SU(2)_L$ doublet, making different rotations on them will complicate their couplings to the W bosons, introducing the CKM matrix there and leading to generation-changing weak-interaction processes.

After symmetry breaking, the fields and their quantum numbers are as follows:

field	Lorentz	massive	$SU(3)_c$	Q
g	real vector	no	8	0
A	real vector	no	1	0
W	complex vector	yes	1	1
Z	real vector	yes	1	0
u_i	Dirac	yes	3	$\frac{2}{3}$
d_i	Dirac	yes	3	$-\frac{1}{3}$
e_i	Dirac	yes	1	-1
ν_i	Weyl	no	1	0
h	real scalar	yes	1	0

(4.43)

5. General relativity

5.1 Dimensions

We set $c = 1$, but not G_{N} or $\hbar$. We are thus left with two units, mass (M), and length (L), in terms of which

$$[ds^2] = L^2 \tag{5.1}$$

$$[d^D x\sqrt{|g|}] = L^D \tag{5.2}$$

$$[S] = [\hbar] = LM \tag{5.3}$$

$$[G_{\mathrm{N}}] = [\kappa^2] = L^{D-3}M^{-1} \tag{5.4}$$

$$[R] = [\Lambda] = L^{-2} \tag{5.5}$$

$$[\mathcal{L}_{\mathrm{m}}] = [\rho_{\mathrm{v}}] = [T^a_a] = L^{1-D}M \tag{5.6}$$

5.2 Fundamentals

The action for gravity coupled to matter is

$$S = \frac{1}{16\pi G_{\mathrm{N}}}\int d^D x\sqrt{|g|}\,(R - 2\Lambda) + S_{\mathrm{m}}, \tag{5.7}$$

which gives rise to the Einstein equation

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G_{\mathrm{N}}T_{ab}, \tag{5.8}$$

where

$$T^{ab} = \frac{2}{\sqrt{|g|}}\frac{\delta S_{\mathrm{m}}}{\delta g_{ab}}. \tag{5.9}$$

If

$$S_{\mathrm{m}} = \int d^D x\sqrt{|g|}\mathcal{L}_{\mathrm{m}}, \tag{5.10}$$

then the stress tensor is given in terms of $\mathcal{L}_{\rm m}$ by

$$T^{ab} = 2\frac{\delta \mathcal{L}_{\rm m}}{\delta g_{ab}} + \mathcal{L}_{\rm m} g^{ab}. \tag{5.11}$$

Having a cosmological constant is equivalent to having a vacuum energy

$$\rho_{\rm v} = \frac{\Lambda}{8\pi G_{\rm N}}, \tag{5.12}$$

which means having a term in $\mathcal{L}_{\rm m}$ equal to $-\rho_{\rm v}$, and thus a term in T_{ab} equal to $-\rho_{\rm v} g_{ab}$. Sometimes in place of $G_{\rm N}$ one uses

$$\kappa^2 = 8\pi G_{\rm N}. \tag{5.13}$$

One can also define the Planck mass $m_{\rm P}$,

$$m_{\rm P}^{D-2} = \frac{\hbar^{D-3}}{G_{\rm N}}. \tag{5.14}$$

In the weak-field limit, $g_{ab} = \eta_{ab} + h_{ab}$, $|h_{ab}| \ll 1$, massive particles moving non-relativistically feel a Newtonian gravitational potential equal to $-h_{00}/2$. Two masses M_1 and M_2 separated by a distance r feel a gravitational attraction of magnitude

$$F = \frac{(D-3)8\pi G_{\rm N} M_1 M_2}{(D-2)\Omega_{D-2} r^{D-3}}\,. \tag{5.15}$$

In our world,

$$8\pi G_{\rm N} = \kappa^2 = 1.9 \times 10^{-27} {\rm cm\,g^{-1}}, \tag{5.16}$$

and

$$m_{\rm P} = 2.2 \times 10^{-5} {\rm g} = 1.3 \times 10^{19} {\rm GeV}. \tag{5.17}$$

A top-form field strength makes a positive contribution to the cosmological constant.

5.3 Spherically symmetric solutions

We work in $D \geq 3$ and look for $SO(D-1)$ invariant vacuum solutions. Birkhoff's theorem states that any such solution is static. Given D and Λ, there is a one-parameter family of such metrics:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{D-2}^2\,, \qquad f(r) = -\frac{2\Lambda r^2}{(D-2)(D-1)} + 1 - \frac{\mu}{r^{D-3}}\,. \tag{5.18}$$

The coordinate r ranges over the region where $r \geq 0$ and $f(r) \geq 0$. Requiring that such a region exist, and that the solution be free of naked singularities, leads to a complicated classification (and nomenclature).

$\mu = 0$ is allowed for any D and for any value of Λ; these are called de Sitter, Minkowski, and anti-de Sitter space for Λ positive, zero, and negative respectively. The curvature radius $R_{\rm (A)dS}$ of (anti-)de Sitter space is related to Λ by

$$\Lambda = \pm\frac{(D-1)(D-2)}{2R_{\rm (A)dS}^2}\,. \tag{5.19}$$

For $D = 3$ the only other non-nakedly singular solutions have $\Lambda < 0$ and $\mu > 1$; they are called BTZ. For $D \geq 4$ and $\Lambda > 0$, we can have any value of μ in the range

$$0 < \mu < \frac{2}{D-1} \left(\frac{(D-3)(D-2)}{2\Lambda} \right)^{(D-3)/2} ; \tag{5.20}$$

these are called dS-Schwarzschild. (In the limit that μ goes to its upper bound, the solution becomes $\mathrm{dS}_2 \times S^{D-2}$.) For $D \geq 4$ and $\Lambda = 0$ we can have any $\mu > 0$; these are called Schwarzschild (for $D > 4$ also sometimes called Schwarzschild-Tangherlini). Similarly, for $D \geq 4$ and $\Lambda < 0$ we can have any $\mu > 0$, and these are called AdS-Schwarzschild.

In the solutions with $\Lambda \leq 0$ the coordinate r is unbounded above, so they are asymptotically flat ($\Lambda = 0$) or AdS ($\Lambda < 0$). The ADM masses of these solutions (relative to AdS or Minkowski space) are

$$M = \frac{(D-2)\Omega_{D-2}\mu}{16\pi G_{\mathrm{N}}} . \tag{5.21}$$

5.4 Cauchy problem

Let's first review the situation for electromagnetism (in flat spacetime, without sources). The equation of motion for the gauge field A_a is

$$\partial^2 A_a - \partial_a \partial^b A_b = 0, \tag{5.22}$$

subject to the gauge freedom

$$A_a \to A_a + \partial_a \lambda. \tag{5.23}$$

In order to obtain a deterministic time evolution, we will obviously need to fix this gauge freedom. Also, we must choose a time slicing of spacetime, breaking the manifest relativistic invariance of the theory. Therefore a natural choice of gauge is the Coulomb gauge:

$$A_0 = 0. \tag{5.24}$$

(Note that to impose this gauge condition it is not necessary to be on-shell.) The remaining degrees of freedom are the spacelike components A_i. These are still subject to the residual gauge freedom

$$A_i \to A_i + \partial_i \lambda, \tag{5.25}$$

where now the gauge parameter λ is required to be time-independent, i.e. this gauge freedom operates only on the initial value surface, not throughout the spacetime. We denote the time derivatives of the A_i, the momenta, by E_i:

$$E_i = \dot{A}_i. \tag{5.26}$$

There are $D-1$ "positions" (values of A_i) to specify at each point on the initial value surface, but due to the residual gauge freedom 5.25, only $D-2$ are physically meaningful. Meanwhile we appear still to have $D-1$ momenta E_i, one too many. Worse, we seem to have D equations of motion 5.22, two too many. On closer inspection, however, one of the equations of motion, the time component of 5.22, turns out not to contain second time derivatives of A_i:

$$\partial_i E_i = 0. \tag{5.27}$$

It is a constraint equation (the Gauss law), reducing the number of independent momenta in the initial conditions to $D-2$. The dynamical equations of motion are the $D-1$ spacelike components of 5.22,

$$-\dot{E}_i + \partial_j(\partial_j A_i - \partial_i A_j) = 0. \tag{5.28}$$

In fact, the divergence of this equation merely serves to conserve the constraint 5.27, so there are actually only $D-2$ dynamical equations of motion, which is the correct number.

The Cauchy problem for general relativity is treated in painful detail in section 10.2 of Wald. Here we give a shortcut to extract some intuition and the most important equations. It will provide a local (in time) formulation of the Cauchy problem.

