Covariant properties of holographic entanglement

Matthew Headrick

based on

arXiv:2208.10507 w/ Veronika Hubeny
work in progress w/ Brianna Grado-White, Guglielmo Grimaldi, Veronika Hubeny
The Ryu-Takayanagi formula & its generalizations have revolutionized our understanding of holography & quantum gravity

\[ S(A) = \frac{1}{4G} \min_{\gamma \sim A} |\gamma| \]

(area (from now on, \(4G = 1\))

\[ = \max_{\nu} \int_{A} \nu \quad (\nabla \cdot \nu = 0, |\nu| \leq 1) \]

convex program; field lines of \(\nu\) are “bit threads”

Generalizations in many directions:

- time dependence
- quantum corrections
- higher-derivative corrections
- Rényi entropies
- reflected entropies
- Python’s lunch
- flat space, de Sitter, cosmology, …
- …
Hubeny-Rangamani-Takayanagi *covariant* entanglement entropy formula:

\[ S(A) = \min_{\gamma \sim A} |\gamma| = \max_{\sigma} \min_{\sigma \supset \gamma \sim A} |\gamma| \]

(maximin) Wall

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**Diagram:**

- The diagram illustrates the bulk Cauchy slice \( \sigma \) and the extremal cross section \( \gamma \) as part of the covariant entanglement entropy formula. The bulk Cauchy slice \( \sigma \) intersects both the region \( A \) and its complement \( A^c \). The extremal cross section \( \gamma \) is shown as a curve that connects points in \( A \) and \( A^c \) across the bulk Cauchy slice. The diagram uses different colors and shading to distinguish between the regions \( D(A) \) and \( D(A^c) \).
HRT maximin

\[ S(A) = \min_{\gamma \sim A} |\gamma| = \max_{\sigma} \min_{\sigma \supseteq \gamma \sim A} |\gamma| \]

Can be used to prove:

- existence of HRT surface
- reduces to RT w/time-reflection symmetry
- competing HRT surfaces are spacelike-separated
- consistency w/boundary causality: \( W(A) \cap \text{bdy} = D(A) \)
- entanglement wedge nesting & complementarity:

\[
W(A) \subset W(AB), \quad W(A) \cap W(A^c) = \gamma(A) = \gamma(A^c)
\]

- entropy inequalities:
  - subadditivity: \( S(AB) \leq S(A) + S(B) \)
  - strong subadditivity: \( S(B) + S(ABC) \leq S(AB) + S(BC) \)

Crucial consistency checks on HRT formula & subregion duality

*Use full dynamics: Einstein eq, null energy condition, AdS boundary conditions*
RT entropy inequalities ("entropy cone")

For RT, many further inequalities, not general properties of quantum states, have been proven.

Examples:

- **MMI**: \( S(A) + S(B) + S(C) + S(ABC) \leq S(AB) + S(BC) + S(AC) \) Hayden-MH-Maloney


Several infinite families + hundreds of isolated examples found, but full set of inequalities (or other characterization of allowed entropies) remains unknown; meaning & implications also unclear.


For MMI, "bipartite dominance" conjecture Cui-Hayden-He-MH-Stoica-Walter (but see Akers-Rath)
Are these inequalities valid for time-dependent states?

Evidence:

- MMI proved using maximin Wall
  - but only MMI can be proved using maximin Rota-Weinberg
- Proved in scaling limit of late times, large distances Bao-Mezei
- Tested numerically in 3d Vaidya Erdmenger-Fernandez-Flory-Megias-Straub-Witkowski, Caginalp
- Proved in 3d, trivial topology:
  - in a given configuration, any inequality is a positive sum of SSAs Czech-Dong
- Tested numerically in 3d, non-trivial topology (rotating BTZ & 3-bdy wormhole, \(~10^8\) random configurations) Grado-White-Grimaldi-MH-Hubeny
- Partial proof, in 3d, non-trivial topology Grado-White-Grimaldi-MH-Hubeny
- General argument from structure of RT entropy cone Hubeny-Rota et al
Is there a covariant dessication for HRT:

weighted graph that encodes all entropies by min cuts?

