Covariant bit threads, minimax surfaces, and entropy inequalities

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(based on work in progress w/ V. Hubeny, G. Giraldi)

Summary:
We will define several new formulas equivalent to the HRT formula. These involve both surfaces and bit threads of various kinds. Using one of them, we will explore entropy inequalities.

\[ \text{QM := bulk, Fix boundary region } A \]
\[ \text{\( \mathcal{M} = D(A) \cup D(A^c) \cup I^+ \cup I^- \)} \]

\[ I^+ := J^+(\partial A) \cup \text{future lightcone} \]
\[ I^- := J^-(\partial A) \cup \text{past lightcone} \]

\[ \Sigma_A := \{ \text{hulk Cauchy slice } \sigma \mid \sigma \cap \mathcal{M} \supset \partial A \} \]
\[ A_r := \sigma \cap D(A) \text{ is a Cauchy slice for } D(A) \]
\[ \Gamma_r := \{ \text{surface } \gamma \in \sigma \mid \gamma \supset A_r \text{ relative to } \partial A \} \]

Maximizing: \( S_\gamma(A) := \sup_{\sigma \in \Sigma_A} \gamma \cap \sigma \) [Wall’92]
Can we put space and time on same footing?

A time-sheet \( \tau \) for \( A \) is an everywhere

\text{timelike or null hypersurface}

homologous to \( D(A) \) relative to \( I^+ \cap I^- \)

\[ T_A := \{ \text{time-sheets for } A \} \]

Hence \( S_-(A) = \sup_{\sigma \in \Sigma_A} \inf_{\tau \in T_A} |\sigma \cdot \tau| \)

Can we switch order of sup and inf?

**Minimax theory** (invented by von Neumann for game theory)

For any function \( f : X \times Y \to \mathbb{R} \),

\[ \inf_{x \in X} \sup_{y \in Y} f(x,y) \geq \sup_{y \in Y} \inf_{x \in X} f(x,y) \]

Equality is guaranteed if:

- \( \exists \) global saddle point \((x_0, y_0)\): \( \forall x \in X, \ f(x, y_0) \geq f(x_0, y_0) \)
  \( \forall y \in Y, \ f(x_0, y) \leq f(x_0, y) \)

or \( X, Y \) are convex sets, \( f(x,y) \) is convex in \( x \), concave in \( y \)

"mixed strategy games"

Define \( \underline{\text{minimax quantity}} \)

\[ S_+(A) := \inf_{\sigma \in \Sigma_A} \sup_{\tau \in T_A} |\sigma \cdot \tau| \]

\[ = \inf_{\sigma \in \Sigma_A} |\sigma| \]
where \( \Gamma_\tau = \{ \text{achronal surfaces in } \tau \} \)

\[ \forall \varepsilon \text{ know } \mathcal{S}_+ (A) > \mathcal{S}_- (A) \]

Equal? Not necessarily:

For any \( \varepsilon \), if \( \inf \{ Y \} = \alpha_\varepsilon \)
\( \mathcal{D}(A^c) \)
\[ \Rightarrow \mathcal{S}_-(A) = \alpha_\varepsilon \]

For any \( \tau \), sup \{ \mathcal{Y} \} = \alpha_+ \)
\( \mathcal{Y} \in \Gamma_\tau \)
\[ \Rightarrow \mathcal{S}_+(A) = \alpha_+ \]

\[ \Rightarrow \mathcal{Y}_{\text{HRT}} \text{ is a global saddle } \Rightarrow \mathcal{S}_+(A) = \mathcal{S}_-(A) = |\mathcal{Y}_{\text{HRT}}| \]

Proof of SSA using minimax:
\[ s(AB) + s(BC) \geq s(\beta) + s(ABC) \]

Given boundary regions \( A, B, C \),

five-sheets \( \tau_1(AB), \tau_2(BC) \) cut each other into partial five-sheets
\[ \tau_1(AB), \tau_2(AB), \tau_1(BC), \tau_2(BC) \]

Note that \( \tau_1(AB) \cup \tau_1(BC) \sim \mathcal{D}(A) \)

\[ \begin{array}{c}
\tau_1(AB) \\
\tau_2(BC)
\end{array} \]

\[ \tau \]

\[ \mathcal{D}(A) \]

However, if QM obeys NEC, AdS boundary conditions then:

- \( \mathcal{Y}_{\text{HRT}} \text{ is maximal on maximal slice} \)
- \( \mathcal{Y}_{\text{HRT}} \text{ is maximal on entanglement horizon, which is a fine-sheet for } A \)

\[ \Rightarrow \mathcal{Y}_{\text{HRT}} \text{ is a global saddle } \Rightarrow \mathcal{S}_+(A) = \mathcal{S}_-(A) = |\mathcal{Y}_{\text{HRT}}| \]
Lemma: \exists \text{ time-sheets } \tau(AB), \tau(BC) \\
containing the HRT surfaces \( \gamma(AB), \gamma(BC) \) \\
such that \( \gamma_1(AB) \) is maximal on \( \tau_1(AB) \), etc.