As in the case of electromagnetism, in order to have a deterministic time evolution we need to eliminate the gauge freedom, in this case local reparametrization invariance. An ideal gauge for treating the Cauchy problem is provided by Gaussian normal coordinates, discussed in subsection 2.13 above. Given a spacelike hypersurface S of a pseudo-Riemannian manifold, and coordinates x^i on S, there is a unique set of coordinates $\{x^0, x^i\}$ (on some neighborhood of S) such that the $x^0 = 0$ surface is S, and such that the metric takes the form

$$ds^2 = -(dx^0)^2 + \gamma_{ij} dx^i dx^j. \tag{5.29}$$

(Note that in making this statement we are not using the equations of motion—this gauge choice is possible even off-shell.) Requiring the metric to be of the form 5.29 fixes all of the local diffeomorphism freedom, except the possibility to choose a different surface S or to reparametrize it (in total, D functions on S, compared to the usual D functions on the full spacetime for general diffeomorphisms.) Of course, for generic choices of S the region covered by the normal coordinates will not be particularly large, meaning that the Cauchy evolution dictated by the equations derived below will typically develop a coordinate singularity well before any real singularity or Cauchy horizon forms in the spacetime—this is why it's local in time.

We'll treat the vacuum Einstein equation, but it's trivial to include a stress tensor or cosmological constant. By our gauge choice, we have gotten rid of all the superfluous components of the metric; the dynamical variables are the components γ_{ij}, which describe the intrinsic geometry of each $x^0 =$ constant surface. Their time derivatives define the extrinsic curvature of the surface:

$$K_{ij} = \frac{1}{2}\partial_0 \gamma_{ij}. \tag{5.30}$$

However, we are not free to choose all of the components of γ_{ij} and K_{ij} independently on the initial value surface S. For each gauge condition we have imposed, one of the equations of motion will turn into a constraint equation. These are the 00 and $i0$ components of the Einstein equation, which do not contain second time derivatives of γ_{ij}:

$$2G_{00} = R' + K^2 - K_{ij}K^{ij} = 0 \tag{5.31}$$

$$G_{i0} = \nabla'_j K^j{}_i - \partial_i K = 0 \tag{5.32}$$

(where R' and ∇' are derived from γ_{ij}). The initial value data thus consists of a total of $D(D-2)$ independent functions (although, given the residual gauge freedom, only $D(D-3)$ of these are meaningful in the sense of giving rise to different spacetimes). The true dynamical equation of motion is $G_{ij} = 0$. This is a bit complicated, but is equivalent to the simpler R_{ij} equation if one assumes that the above constraints are satisfied:

$$R_{ij} = \partial_0 K_{ij} + R'_{ij} + KK_{ij} - 2K_{ik}K^k{}_j = 0. \tag{5.33}$$

Together with 5.30, this equation determines the evolution of K_{ij} and γ_{ij}. Although I haven't checked it myself, it should be the case that if the constraints 5.31 are satisfied in the initial conditions, then they will automatically continue to be satisfied.

5.5 Gravitational waves

Let us first recall the situation for electromagnetic waves (in flat space). We wish to solve the equation of motion

$$\partial^2 A_a - \partial_a \partial^b A_b = 0 \tag{5.34}$$

modulo the gauge freedom

$$A_a \to A_a + \partial_a \lambda. \tag{5.35}$$

It is easy to show using the equation of motion that any wave with spacelike or timelike momentum must have a polarization parallel to its momentum and must therefore be pure gauge. Therefore, on-shell one may impose the gauge condition

$$\partial^2 A_a = 0, \tag{5.36}$$

which is equivalent (on-shell) to

$$\partial^a A_a = 0 \tag{5.37}$$

(assuming A_a is nicely behaved at infinity). This condition, which partially fixes the gauge, is known as Lorenz gauge. (Off-shell, neither 5.36 nor 5.37 can be imposed.) Since 5.36 and 5.37 together imply the equation of motion, one may take them together as equation of motion-cum-gauge condition.

Working in Lorenz gauge, we can fix a null momentum k_a and choose light-cone coordinates $x^\pm, x^m$ such that

$$ds^2 = -2dx^+dx^- + dx^m dx^m \tag{5.38}$$

$$k^- = k_+ = 0, \qquad k^m = k_m = 0. \tag{5.39}$$

In these coordinates the component A_- is pure gauge, and the equation of motion is

$$A_+ = 0. \tag{5.40}$$

The on-shell physical degrees of freedom are the components A_m, which form a vector of $SO(D-2)$.

We can fix the remaining gauge freedom by setting $A_- = 0$. If we have chosen light-cone coordinates such that $(x^+ + x^-)/\sqrt{2}$ coincides with the original time coordinate x^0, then this condition is equivalent to $A_0 = 0$ (Coulomb gauge).

Now let us turn to gravitational waves in flat space. The equation of motion for a metric fluctuation h_{ab} is

$$\delta R_{cd} = \partial_a \partial_{(c} h^a_{d)} - \frac{1}{2}\partial_c \partial_d h - \frac{1}{2}\partial^2 h_{cd} = 0. \tag{5.41}$$

This must be solved modulo the gauge freedom

$$h_{ab} \to h_{ab} + \partial_{(a}\xi_{b)}. \tag{5.42}$$

The analysis is similar to the case of electromagnetism. Again, waves with spacelike or timelike momentum are easily seen using the equation of motion to be pure gauge. Thus on-shell we may partially fix the gauge by imposing the condition

$$\partial^2 h_{ab} = 0, \tag{5.43}$$

or equivalently

$$\partial^a h_{ab} - \frac{1}{2}\partial_b h = 0. \tag{5.44}$$

(Again, neither condition may be imposed off-shell, and again, together they imply the equation of motion and may therefore be taken together as a combined equation of motion and gauge fixing condition.) This is the analog of Lorenz gauge, and it is sometimes called de Donder gauge. In the coordinate system 5.38, the components h_{-a}, h_{a-} are pure gauge, and the equation of motion is

$$h_{++} = h_{m+} = h_{+m} = h_{mm} = 0. \tag{5.45}$$

So the on-shell physical degrees of freedom are h_{mn}, with $h_{mm} = 0$, i.e. a symmetric traceless tensor of $SO(D-2)$.

We can fix the gauge a little more, but still not completely, by also setting $h_{+-} = h_{-+} = 0$. In position space this amounts to setting

$$h = 0, \qquad \partial^a h_{ab} = 0, \qquad \partial^2 h_{ab} = 0. \tag{5.46}$$

This is transverse traceless gauge. (Among the four equations 5.41, 5.46, any three imply the fourth with suitably nice behavior at infinity.)

Finally, we can completely fix the gauge by also setting $h_{-a} = h_{a-} = 0$, or (in the original coordinate system, in position space) $h_{0a} = h_{a0} = 0$.

The transverse traceless gauge can be generalized to curved backgrounds as follows (see section 7.5 of Wald):

$$h = 0, \qquad \nabla^a h_{ab} = 0, \qquad \nabla^2 h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} = 0. \tag{5.47}$$

The wave operator acting on h_{ab} in the last equation of 5.47 is known as the Lichnerowicz operator.

The quadratic approximation to the Einstein-Hilbert action $S = \frac{1}{2\kappa^2}\int \sqrt{|g|}R$ for a transverse traceless fluctuation h_{ab} about a solution, is

$$\begin{aligned} S &= \frac{1}{4\kappa^2}\int d^D x \sqrt{|g|}\, h^{ab}\left(\frac{1}{2}\nabla^2 h_{ab} + R_a{}^c{}_b{}^d h_{cd}\right) + O(h^3) \\ &= -\frac{1}{4\kappa^2}\int d^D x \sqrt{|g|}\, \nabla^c h^{ab}\left(\frac{1}{2}\nabla_c h_{ab} - \nabla_a h_{bc}\right) + O(h^3). \end{aligned} \tag{5.48}$$

5.6 Israel junction condition

Suppose that the metric is continuous but has components with discontinuous first derivatives across a timelike hypersurface S. Work in Gaussian normal coordinates (see subsection 2.13), so the metric is

$$ds^2 = \gamma_{ij}(y,\lambda) dy^i dy^j + d\lambda^2\,, \tag{5.49}$$

with S at $\lambda = 0$. The extrinsic curvature of the constant-λ hypersurfaces is

$$K_{ij} = \frac{1}{2}\frac{\partial \gamma_{ij}}{\partial \lambda}\,. \tag{5.50}$$

We assume that γ_{ij} is continuous but has a discontinuous first derivative at $\lambda = 0$, so K_{ij} is double-valued there:

$$K_{ij}^{\pm} := K_{ij}(\lambda = 0^{\pm})\,. \tag{5.51}$$

Note that K_{ij}^- is the extrinsic curvature with respect to the normal pointing *into* S on the $\lambda < 0$ side, while K_{ij}^+ is the extrinsic curvature with respect to the normal pointing *out of* S on the $\lambda > 0$ side. Let ΔK_{ij} be their difference:

$$\Delta K_{ij} := K_{ij}^+ - K_{ij}^-\,. \tag{5.52}$$

From (2.176)–(2.179), we see that the Einstein tensor has a delta-function with components tangent to S:

$$G_{ij} = (-\Delta K_{ij} + \Delta K \gamma_{ij})\,\delta(\lambda) + \text{finite} \tag{5.53}$$

$$G_{i\lambda} = \text{finite} \tag{5.54}$$

$$G_{\lambda\lambda} = \text{finite}\,. \tag{5.55}$$

For the spacetime to be a solution to the Einstein equation requires the stress tensor to include a piece localized on S with components tangents to it:

$$T_{ij} = t_{ij}\delta(\lambda) + \text{finite} \tag{5.56}$$

$$T_{i\lambda} = \text{finite} \tag{5.57}$$

$$T_{\lambda\lambda} = \text{finite}\,, \tag{5.58}$$

where

$$t_{ij} = \frac{1}{8\pi G_{\rm N}}\left(-\Delta K_{ij} + \Delta K \gamma_{ij}\right). \tag{5.59}$$

Such a localized stress tensor would result from an action localized on S: $S_S = \int_S |\gamma|^{1/2}\mathcal{L}_S$, where $\mathcal{L}_{\rm m}$ depends on the metric only through the induced metric on S.

