Quantum Entanglement and the Geometry of Spacetime

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Entropy and area

Bekenstein-Hawking '74:

$$S = \frac{k_B c^3 \text{area(horizon)}}{4 G_N \hbar} = k_B \frac{\text{area(horizon)}}{4 l_P^2}$$

Planck length
(≈10^{-33} cm)

Event Horizon Telescope '19

G_N → gravity

$\hbar$ → quantum mechanics

k_B → statistical mechanics

What are the "atoms" of the black hole?
Why is $S \propto$ area?

Entropy and area

If space has d dimensions, $G_{\mathrm{N}}\hbar$ has units of area (L^{d-1})

Planck area: basic unit in quantum gravity, translates into unit of *entropy*

Generalizations of Bekenstein-Hawking:

De Sitter spacetime (Gibbons-Hawking '77):

$$S = \frac{\text{area(horizon)}}{4G_{\mathrm{N}}\hbar}$$

Holographic entropy bounds (Bekenstein '81, Bousso '99): for arbitrary closed surface in arbitrary spacetime

$$S \leq \frac{\text{area}}{4G_{\mathrm{N}}\hbar}$$

Jacobson '95: area-entropy relation *implies* Einstein equation

How general is the area-entropy relation? What is its origin?

A clue: *Holographic entanglement entropy* (Ryu-Takayanagi '06)

Vast (but also limited) generalization of Bekenstein-Hawking

To understand it, we first need to extend our notion of entropy…

Entanglement Entropy

Classical mechanics:

definite state → certain outcome for any measurement

Quantum mechanics:

definite state → uncertain outcomes for some measurements

Example: $|\uparrow\rangle$

measurement of S_z definitely gives $+\frac{1}{2}\hbar$

measurement of S_x gives $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$ with equal probability

When only certain kinds of measurements are allowed, a definite (pure) state will *effectively* be indefinite (mixed)

Suppose a system has two parts, but we can only measure one part

Spin singlet state: $$|0,0\rangle = \frac{1}{\sqrt{2}}\left(\underbrace{|\uparrow\rangle}_{A}\underbrace{|\downarrow\rangle}_{B} - |\downarrow\rangle|\uparrow\rangle\right) \qquad S_{AB} = 0$$

To see that this is a pure state (superposition, not mixture, of $|\uparrow\rangle|\downarrow\rangle$ and $|\downarrow\rangle|\uparrow\rangle$) requires access to both A and B

For an observer who only sees A, effective state is mixed:

$$\rho_A = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) \qquad S_A = \ln 2$$

Indefiniteness from *entanglement*

Entanglement Entropy

In general: if ρ is state of full system, then effective state for subsystem A is

$$\rho_A = \mathrm{Tr}_{A^c}\,\rho$$

Entanglement entropy is defined as von Neumann entropy of subsystem:

$$S_A = -\mathrm{Tr}\rho_A \ln \rho_A$$

(*Any* entropy can be viewed as due to entanglement with environment)

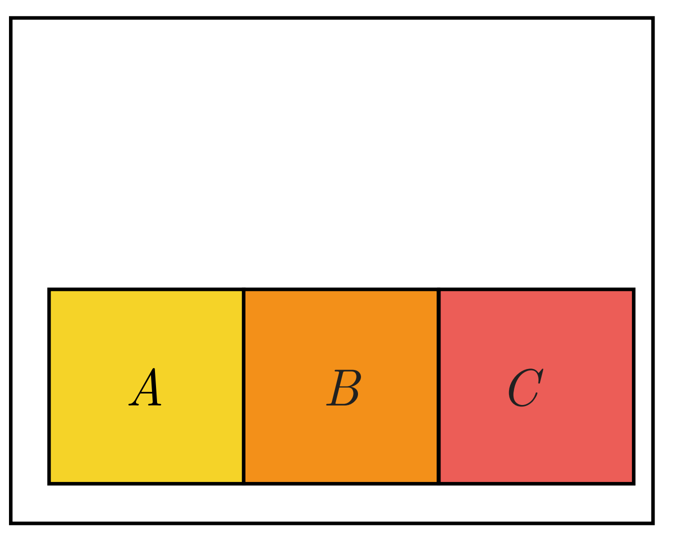
Entanglement entropies obey many important properties, such as:

Subadditivity: $S_{AB} \leq S_A + S_B$

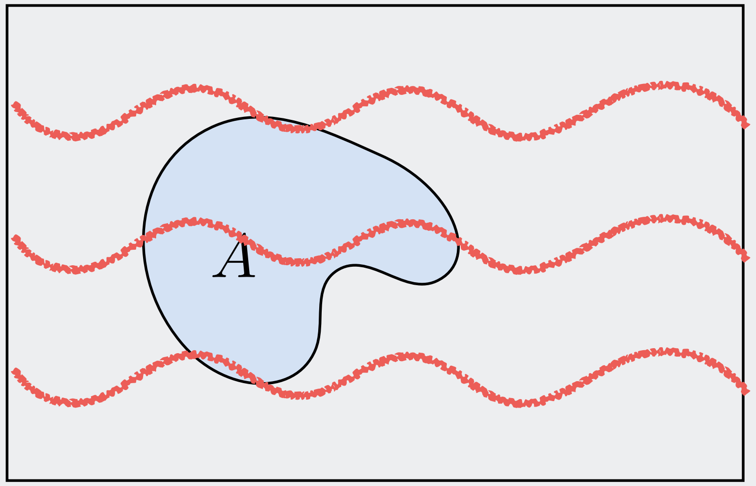
Mutual information: $I_{A:B} := S_A + S_B - S_{AB} \geq 0$

Strong subadditivity: $S_{AB} + S_{BC} \geq S_B + S_{ABC}$

(Lieb-Ruskai '73) $(I_{A:BC} \geq I_{A:B})$



Entanglement Entropy in QFT

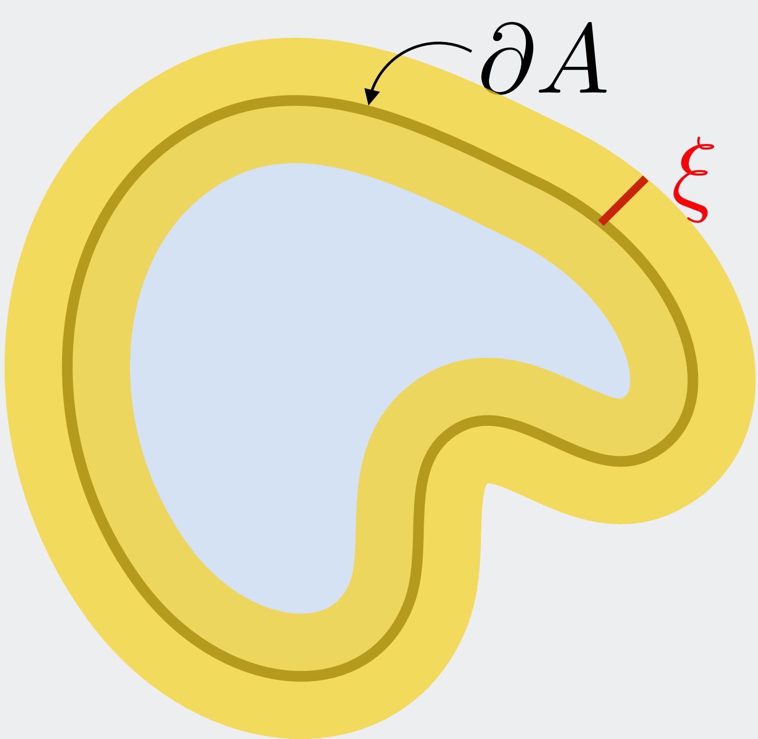


In quantum field theories (& many-body systems),
spatial regions are highly entangled with each other

Consider microwave cavity
Even in vacuum, electromagnetic field fluctuates:
zero-point quantum fluctuations of modes
Each mode is distributed in space
=> fluctuations are spatially correlated
=> any part A of cavity is entangled with rest

S_A is ultraviolet divergent due to entanglement of short-wavelength modes across ∂A