Define \( \tilde{\gamma}(B) \) as maximal surface on \( \tau_1(AB) \cup \tau_1(BC) \) \\
\( \tilde{\gamma}(ABC) \) as maximal surface on \( \tau_1(AB) \cup \tau_1(BC) \)

In general, \( \tilde{\gamma}(B) \neq \gamma_1(AB) \cup \gamma_1(BC) \), \( \tilde{\gamma}(ABC) \neq \gamma_2(AB) \cup \gamma_2(BC) \)

\[ \text{do not connect to form a spacelike surface} \]

But \( |\tilde{\gamma}(B)| < |\gamma_1(AB) \cup \gamma_1(BC)| \), \( |\tilde{\gamma}(ABC)| < |\gamma_2(AB) \cup \gamma_2(BC)| \)

maximized under a weaker constraint

Hence

\[ S(B) < |\tilde{\gamma}(B)| < |\gamma_1(AB)| + |\gamma_1(BC)| \]

\[ S(ABC) < |\tilde{\gamma}(ABC)| < |\gamma_1(AB)| + |\gamma_1(BC)| + |\gamma_2(BC)| \]

\[ \Rightarrow S(B) + S(ABC) < |\gamma_1(AB)| + |\gamma_2(AB)| + |\gamma_1(BC)| + |\gamma_2(BC)| \]

\[ = |\gamma(AB)| + |\gamma(BC)| \]

\[ = S(AB) + S(BC) \]
Proof generalizes to higher entropy inequalities [Boo et al] (proven for $d=2+1$ by Czech-Dong)

<table>
<thead>
<tr>
<th>RT entropy cone</th>
<th>RT entropy cone</th>
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</thead>
<tbody>
<tr>
<td>$\geq S(AB) + S(BC) + S(AD) + \cdots$</td>
<td></td>
</tr>
<tr>
<td>$S(AD) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC)$</td>
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Bit threads from convex relaxation + duality:

We can get even more interesting formulas by convex-relaxing the space of slices + fine-sheets

Smeared slice: function $\varphi : M \to [-\frac{1}{2}, \frac{1}{2}]$ s.t.

- $\varphi$ is future-directed causal
- $\varphi_i |_{(q_{\text{fin}})_v} = \pm \frac{1}{2}$

$\Rightarrow$ every level set of $\varphi$ is a Cauchy slice $\in \Sigma_A$

This function set is convex

Smeared fine-sheet: function $\Psi : M \to [-\frac{1}{2}, \frac{1}{2}]$ s.t.

- $\Psi_{\text{fin}}$ is spacelike or $O$
- $\Psi_i |_{(q_{\text{fin}})_v} = \pm \frac{1}{2}$

$\Rightarrow$ every level set of $\Psi$ is a fine-sheet $\in \Sigma_A$

This function set is not convex!
double coarea formula
\[ f(\varphi, \psi) := \text{average over level sets of } |\varphi - \psi| = \int_G |d\varphi \land d\psi| \]

Drop condition on \( \partial \psi \) to make \( \varphi \) convex;
extend \( f \) to convex-concave function:
\[ |\mathbf{u} \land \mathbf{v}| := \max \left( |\mathbf{u} \land \mathbf{v}|, |\mathbf{u} \cdot \mathbf{v}| \right) \quad f(\varphi, \psi) := \int_G |d\varphi \land d\psi| \]

(\text{convex envelope function of } |\mathbf{u} \land \mathbf{v}| \text{ with respect to } \psi)

\[ S_c(A) := \sup_{\varphi} \inf_\psi f(\varphi, \psi) = \inf_\psi \sup_{\varphi} f(\varphi, \psi) \]

\text{minmax theorem}

Now we can dualize on \( \varphi \) or on \( \psi \)
\[ S_c(A) = \sup_V \int_V \mathbf{u} \cdot \mathbf{v} = \inf_u \int_u \mathbf{v} \]

\text{\textquotedblleft} \text{V-flow} \text{\textquotedblright:} 1-form \( \mathbf{v} \) s.t.
\begin{itemize}
  \item \( d^* \mathbf{v} = 0 \) (divergenceless)
  \item \( \varphi |_{\mathbb{T}_G} = 0 \) \quad (\text{no flux})
  \item \( \exists \varphi \text{ s.t. } \varphi |_{\mathbb{T}_G} = \pm \frac{1}{2}, \ d\varphi \pm \mathbf{v} \text{ future-directed causal} \)
\end{itemize}
\( \text{V} \text{ timelike curve } \gamma, \int_0^1 \gamma' \leq 1 \quad \text{orthogonal projection} \)

\text{\textquotedblleft} \text{U-flow} \text{\textquotedblright:} 1-form \( \mathbf{u} \) s.t.
\begin{itemize}
  \item \( d^* \mathbf{u} = 0 \) \quad (\text{divergenceless})
  \item \( \mathbf{u} |_{\mathbb{T}_G} = 0 \) \quad (\text{no flux})
\end{itemize}

\text{\textit{(norm bound)}}
\[ \exists \Psi \text{ s.t. } \Psi|_{D(A)} = -\frac{1}{2}, \quad \Psi|_{D(A^c)} = \frac{1}{2}, \quad U = \text{future-directed causal} \]

\[ \forall \text{ spacelike curve } p \text{ from } D(A) \text{ to } D(A^c), \quad \int ds |U| \geq 1 \]

In general, \( S_- (A) \leq S_c (A) \leq S_+ (A) \)

\[ \text{If } S_- (A) = S_+ (A) \text{ then } S_c (A) = S_c (A) \]

Optimal U-flow, V-flow find HRT surface + entanglement wedges
\[ I^+ \]

\[ I^- \]

\[ D(A) \]

\[ D(A^c) \]

\[ H(A) \]

\[ H(A^c) \]

\[ \gamma(A) \]

\[ H(A) \]

\[ H(A^c) \]

\[ H(A^c) \]

\[ D(A) \]

\[ D(A^c) \]

\[ I^+ \]

\[ I^- \]

\[ D(A) \]

\[ D(A^c) \]

\[ H(A) \]

\[ H(A^c) \]

\[ \gamma(A) \]