The finite components of the stress tensor need not be continuous across S. The continuity equation $\nabla_a T^{ab} = 0$ relates the discontinuity $\Delta T_{ab} := T_{ab}(\lambda = 0^+) - T_{ab}(\lambda = 0^-)$ to t_{ij}:

$$\nabla'_i t^i{}_j = -\Delta T_{j\lambda}\,, \qquad K_{ij}t^{ij} = \Delta T_{\lambda\lambda}\,, \tag{5.60}$$

where ∇' is the covariant derivative with respect to γ_{ij}, and $K^{\rm ave}_{ij} := \frac{1}{2}(K^+_{ij} + K^-_{ij})$. (This can be derived by thickening S into a thin slab.) The first equation in (5.60) allows S to exchange energy and momentum with the surroundings, while the second relates a pressure difference on the two sides of S to a bending or acceleration of a brane.

If S is spacelike, then all the above equations still hold except (5.59), which is replaced by

$$t_{ij} = -\frac{1}{8\pi G_{\rm N}}\left(-\Delta K_{ij} + \Delta K \gamma_{ij}\right), \tag{5.61}$$

and the second equation in (5.60), which becomes

$$K_{ij}t^{ij} = \Delta T_{\lambda\lambda}\,. \tag{5.62}$$

However, such a stress tensor can never obey the null energy condition.

5.7 Boundary term

Let us consider the Einstein-Hilbert action (for pure gravity):

$$S_{\rm bulk} = \frac{1}{2\kappa^2}\int d^D x\sqrt{|g|}(R - 2\Lambda). \tag{5.63}$$

If the manifold on which we define this action has a boundary ∂, then the variation of the action under a small change in the metric $\delta g_{ab} = h_{ab}$ is

$$\delta S_{\rm bulk} = -\frac{1}{2\kappa^2}\int d^D x\sqrt{|g|}\left(R^{ab} - \frac{1}{2}Rg^{ab} + \Lambda g^{ab}\right)h_{ab} + \frac{1}{2\kappa^2}\int_\partial d^{D-1}x\sqrt{|\gamma|}n^a\left(\nabla^b h_{ab} - \partial_a h\right), \tag{5.64}$$

where γ_{ab} is the induced metric on the boundary and n^a is the unit normal vector to the boundary. In Gaussian normal coordinates (see subsection 2.13) the integrand of the surface term is

$$n^a\left(\nabla^b h_{ab} - \partial_a h\right) = \nabla'_i h^i{}_\lambda + K_{ij}h^{ij} + \sigma K h_{\lambda\lambda} - \gamma^{ij}\dot{h}_{ij} \tag{5.65}$$

$$= K^{ij}h_{ij} - 2\gamma^{ij}\delta K_{ij} - \nabla'_i h^i{}_\lambda\,, \tag{5.66}$$

where i, j index directions along the boundary, $h_{ij} = \delta\gamma_{ij}$, and K_{ij} is the extrinsic curvature of the boundary.[2] The extrinsic curvature is roughly speaking the normal derivative of the tangential components of the metric (see (2.163)). Its appearance here can be traced to the fact that the action (5.63) contains second derivatives of the metric. Since the last term in (5.66) is the divergence of the boundary vector field $h^i{}_\lambda$, its integral over the boundary vanishes, so the surface term can be written as

$$\frac{1}{2\kappa^2}\int_\partial d^{D-1}x\sqrt{|\gamma|}\left(K^{ij}h_{ij} - 2\gamma^{ij}\delta K_{ij}\right), \tag{5.67}$$

If we wish to have a well-posed variational problem, we will need to impose some boundary conditions on the metric. Let us first impose Dirichlet boundary conditions, i.e. fix γ_{ij}. This implies $h_{ij} = 0$, so the first term in 5.67 vanishes. The second term may be cancelled by adding the following boundary term to the action:[3]

$$S_{\text{boundary}} = \frac{1}{\kappa^2}\int_\partial d^{D-1}x\sqrt{|\gamma|}K\,. \tag{5.69}$$

This boundary term may be justified for another reason, namely that it makes the action extensive. Specifically, suppose we divide the spacetime volume V on which we are evaluating the action into two pieces V_1 and V_2 along the hypersurface S. If we demand that the metric be continuous across S, but not its first derivative (i.e. S may have different a extrinsic curvature on the V_1 side than on the V_2 side), then the Ricci scalar may have a delta-function singularity on S (see 2.179). Without the boundary term, the Einstein-Hilbert action evaluated on V_1 alone will miss this contribution, as will the action evaluated on V_2 alone. Hence the action evaluated on all of V, which includes this contribution, will not be the sum of the actions evaluated on V_1 and V_2. Inclusion of the boundary term remedies this defect.

[2] To get the sign of K_{ij} right: For a timelike boundary K_{ij} is defined as the outward-directed derivative of γ_{ij} (so that, using 2.163, $\lambda < 0$ inside the region being integrated). For a spacelike boundary K_{ij} is the inward-directed derivative of γ_{ij} (the usual time derivative for a past boundary, minus the usual one for a future boundary; using 2.163, $\lambda > 0$ inside the region being integrated).

[3] In Euclidean signature, the Einstein-Hilbert action, including boundary term, is

$$S = -\frac{1}{2\kappa^2}\int d^Dx\,\sqrt{g}R - \frac{1}{\kappa^2}\int d^{D-1}x\,\sqrt{\gamma}K\,, \tag{5.68}$$

where K_{ij} is the outward-directed derivative of γ_{ij} at the boundary. (For example, for a flat disk, $\int\sqrt{\gamma}K = 2\pi$.)

Now let us consider the effect of matter. If the matter Lagrangian depends on the first derivative of the metric (e.g. on Γ^a_{bc}), then the surface term will pick up additional contributions proportional to h_{ij}, which will vanish with Dirichlet boundary conditions. Therefore we need not include any further boundary term in the action.

One can also consider other consistent boundary conditions, such as Neuman, $K_{ij} = 0$, or mixed (or umbilic), $K_{ij} = \alpha\gamma_{ij}$ where α is a constant. For pure gravity, the former does not require a boundary term, as (5.67) vanishes identically, while the latter requires the boundary term[4]

$$S_{\text{boundary}} = \frac{\alpha}{\kappa^2}\int_\partial d^{D-1}x\sqrt{|\gamma|}\,. \tag{5.70}$$

In the presence of matter, additional boundary terms may be required for these boundary conditions.

5.8 Dimensional reductions

If we dimensionally reduce $d + k$ dimensions down to d dimensions on a manifold ds^2_k of vanishing Ricci scalar, then the following ansatz relates the (Einstein-frame) metric ds^2_{d+k} of the higher-dimensional manifold to the (Einstein-frame) metric ds^2_d of the lower-dimensional one:

$$ds^2_{d+k} = e^{-2k\tau/(d-2)}ds^2_d + e^{2\tau}ds^2_k. \tag{5.71}$$

The Einstein-Hilbert action is

$$\frac{1}{2\kappa^2_{d+k}}\int d^{d+k}x\,\sqrt{|g_{d+k}|}R_{d+k} = \frac{1}{2\kappa^2_d}\int d^dx\,\sqrt{|g_d|}\left(R_d - k\left(\frac{k}{d-2}+1\right)\partial_m\tau\partial^m\tau\right), \tag{5.72}$$

where $\kappa^2_d = \kappa^2_{d+k}/V_k$. This is a consistent truncation of $(d + k)$-dimensional gravity only if the metric ds^2_k is fixed (up to the overall scale) by symmetries, for example for a circle or a square torus. In that case, the equations of motion derived from (5.72) are equivalent to the full $d+k$ dimensional Einstein equation. (See the discussion near the beginning of subsection 3.2.) More generally, any parameters of the compact space which are not determined by symmetries (for example, the moduli of a general torus) must be included.