Would imply all higher inequalities;

also clarify tensor networks for time-dependent states
New formulations of HRT

In the belief that it’s useful to have more perspectives on this crucial entry in the holographic dictionary, I’ll describe 3 more equivalent formulations of the HRT formula:

1) V-threads
2) U-threads
3) Minimax

(Equivalence to HRT requires NEC, AdS boundary conditions)

I’ll then give two applications of minimax:
   a) New SSA proof
   b) Entangled universes

(and ask me about c) Python’s lunch)
\[ S(A) = \max_V \int_{D(A)} V \quad \text{where: } \nabla \cdot V = 0, \quad V\big|_{(D(A) \cup D(A^c))^c} = 0 \]

\[ \int_{\mathcal{C}} d\tau |V_\perp| \leq 1 \quad \text{for any timelike curve } \mathcal{C} \]

(can be written as local constraint by adding clock function)

V-threads

curves \( \mathcal{C} \)

\( D(A^c) \)

\( D(A) \)

allowed \( V \)

\( \gamma(A) \)

optimal \( V \)

finds \( \gamma(A) \), entanglement wedges

Morally, V-threads are Bell pairs

MH-Hubeny

convex program

Bell pairs
where:\n\[ S(A) = \min_U \int_\sigma U \text{ where: } U \text{ timelike, } \nabla \cdot U = 0, \ U|_{D(A) \cup D(A^c)} = 0 \]

(bulk Cauchy slice containing $A, A^c$)

convex program, dual of V-threads

\[
\int_\mathcal{C} ds \ |U_\perp| \geq 1 \text{ for any curve } \mathcal{C} \text{ from } D(A) \text{ to } D(A^c)
\]

(can be written as local constraint by adding function interpolating between $D(A) \& D(A^c)$)

U-threads

Morally, U-threads are disentanglers

MH-Hubeny
Minimax timesheet is highly non-unique (floppy) away from HRT

Using minimax, can define $W(A)$ as the smallest spacetime homology region, prove nesting etc.
Intersecting timesheets cut HRT surfaces

Timesheets **cooperate** if every partial HRT surface $\gamma_i$ is maximal on partial timesheet $\tau_i$

**Lemma:** Given boundary regions $A, B, C$, cooperating timesheets exist for $AB$ & $BC$

Let $\tilde{\gamma}_B$ be maximal surface on $\tau_2 \cup \tau_3$

Then $S(B) \leq |\tilde{\gamma}_B| \leq |\gamma_2| + |\gamma_3|$

Similarly, $S(ABC) \leq |\tilde{\gamma}_{ABC}| \leq |\gamma_1| + |\gamma_4|$

Hence $S(B) + S(ABC) \leq |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4|$

$$= S(AB) + S(BC)$$

Unclear if lemma holds for $\geq 3$ timesheets

If it does, then we have **graph model & all higher inequalities**
Cosmological generalization of HRT

Two-sided asymp. AdS black hole is an entangled state of asymp. AdS universes ( = CFTs)

HRT: entanglement entropy = $|\gamma|$  
$\gamma = \text{closed extremal surface, in causal shadow}$

Asymp. flat two-sided black hole:
By maximin or minimax, $\exists$ closed extremal surface $\gamma$ in causal shadow

Asymp. dS two-sided black hole:
By minimax, $\exists$ closed extremal surface $\gamma$ in causal shadow

Claims:

- Each asymp. region has a Hilbert space
- Two-sided black hole is an entangled state
- Entanglement entropy = $|\gamma|$
Cosmological generalization of HRT

An asymp. region can have several black holes
A black hole can have several asymp. regions — not necessarily with same asymptotics

In general, universes & wormholes define bipartite graph

Entropies computed via minimax, spacetime homology w/asymptotic boundaries
Three new formulations of HRT:

**V-threads**

**U-threads**

**minimax**

Minimax applications:
(a) new SSA proof;  (b) entanglement between universes;  (c) Python’s lunch

Open problem: *graph model & higher entropy inequalities* for HRT

THANK YOU!