We can obtain dilaton gravity by dimensionally reducing pure Einstein gravity on a circle,

$$\int d^{d+1}x\,\sqrt{|g_{d+1}|}R_{d+1} = 2\pi r\int d^dx\,\sqrt{|g|}e^{-2\Phi}\left(R + 4\partial_m\Phi\partial^m\Phi\right), \tag{5.73}$$

[4]One way to impose the boundary condition $K_{ij} = \alpha\gamma_{ij}$ is to double the manifold along the boundary, impose a $\mathbf{Z}_2$ symmetry on the metric, and insert a brane of tension $-2(D-2)\alpha/\kappa^2$ at the (former) boundary. In the notation of subsection 5.6, the localized stress tensor due to the brane is $t_{ij} = -2(D-2)\alpha\gamma_{ij}/\kappa^2$, and the Israel junction condition (**??**) requires (in view of the $\mathbf{Z}_2$ symmetry) $K_{ij} = -\Delta K_{ij}/2 = \alpha\gamma_{ij}$. In this setup the boundary condition thus becomes an equation of motion. Imposing this equation, there is, in addition to the brane action $-2(D-2)\alpha/\kappa^2\int_\partial\sqrt{|\gamma|}$, a localized piece of the Einstein-Hilbert action $2\alpha(D-1)/\kappa^2\int_\partial\sqrt{|\gamma|}$ (see (2.179)). Adding them, we obtain twice the boundary action (5.70) (recall that we've doubled the bulk action).

with the ansatz

$$ds_{d+1}^2 = \exp\left(\frac{4}{\sqrt{d-1}}\Phi\right)dy^2 + \exp\left(\frac{4}{\sqrt{d-1}-(d-1)}\Phi\right)ds^2, \tag{5.74}$$

where y is the coordinate on the circle (with coordinate radius r) and ds^2 is the d-dimensional (string frame) metric. The case $d = 10$ gives the usual reduction of M-theory to 10-dimensional supergravity.

6. String theory & CFT

6.1 Complex coordinates

On a Euclidean worldsheet we can use real coordinates σ^1, σ^2 or complex coordinates (Polchinski's conventions)

$$z = \sigma^1 + i\sigma^2, \qquad \bar{z} = \sigma^1 - i\sigma^2. \tag{6.1}$$

In these coordinates we have

$$\partial = \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \qquad \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2), \tag{6.2}$$

and for a general vector v^a

$$v^z = v^1 + iv^2, \qquad v^{\bar{z}} = v^1 - iv^2, \qquad v_z = \frac{1}{2}(v_1 - iv_2) \qquad v_{\bar{z}} = \frac{1}{2}(v_1 + iv_2). \tag{6.3}$$

Note that

$$d^2\sigma = \frac{1}{2}d^2z, \tag{6.4}$$

so we define the delta function

$$\delta^2(z,\bar{z}) = \frac{1}{2}\delta^2(\sigma^1,\sigma^2). \tag{6.5}$$

If the worldsheet is flat, $g_{ab} = \delta_{ab}$, then

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}, \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0, \qquad g^{z\bar{z}} = g^{\bar{z}z} = 2, \qquad g^{zz} = g^{\bar{z}\bar{z}} = 0, \tag{6.6}$$

and

$$\epsilon_{z\bar{z}} = -\epsilon_{\bar{z}z} = \frac{i}{2}, \qquad \epsilon^{z\bar{z}} = -\epsilon^{\bar{z}z} = -2i. \tag{6.7}$$

The divergence theorem is

$$\int_R d^2z(\partial_z v^z + \partial_{\bar{z}} v^{\bar{z}}) = i\oint_{\partial R}(d\bar{z}v^z - dzv^{\bar{z}}). \tag{6.8}$$

6.2 CFT stress tensor

By convention in 2d CFT, the stress tensor is defined with an extra factor of 2π compared to the standard definition (5.9) in GR:

$$T^{ab} := \frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g_{ab}}\,. \tag{6.9}$$

6.3 Liouville action

The stress tensor of a CFT is traceless on a flat world-sheet, but the Weyl anomaly implies the following value for the trace of the stress tensor (where c is the central charge):

$$T^a{}_a = -\frac{c}{12} R \,. \tag{6.10}$$

Note that, whereas the other stress tensor components are operators, the trace is a c-number. Writing the metric on a closed surface as $e^{2\omega} g_{ab}$, where g_{ab} is a fixed fiducial metric, we then have the following functional differential equation for the partition function as a functional of the metric:

$$-\frac{2\pi}{e^{2\omega}\sqrt{g}} \frac{\delta \ln Z[e^{2\omega} g_{ab}]}{\delta\omega} = \langle T^a{}_a \rangle = T^a{}_a = -\frac{c}{12} R[e^{2\omega} g_{ab}] = -\frac{c}{12} e^{-2\omega} \left(R - 2\nabla^2\omega\right) \tag{6.11}$$

(where $R := R[g_{ab}]$), in other words

$$\frac{\delta \ln Z[e^{2\omega} g_{ab}]}{\delta\omega} = \frac{c}{24\pi} \left(R - 2\nabla^2\omega\right). \tag{6.12}$$

The solution to (6.12) is

$$\ln Z[e^{2\omega} g_{ab}] = \ln Z[g_{ab}] + \frac{c}{6} S_L[\omega, g_{ab}] \,, \tag{6.13}$$

where S_L is the Liouville action:

$$S_L[\omega, g_{ab}] := \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \left(g^{ab} \partial_a\omega \partial_b\omega + R\omega \right). \tag{6.14}$$

The Liouville action obeys the following composition rule, as it must for consistency of (6.13):

$$S_L[\omega_1 + \omega_2, g_{ab}] = S_L[\omega_1, g_{ab}] + S_L[\omega_2, e^{2\omega_1} g_{ab}] \,. \tag{6.15}$$

If ω is constant then

$$S_L[\omega, g_{ab}] = \chi\omega \,, \tag{6.16}$$

where χ is the Euler character of the surface.

6.4 Sigma model

The bosonic sigma model action is (for a Euclidean worldsheet)

$$S_\sigma = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left[\left(g^{ab} G_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu}(X) \right) \partial_a X^\mu \partial_b X^\nu + \alpha' R_{(2)} \Phi(X) \right]. \tag{6.17}$$

In conformal gauge, $g_{ab} = e^{2\omega}\delta_{ab}$, we can use complex coordinates, in which case the action becomes:

$$S_\sigma = \frac{1}{2\pi\alpha'} \int d^2z \left[\left(G_{\mu\nu}(X) + B_{\mu\nu}(X) \right) \partial X^\mu \bar\partial X^\nu - 2\alpha' \Phi(X) \partial\bar\partial\omega \right]. \tag{6.18}$$

The trace of the worldsheet stress tensor for this model is

$$T^a{}_a = -\frac{1}{2\alpha'}\left(g^{ab}\beta^G_{\mu\nu} + i\epsilon^{ab}\beta^B_{\mu\nu}\right)\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R_{(2)}, \tag{6.19}$$

where the beta functions are

$$\frac{1}{\alpha'}\beta^G_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu\partial_\nu\Phi - \frac{1}{4}H_{\mu\lambda\xi}H_\nu{}^{\lambda\xi} + O(\alpha') \tag{6.20}$$

$$\frac{1}{\alpha'}\beta^B_{\mu\nu} = -\frac{1}{2}\nabla^\xi H_{\xi\mu\nu} + \partial^\xi\Phi H_{\xi\mu\nu} + O(\alpha') \tag{6.21}$$

$$\frac{1}{\alpha'}\beta^\Phi = \frac{D+c'}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \partial_\xi\Phi\partial^\xi\Phi - \frac{1}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + O(\alpha'). \tag{6.22}$$

Here c' is the total central charge of any other fields on the world-sheet, which are assumed to be CFTs (for example -26 for the ghosts). To this order in α', the beta functions obey the following identities:[5]

$$\nabla_\mu\beta^{B\mu\nu} - 2\partial_\mu\Phi\beta^{B\mu\nu} = 0 \tag{6.23}$$

$$-2\nabla_\mu\beta^{G\mu}{}_\alpha + \partial_\alpha\beta^{G\mu}{}_\mu + 4\partial_\mu\Phi\beta^{G\mu}{}_\alpha + H_{\mu\nu\alpha}\beta^{B\mu\nu} - 4\partial_\alpha\beta^\Phi = 0\,. \tag{6.24}$$

The sigma model defines a CFT as long as G, B, and Φ satisfy $\beta^G = \beta^B = 0$.[6] Under these conditions, the Wess-Zumino consistency condition requires β^Φ to be a c-number rather than an operator, i.e. to be constant in the target space; this can also be seen from (6.24) at the corresponding order in α'. This constant is, up to a factor of 6, the central charge:

$$c = 6\beta^\Phi. \tag{6.25}$$

Thus $\beta^\Phi = 0$ is the further condition for having a good string background.

6.5 Spacetime action

The equations $\beta^G = \beta^B = \beta^\Phi = 0$ for G, B, and Φ can be derived as the equations of motion of the following spacetime action:[7]

$$S = \frac{1}{2\kappa^2}\int d^Dx\sqrt{|G|}e^{-2\Phi}\left(-2\frac{D+c'}{3\alpha'} + R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + O(\alpha')\right), \tag{6.26}$$

[5]As we will discuss in the next subsection, the beta functions can be derived as equations of motion from a target-space action. (6.23) follows from the fact that this action is invariant under B-field gauge transformations, and (using (6.23)) (6.24) follows from the fact that it is invariant under diffeomorphisms. Since the derivability of the beta functions from a gauge- and diffeomorphism-invariant action is believed to hold to all orders in α', the identities (6.23), (6.24), possibly with explicit α'-corrections, should also hold to all orders.

[6]Note that it is necessary to specify Φ to define the sigma model even if one is working on a flat worldsheet, because it enters into the definition of the stress tensor.

[7]The boundary term for this action is $\frac{1}{\kappa^2}\int_\partial d^{D-1}x\sqrt{|\gamma|}e^{-2\Phi}K$.

Specifically,

$$\delta S = \\ -\frac{1}{2\kappa^2\alpha'}\int d^Dx\sqrt{|G|}e^{-2\Phi}\left(\delta G_{\mu\nu}\beta^{G\mu\nu}+\delta B_{\mu\nu}\beta^{B\mu\nu}-8(\delta\Phi-\frac{1}{4}G^{\mu\nu}\delta G_{\mu\nu})(\beta^\Phi-\frac{1}{4}\beta^{G\xi}_\xi)\right). \tag{6.27}$$

The RHS of this equation defines a metric on the space of fluctuations of the fields G, B, Φ. In other words, using I, J to index both the field and the spacetime point, we have

$$\delta S = -\delta\phi^I g_{IJ}\beta^J. \tag{6.28}$$

It follows that RG flow is gradient flow (albeit with an indefinite metric) with respect to a potential given by the spacetime action

$$-\frac{d\phi^I}{d\ln\Lambda_{\text{ren}}} = -\beta^I = g^{IJ}\frac{\delta S}{\delta\phi^J}. \tag{6.29}$$

(Note that g_{IJ} is not the Zamolodchikov metric for the vertex operators corresponding to the fluctuations, since it does not vanish for pure gauge fluctuations. However, perhaps there is some ambiguity in its definition, and one could choose a metric which vanishes on pure gauge fluctuations and is positive definite on physical ones. This would then presumably equal the Zamolodchikov metric. If this is the case, it would be interesting to consider whether it has any implications for RG flows.)

The action (6.26) is also the NS-NS part of the action for the superstring in $D = 10$, with the first (α'^{-1}) term absent and

$$2\kappa^2 = (2\pi)^7\alpha'^4. \tag{6.30}$$

6.6 Dimensional reduction and Buscher duality

We consider configurations of the metric, B-field, and dilaton that have a $U(1)$ invariance. Call the invariant direction y, the other directions x^μ, and all of them x^M. The coordinate radius of y is R.

The Buscher rules are as follows:

$$R' = \frac{\alpha'}{R} \tag{6.31}$$

$$G'_{\mu\nu} = G_{\mu\nu} - G^{-1}_{yy}G_{y\mu}G_{y\nu} + G^{-1}_{yy}B_{y\mu}B_{y\nu} \tag{6.32}$$

$$B'_{\mu\nu} = B_{\mu\nu} - G^{-1}_{yy}G_{y\mu}B_{y\nu} + G^{-1}_{yy}B_{y\mu}G_{y\nu} \tag{6.33}$$

$$G'_{y\mu} = G^{-1}_{yy}B_{y\mu} \tag{6.34}$$

$$B'_{y\mu} = G^{-1}_{yy}G_{y\mu} \tag{6.35}$$

$$G'_{yy} = G^{-1}_{yy} \tag{6.36}$$

$$e^{2\Phi'} = \frac{\alpha'}{G_{yy}R^2}e^{2\Phi} \tag{6.37}$$

These can be more usefully organized in terms of the fields of the dimensionally reduced theory. These are: a metric $g_{\mu\nu}$; a two-form $b_{\mu\nu}$; two gauge fields A_μ and $\tilde{A}_\mu$; and two scalars σ and ϕ. The fields of the original theory are given in terms of these as follows (we set the coordinate radius R equal to $\sqrt{\alpha'}$):

$$
\begin{aligned}
ds^2 = G_{MN}dx^M dx^N &= g_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}(dy + A_\mu dx^\mu)^2 && (6.38)\\
B &= b + (A + 2dy) \wedge \tilde{A} && (6.39)\\
\Phi &= \phi + \frac{1}{2}\sigma. && (6.40)
\end{aligned}
$$

Other useful formulas include:

$$
\begin{aligned}
G_{\mu\nu} &= g_{\mu\nu} + e^{2\sigma}A_\mu A_\nu && (6.41)\\
G_{y\mu} &= e^{2\sigma}A_\mu && (6.42)\\
G_{yy} &= e^{2\sigma} && (6.43)\\
G^{\mu\nu} &= g^{\mu\nu} && (6.44)\\
G^{\mu y} &= -A^\mu && (6.45)\\
G^{yy} &= e^{-2\sigma} + A^2 && (6.46)\\
\sqrt{G} &= e^\sigma \sqrt{g} && (6.47)\\
R^{(G)} &= R^{(g)} - 2e^{-\sigma}\nabla^2 e^\sigma - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} && (6.48)\\
B_{\mu\nu} &= b_{\mu\nu} + A_{[\mu}\tilde{A}_{\nu]} && (6.49)\\
B_{y\mu} &= \tilde{A}_\mu. && (6.50)
\end{aligned}
$$

The spacetime action is

$$
\begin{aligned}
S &= \frac{1}{2\kappa^2}\int \sqrt{|G|}e^{-2\Phi}\left(R^{(G)} + 4\partial_M\Phi\partial^M\Phi - \frac{1}{12}H_{MNL}H^{MNL}\right) && (6.51)\\
&= \frac{2\pi\sqrt{\alpha'}}{2\kappa^2}\int \sqrt{|g|}e^{-2\phi} && (6.52)\\
&\quad\times\left(R^{(g)} - \partial_\mu\sigma\partial^\mu\sigma + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}e^{-2\sigma}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{12}h_{\mu\nu\lambda}h^{\mu\nu\lambda}\right). && (6.53)
\end{aligned}
$$

We raise and lower M, N with G_{MN}, and μ, ν with $g_{\mu\nu}$. The field strengths $F, \tilde{F}$ are defined in the usual way ($F = dA$, $\tilde{F} = d\tilde{A}$), while h also has Chern-Simons contributions:

$$
h = db + \frac{1}{2}A \wedge \tilde{F} + \frac{1}{2}\tilde{A} \wedge F. \tag{6.54}
$$

This action has several gauge invariances (the higher dimensional origin of each is indicated in parentheses):

- Diffeomorphisms (diffeomorphisms $x'^\mu = x'^\mu(x^\mu)$, $y' = y$).

- Gauge transformations on b: $b' = b + d\lambda$ (B-field transformations where the gauge parameter does not have a y component).

- Gauge transformations on A: $A' = A + d\lambda$, $b' = b + \frac{1}{2}d\lambda \wedge \tilde{A}$ (diffeomorphisms $x'^\mu = x^\mu$, $y' = y + \lambda(x^\mu)$).

- Gauge transformations on $\tilde{A}$: $\tilde{A}' = \tilde{A} + d\lambda$, $b' = b - \frac{1}{2}A \wedge d\lambda$ (B-field transformations where the gauge parameter is proportional to dy).

- The $\mathbf{Z}_2$ transformation $\sigma' = -\sigma$, $A' = \tilde{A}$, $\tilde{A}' = A$ (T-duality).

The definitions of the lower-dimensional fields have been carefully chosen so as to give the simplest possible transformation rules. In particular, $g_{\mu\nu}$ is defined the way it is in order to be invariant under the gauge transformations of A; as a bonus it also ends up being invariant under T-duality. ϕ and b are defined as they are in order to be invariant under T-duality. Note that the 3-form field strength h is invariant under all of these transformations (actually, covariant under diffeomorphisms).

The metric (6.27) on the space of field configurations becomes, in terms of these variables,

$$dS^2 = \frac{1}{2\kappa^2\alpha'}\int\sqrt{|G|}e^{-2\Phi}\left(dG_{MN}^2 + dB_{MN}^2 - \frac{1}{2}\left(dG_M{}^M - d\Phi\right)^2\right) \tag{6.55}$$

$$\begin{aligned} &= \frac{2\pi}{2\kappa^2\sqrt{\alpha'}}\int\sqrt{|g|}e^{-2\phi}\left(dg_{\mu\nu}^2 + \left(db_{\mu\nu} - A_{[\mu}d\tilde{A}_{\nu]} - \tilde{A}_{[\mu}dA_{\nu]}\right)^2 + 2e^{2\sigma}dA_\mu^2 + 2e^{-2\sigma}d\tilde{A}_\mu^2\right. \\ &\qquad\qquad \left. + 4d\sigma^2 - \frac{1}{2}\left(dg_\mu{}^\mu - 4d\phi\right)^2\right) \end{aligned} \tag{6.56}$$

This is manifestly invariant under T-duality. It follows that the beta functions are also invariant under T-duality. This is because T-duality is an equivalence between sigma models, irrespective of whether they are conformal.

6.7 More dualities

The NS-NS 2-form potential B has mass dimension -2, and the R-R p-form potentials, denoted C_p, have dimension $-p$. All metrics are in string frame. In the first two dualities below, all fields are assumed independent of the coordinate y, on which the dualities are performed.

M-theory–IIA:

$$M_{11}^2 ds_{\rm M}^2 = G_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}\left(dy + A_\mu dx^\mu\right)^2, \qquad y \sim y + 2\pi R \tag{6.57}$$

$$\Updownarrow \tag{6.58}$$

$$\frac{1}{\alpha'}ds_{\rm IIA}^2 = Re^\sigma G_{\mu\nu}dx^\mu dx^\nu, \qquad e^\Phi = (Re^\sigma)^{3/2}, \qquad \frac{1}{\sqrt{\alpha'}}C_1 = \frac{1}{R}A_\mu dx^\mu \tag{6.59}$$

T-duality:

$$ds^2_{\text{IIA}} = G_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}\left(dy + A_\mu dx^\mu\right)^2, \qquad y \sim y + 2\pi R, \tag{6.60}$$

$$B_{\text{IIA}} = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu + b_\mu dy \wedge dx^\mu, \qquad e^{\Phi_{\text{IIA}}}, \qquad C_1 = C_\mu dx^\mu + C_y dy \tag{6.61}$$

$$\Updownarrow \tag{6.62}$$

$$ds^2_{\text{IIB}} = G_{\mu\nu}dx^\mu dx^\nu + e^{-2\sigma}\left(dy + b_\mu dx^\mu\right)^2, \qquad y \sim y + 2\pi\frac{\alpha'}{R}, \tag{6.63}$$

$$B_{\text{IIB}} = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu + b_\mu A_\nu dx^\mu \wedge dx^\nu + A_\mu dy \wedge dx^\mu, \qquad e^{\Phi_{\text{IIB}}} = \frac{\sqrt{\alpha'}}{Re^\sigma}e^{\Phi_{\text{IIA}}}, \tag{6.64}$$

$$C_2 = \frac{R}{\sqrt{\alpha'}}C_\mu dx^\mu \wedge dy, \qquad C_0 = \frac{R}{\sqrt{\alpha'}}C_y \tag{6.65}$$

S-duality:

$$ds'^2 = e^{-\Phi}ds^2, \qquad \Phi' = -\Phi, \qquad B' = C_2, \qquad C'_2 = -B \tag{6.66}$$

6.8 Orbifolds and quotient groups

Let G be a group acting on a space (or a CFT), and H a normal subgroup of G. Then the space (or CFT) obtained by orbifolding by G is the same as that obtained by orbifolding first by H and then by G/H.

7. Quantum information theory

In the following, ρ, σ denote density matrices and $\{\lambda_i\}$ is a probability distribution (i.e. $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$). The Shannon entropy is

$$H(\{\lambda_i\}) := -\sum_i \lambda_i \ln \lambda_i\,, \tag{7.1}$$

where $0 \ln 0$ is defined to equal 0.

7.1 Von Neumann entropy

Definition:

$$S(\rho) := -\operatorname{Tr}\rho\ln\rho = \langle -\ln\rho\rangle_\rho = H(\{p_a\})\,, \tag{7.2}$$

where $\{p_a\}$ is the spectrum of ρ.

Positivity:

$$S(\rho) \geq 0 \tag{7.3}$$

with equality if and only if ρ is pure.

Maximal value:

$$S(\rho) \leq \ln d \tag{7.4}$$

(d is the dimension of the Hilbert space). For d finite, there is equality if and only if $\rho = I/d$. For d infinite, $S(\rho)$ can be arbitrarily large or infinite; e.g. if the nth eigenvalue of ρ (in decreasing order) goes like $1/(n(\ln n)^2)$, then $S(\rho) = \infty$.

Invariance:

$$S(U\rho U^{-1}) = S(\rho) \tag{7.5}$$

(U unitary).

Expansibility:

$$S(\rho \oplus 0) = S(\rho) \tag{7.6}$$

Concavity:

$$S\left(\lambda\rho + (1-\lambda)\sigma\right) \geq \lambda S(\rho) + (1-\lambda)S(\sigma) \tag{7.7}$$

for $0 \leq \lambda \leq 1$, with equality if and only if $\lambda = 0$, $\lambda = 1$, $\rho = \sigma$, $S(\rho) = \infty$, or $S(\sigma) = \infty$. For a general convex combination $\rho = \sum_i \lambda_i \rho_i$,

$$\sum_i \lambda_i S(\rho_i) \leq S(\rho) \leq \sum_i \lambda_i S(\rho_i) + H(\{\lambda_i\})\,. \tag{7.8}$$

The term $H(\{\lambda_i\})$ is called the *mixing entropy*. The difference between $S(\rho)$ and the left-hand side is called the *Holevo information*:

$$\chi(\{\lambda_i, \rho_i\}) := S(\rho) - \sum_i \lambda_i S(\rho_i)\,. \tag{7.9}$$

So

$$0 \leq \chi(\{\lambda_i, \rho_i\}) \leq H(\{\lambda_i\})\,. \tag{7.10}$$

If the ρ_i are pairwise orthogonal then the second inequality is saturated:

$$S(\rho) = \sum_i \lambda_i S(\rho_i) + H(\{\lambda_i\})\,, \qquad \chi(\{\lambda_i, \rho_i\}) = H(\{\lambda_i\})\,. \tag{7.11}$$

Canonical ensemble: Given a Hamiltonian H and energy E, $S(\rho)$ is maximized within the set of density matrices satisfying $\langle H \rangle_\rho = E$ by

$$\rho_\beta = e^{-\beta H}/Z\,, \qquad Z = \operatorname{Tr} e^{-\beta H}\,, \tag{7.12}$$

where β is chosen to satisfy $\langle H \rangle_\rho = E$. For fixed H, $S(\rho_\beta)$ is a decreasing function of β.

7.2 Relative entropy

Definition:

$$S(\sigma|\rho) := \operatorname{Tr} \rho \left(\ln \rho - \ln \sigma\right) = \langle -\ln \sigma \rangle_\rho - S(\rho) \tag{7.13}$$

Positivity:

$$S(\sigma|\rho) \geq 0\,, \tag{7.14}$$

with equality if and only if $\sigma = \rho$.

Norm bound:

$$S(\sigma|\rho) \geq \frac{1}{2}\|\sigma - \rho\|_1^2 \tag{7.15}$$

Relative entropy in terms of entropy: Define

$$S_\lambda(\sigma|\rho) = S(\lambda\sigma + (1-\lambda)\rho) - \lambda S(\sigma) - (1-\lambda)S(\rho) \qquad (0 \leq \lambda \leq 1)\,. \tag{7.16}$$

$S_\lambda(\sigma|\rho)$ is concave in λ and satisfies

$$0 \leq S_\lambda(\sigma|\rho) \leq -\lambda \ln \lambda - (1-\lambda)\ln(1-\lambda) \tag{7.17}$$

(so in particular $S_0(\sigma|\rho) = S_1(\sigma|\rho) = 0$). Then

$$S(\sigma|\rho) = \frac{d}{d\lambda} S_\lambda(\sigma|\rho)\bigg|_{\lambda=0}\,. \tag{7.18}$$

Joint convexity:

$$S(\sigma|\rho) \leq \lambda S(\sigma_1|\rho_1) + (1-\lambda)S(\sigma_2|\rho_2) \tag{7.19}$$

where $\sigma = \lambda\sigma_1 + (1-\lambda)\sigma_2$, $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$.

7.3 Rényi entropy

The Rényi entropy $S_\alpha(\rho)$ is defined for $0 \leq \alpha \leq \infty$:

$$S_\alpha(\rho) = \frac{1}{1-\alpha}\ln \operatorname{Tr}\rho^\alpha \qquad (\alpha \neq 0, 1, \infty) \tag{7.20}$$

$$S_0(\rho) = \ln \operatorname{rank}\rho = \lim_{\alpha\to 0} S_\alpha(\rho) \tag{7.21}$$

$$S_1(\rho) = S(\rho) = \lim_{\alpha\to 1} S_\alpha(\rho) \tag{7.22}$$

$$S_\infty(\rho) = -\ln p_1 = \lim_{\alpha\to\infty} S_\alpha(\rho) \tag{7.23}$$

where p_1 is the largest eigenvalue of ρ. S_0 is also called the Hartley entropy. Properties (for fixed ρ):

$$S_\alpha \geq 0 \qquad (= 0 \text{ iff } \rho \text{ is pure}) \tag{7.24}$$

$$\frac{d}{d\alpha} S_\alpha \leq 0 \qquad (= 0 \text{ iff } \rho \propto \text{identity on its image}) \tag{7.25}$$

$$\frac{d^2}{d\alpha^2} S_\alpha \geq 0 \tag{7.26}$$

$$S_\alpha \leq \frac{\alpha}{\alpha - 1} S_\infty \qquad (\alpha > 1) \tag{7.27}$$

$$\frac{d}{d\alpha}\left(\frac{\alpha - 1}{\alpha} S_\alpha\right) \geq 0 \tag{7.28}$$

$$\frac{d}{d\alpha}\left((\alpha - 1) S_\alpha\right) \geq 0 \tag{7.29}$$

$$\frac{d^2}{d\alpha^2}\left((\alpha - 1) S_\alpha\right) \leq 0 \tag{7.30}$$

7.4 Mixing-enhancing

Let $\{p_a\}$ be the eigenvalues of ρ, arranged in decreasing order, and similarly $\{p'_a\}$ for σ. If, for all n less than or equal to the smaller of their dimensions,

$$\sum_{a=1}^{n} p_a \leq \sum_{a=1}^{n} p'_a \tag{7.31}$$

then we write $\rho \succ \sigma$ and say that ρ is *more mixed* than σ and that σ is *purer* than ρ. If $\rho \succ \sigma$ then

$$S(\rho) \geq S(\sigma)\,. \tag{7.32}$$

More generally

$$\operatorname{Tr} f(\rho) \geq \operatorname{Tr} f(\sigma) \tag{7.33}$$

for any concave f, and

$$S_\alpha(\rho) \geq S_\alpha(\sigma) \tag{7.34}$$

for all α.

A transformation F on density matrices such that $F(\rho) \succ \rho$ for all ρ is called *mixing-enhancing*. Examples:

1. $$\rho \mapsto \frac{f(\rho)}{\operatorname{Tr} f(\rho)} \tag{7.35}$$

   for f concave, positive on $(0, 1]$, and with $f(0) = 0$.

2. $$\rho \mapsto \sum_i P_i \rho P_i\,, \tag{7.36}$$

where $\{P_i\}$ is a set of orthogonal projectors such that $\sum_i P_i = I$ (representing possible outcomes of a measurement, for example). If the P_i are rank 1, then this corresponds to deleting the off-diagonal elements in the matrix representing ρ in the basis corresponding to the P_i; more generally it corresponds to deleting the off-block-diagonal elements.

3.
$$\rho \mapsto \sum_i \lambda_i U_i^{-1} \rho U_i \,, \tag{7.37}$$

where the U_i are unitary operators.

7.5 Joint systems

A, B, C denote disjoint systems. AB denotes $A \cup B$, etc. Partial traces are implied by subscripts, e.g. $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$. $S(A)$ denotes $S(\rho_A)$. Note that $\ln(\rho_A \otimes \rho_B) = \ln \rho_A \otimes I_B + I_A \otimes \ln \rho_B$.

Separable and entangled states: A *separable state* on AB is one that can be written in the form
$$\rho_{AB} = \sum_i \lambda_i \rho_A^{(i)} \otimes \rho_B^{(i)} \,, \tag{7.38}$$

where $\rho_{A,B}^{(i)}$ are density operators. A state that is not separable is called *entangled*.

Additivity:
$$S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) \,. \tag{7.39}$$

More generally,
$$S_\alpha(\rho_A \otimes \rho_B) = S_\alpha(\rho_A) + S_\alpha(\rho_B) \,. \tag{7.40}$$

Subadditivity:
$$S(AB) \leq S(A) + S(B) \tag{7.41}$$

with equality if and only if $\rho_{AB} = \rho_A \otimes \rho_B$.

Among the Rényi entropies, only the Hartley is also subadditive: $S_0(AB) \leq S_0(A) + S_0(B)$. This is true even for classical distributions. A simple counterexample for the $\alpha > 1$ Rényis is the following distribution on two bits, $p_{00} = 1 - 2x, p_{01} = p_{10} = x, p_{11} = 0$. This violates subadditivity for x in the range $0 \leq x \leq x_\alpha$ (where $0 < x_\alpha < 1/2$).

S and S_0 are the only functions of ρ obeying invariance, additivity, and subadditivity.

Araki-Lieb (or triangle):
$$|S(A) - S(B)| \leq S(AB) \tag{7.42}$$

$S(A) - S(B) = S(AB)$ if and only if A can be decomposed as $A = A_L \otimes A_R$ such that $\rho_{AB} = \rho_{A_L} \otimes \rho_{A_R B}$ where $\rho_{A_R B}$ is pure (see arXiv:1105.2993).

Strong subadditivity:

$$S(ABC) + S(B) \leq S(AB) + S(BC) \tag{7.43}$$
$$S(A) + S(C) \leq S(AB) + S(BC) \tag{7.44}$$

The first inequality if saturated if and only if B can be decomposed as $B = \bigoplus_i B_{Li} \otimes B_{Ri}$ such that

$$\rho_{ABC} = \bigoplus_i \lambda_i \rho^i_{AB_{Li}} \otimes \rho^i_{B_{Ri}C} \tag{7.45}$$

(see arXiv:quant-ph/0304007).

Additivity of relative entropy:

$$S(\sigma_A \otimes \sigma_B | \rho_A \otimes \rho_B) = S(\sigma_A|\rho_A) + S(\sigma_B|\rho_B) \tag{7.46}$$

Monotonicity of relative entropy:

$$S(\sigma_{AB}|\rho_{AB}) \geq S(\sigma_A|\rho_A) \tag{7.47}$$

Mutual information:

$$I(A:B) := S(A) + S(B) - S(AB) = S(\rho_A \otimes \rho_B|\rho_{AB}) \tag{7.48}$$

Properties of mutual information:

$$I(A:B) = I(B:A) \tag{7.49}$$
$$I(A:B) \geq 0 \qquad \text{(with equality if and only if } \rho_{AB} = \rho_A \otimes \rho_B) \tag{7.50}$$
$$I(A:B) + 2S(C) \geq I(A:BC) \geq I(A:B) \tag{7.51}$$

Bound on correlators: Let M_A, M_B be bounded operators in A, B respectively. Then

$$\frac{1}{2}\left(\frac{\langle M_A M_B\rangle - \langle M_A\rangle\langle M_B\rangle}{\|M_A\|\|M_B\|}\right)^2 \leq I(A:B) \tag{7.52}$$

Conditional entropy:

$$H(A|B) := S(AB) - S(B) = \ln d_B - S(\rho_A \otimes I_B/d_B|\rho_{AB}) \tag{7.53}$$

where d_B is the dimension of the B Hilbert space. Concavity of conditional entropy:

$$H(A|B)_{\lambda\rho_{AB}+(1-\lambda)\sigma_{AB}} \geq \lambda H(A|B)_{\rho_{AB}} + (1-\lambda)H(A|B)_{\sigma_{AB}} \tag{7.54}$$

Purification: A *purification* of ρ_A, is a Hilbert space B and a pure joint density matrix ρ_{AB} such that $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$. Then $S(B) = S(A)$. In general there exist many purifications of a given density matrix. One construction, sometimes called the *canonical purification*, is as follows. Let p_a, $|a\rangle_A$ be the eigenvalues and corresponding eigenvectors of ρ_A, so $\rho_A = \sum_a p_a |a\rangle_{AA}\langle a|$. Let the B Hilbert space have dimension equal to the dimension of A, and let $|a\rangle_B$ be an orthonormal basis for it. Then let

$$\rho_{AB} = |\psi\rangle\langle\psi| \tag{7.55}$$

where $|\psi\rangle$ is the following state on AB:

$$|\psi\rangle = \sum_a p_a^{1/2} |a\rangle_A |a\rangle_B \,. \tag{7.56}$$

8. Superselection rules

See arXiv:0710.1516 for a review of superselection rules. Here we will be brief.

A superselection rule is defined by some observable R, for example electric charge. The rule says that we consider only eigenstates of R, i.e. we do not allow superpositions of states with different eigenvalues. However, we *do* allow mixtures of states with different eigenvalues. This is equivalent to the requirement that the density matrix ρ commutes with R. We also consider only operators that carry a definite R quantum number, i.e.

$$[R, M] = r_M M \tag{8.1}$$

where r_M is a c-number. (Alternatively, if R is unitary, for example $R = (-1)^F$, where F is fermion number, then we would require

$$R^{-1} M R = r_M M \,.) \tag{8.2}$$

This guarantees that M preserves the condition of being an eigenstate of R. Only the operators that commute with R ($r_M = 0$ in (8.1) or $r_M = 1$ in (8.2)) represent physical observables (i.e. have non-zero expectation values on physical states, and have eigenvectors that are physical states); these form a subalgebra of the full algebra of operators.

8.1 Joint systems

On a joint system AB, the superselection rule will typically be defined by an observable of the form

$$R = R_A \otimes I_B + I_A \otimes R_B \tag{8.3}$$

(or, in the unitary case, $R = R_A \otimes R_B$, for example $(-1)^F = (-1)^{F_A} \otimes (-1)^{F_B}$). Two interesting points arise.

The first is that superselection rules are preserved by restriction to a subsystem. For example, we can consider a state with $R = 1$ that is a superposition of an $R_A = 1$, $R_B = 0$

state and an $R_A = 0$, $R_=1$ state. But when we trace over B, we will get a *mixture* of states on A with $R_A = 0$ and 1. More generally, if ρ commutes with R, then ρ_A commutes with R_A. Furthermore, if an operator or observable is allowed by R and localized on A, i.e. is of the form $M_A \otimes I_B$, then M_A is allowed by R_A.

On the other hand, an observable on AB may be formed by tensoring operators on A and B that are not themselves observables, as long as their charges cancel. Thus $M = M_A \otimes M_B$ is an observable if $r_M = r_{M_A} + r_{M_B} = 0$, even if $r_{M_A} \neq 0$. Hence the algebra of observables on the full system is larger than the tensor product of observables on the subsystems, something that does not happen in the absence of superselection rules.

8.2 Joint systems and fermionic operators

If two subsystems A, B contain fermionic operators (e.g. if A, B are spatial regions in a field theory with fermionic fields), then in the joint systems those operators must anticommute. This is confusing, since in the tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, operators on A would seem to always commute with operators on B. Is the joint system really described by the tensor product, or by some more exotic structure?

The solution to the puzzle is simple. The joint system is indeed described by the usual tensor product. However, the statement above about operators on $\mathcal{H}_A$ and $\mathcal{H}_B$ was too fast. Given an operator M_A on A, we need a way to associate to it an operator on AB. This mapping T_A should be an algebra homomorphism and should preserve the expectation value for an arbitrary product state, i.e.

$$\langle T_A(M_A) \rangle_{\rho_A \otimes \rho_B} = \langle M_A \rangle_{\rho_A} \qquad \forall\, \rho_A, \rho_B \,. \tag{8.4}$$

The simplest option (which is usually assumed implicitly) is $T_A(M_A) = M_A \otimes I_B$. In fact, in the absence of superselection rules, this is the unique homomorphism obeying (8.4). However, if we do the same for the B operators, $T_B(M_B) = I_A \otimes M_B$, then indeed $T_A(M_A)$ and $T_B(M_B)$ will always commute. So we need to find a different homomorphism.

For bosonic operators, the mapping T_A is fixed by the requirement that the expectation value in an arbitrary product physical state should be preserved, i.e. (8.4). However, the fact that the expectation value of a fermionic operator vanishes in any physical state gives us more freedom in choosing the homomorphism. Let us denote the statistics of an operator by $s(M)$, where $s(M) = 0$ for bosonic operators and $s(M) = 1$ for fermionic ones. Then it is easy to see that the following is a homomorphism:

$$T_A(M_A) = M_A \otimes (-1)^{s(M_A) F_B} \,. \tag{8.5}$$

Furthermore, if we map the B operators to AB in the usual way, $T_B(M_B) = I_A \otimes M_B$, then $T_A(M_A)$ and $T_B(M_B)$ will anticommute when M_A and M_B are both fermionic and commute otherwise, which is what we wanted. Another option is to put the cocyle on T_B, i.e.

$$T_A(M_A) = M_A \otimes I_B \,. \qquad T_B(M_B) = (-1)^{s(M_B) F_A} \otimes M_B \,. \tag{8.6}$$

(We can put cocycles on both T_A and T_B, but then the fermionic operators will again commute.) These two choices are equivalent, since they are related by conjugation by $(-1)^{F_A F_B}$. For multiple systems $A_1, \dots, A_n$, we can for example take

$$T_{A_i}(M_{A_i}) = I_{A_1} \otimes \cdots \otimes I_{A_{i-1}} \otimes M_{A_i} \otimes (-1)^{s(M_{A_i})F_{A_{i+1}}} \otimes \cdots \otimes (-1)^{s(M_{A_i})F_{A_n}} . \tag{8.7}$$

It is easy to see that this prescription reproduces the usual action of the creation and annihilation operators on the Fock space of a free fermionic field theory.

The freedom in choosing the homomorphism presumably extends to the case with more than two superselection sectors (e.g. with anyons). It might even be possible for the cocycles to be non-abelian, although I haven't thought through that case.

9. Dimensions of physical quantities

The tables below follow the SI system. I have chosen charge, mass, length, and time as the basic dimensions. This is obviously an arbitrary choice but seems to be conventional, at least at the undergraduate level. In particular, in undergraduate physics, mass is taken as a fundamental concept, with concepts like energy and action derived; whereas from a more advanced viewpoint (based on QFT for example), action is fundamental and mass is derived. One interesting observation from the table is that time almost always appears with a negative power.

quantity	dimensions			unit	constant	value
time			T	s	$(G\hbar c^{-5})^{1/2}$	5.4×10^{-44}
frequency, angular velocity, $\delta(t)$, $\dot{\ }$			T^{-1}	Hz		
angular acceleration			T^{-2}			
length		L		m	$(G\hbar c^{-3})^{1/2}$	1.6×10^{-35}
area		L^2			$G\hbar c^{-3}$	2.6×10^{-70}
volume		L^3				
wave number/vector, $\delta(x)$, ∇		L^{-1}				
number density, $\delta^3(\mathbf{r})$		L^{-3}				
spacetime volume		L^3	T			
speed, velocity		L	T^{-1}		c	3.0×10^8
acceleration		L	T^{-2}			
diffusion constant, kinematic viscosity		L^2	T^{-1}			
probability/number current/flux		L^{-2}	T^{-1}			
mass	M			kg	$(G^{-1}\hbar c)^{1/2}$	2.2×10^{-8}
moment of inertia	M	L^2				
mass density	M	L^{-3}				
momentum	M	L	T^{-1}			
energy, torque	M	L^2	T^{-2}	J	$(G^{-1}\hbar c^5)^{1/2}$	2.0×10^9
force	M	L	T^{-2}	N		
power	M	L^2	T^{-3}	W	$G^{-1}c^5$	3.6×10^{52}
energy flux	M		T^{-3}			
angular momentum, action	M	L^2	T^{-1}		$\hbar$	1.1×10^{-34}
Young/shear/bulk modulus, energy density, pressure, stress	M	L^{-1}	T^{-2}	Pa		
momentum density	M	L^{-2}	T^{-1}			
dynamic viscosity	M	L^{-1}	T^{-1}			
surface tension, spring constant	M		T^{-2}			
	M^{-1}	L^3	T^{-2}		G	6.7×10^{-11}

quantity	dimensions				unit	constant	value
charge	Q				C	$e = (\alpha 4\pi\epsilon_0\hbar c)^{1/2}$	1.6×10^{-19}
electric dipole moment	Q		L				
electric 2^n-pole moment	Q		L^n				
charge density	Q		L^{-3}				
current	Q			T^{-1}	A		
current density	Q		L^{-2}	T^{-1}			
electric field	Q^{-1}	M	L	T^{-2}			
electric potential	Q^{-1}	M	L^2	T^{-2}	V		
electric flux	Q^{-1}	M	L^3	T^{-2}			
electric permittivity	Q^2	M^{-1}	L^{-3}	T^2		ϵ_0	8.9×10^{-12}
	Q^{-2}	M	L^3	T^{-2}		$(4\pi\epsilon_0)^{-1}$	9.0×10^9
capacitance	Q^2	M^{-1}	L^{-2}	T^2	F		
resistance, impedance	Q^{-2}	M	L^2	T^{-1}	Ω		
resistivity	Q^{-2}	M	L^3	T^{-1}			
conductivity	Q^2	M^{-1}	L^{-3}	T			
displacement **D**, polarization	Q		L^{-2}				
magnetic dipole moment	Q		L^2	T^{-1}			
magnetic field **B**	Q^{-1}	M		T^{-1}	T		
magnetic flux	Q^{-1}	M	L^2	T^{-1}	Wb	$\Phi_0 = \pi\hbar e^{-1}$	2.1×10^{-15}
magnetic permeability	Q^{-2}	M	L			$\mu_0 = (\epsilon_0 c^2)^{-1}$	1.3×10^{-6}
inductance	Q^{-2}	M	L^2		H		
vector potential	Q^{-1}	M	L	T^{-1}			
H, magnetization	Q		L^{-1}	T^{-1}			