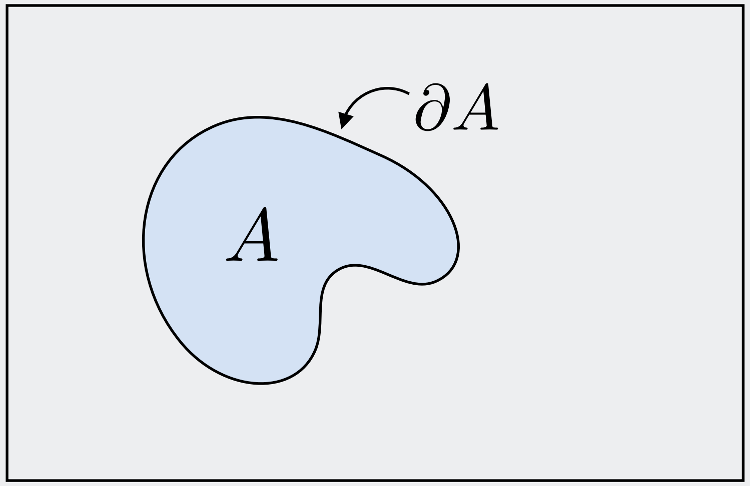
Massive (gapped) field: entanglement extends out to *correlation/Compton length* ξ



$$S_A = \text{area}(\partial A)\left(\frac{1}{\epsilon^{d-1}} - \frac{1}{\xi^{d-1}}\right) + \cdots$$

short-distance cut-off

Entanglement Entropy in QFT



In quantum field theories (& many-body systems), spatial regions are highly entangled with each other

S_A depends on:

- parameters of theory (including ϵ)
- state
- size and shape of region A

Contains a lot important physics

Examples:

- Gapped theory in $d = 2$: $S_A = \frac{L}{\epsilon} - \frac{L}{\xi} - \gamma$

  $\mathrm{length}(\partial A)$

  topological entanglement entropy (Kitaev-Preskill '05; Levin-Wen '05)

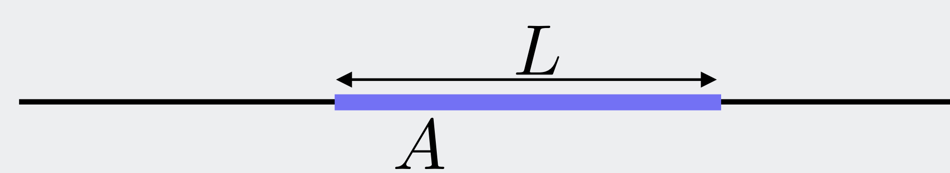
  correlation length

- Critical (conformal) theory in $d = 1$: $S_A = \frac{c}{3} \ln \frac{L}{\epsilon}$

  central charge

  short-distance cutoff

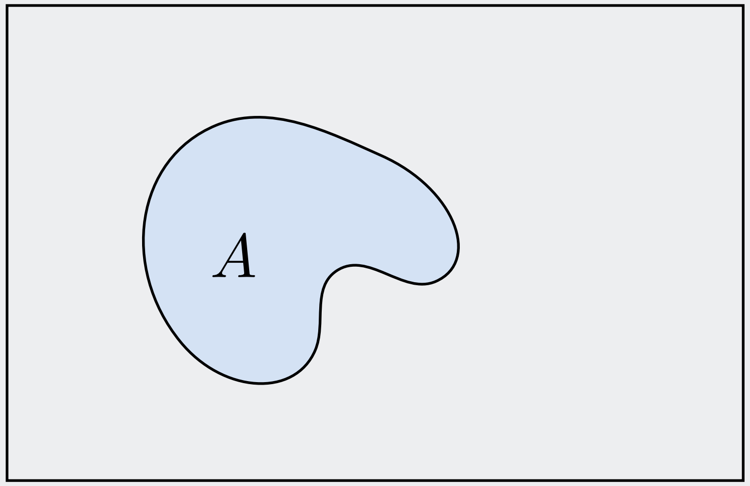
  (Holzhey-Larsen-Wilczek '94; Calabrese-Cardy '03)



- At finite temperature, also usual extensive entropy: $s(T) \times \mathrm{volume}(A)$

  thermal entropy density

Entanglement Entropy in QFT



Powerful probe of QFTs and many-body systems:

- quantum criticality
- topological order
- renormalization-group flows
- energy conditions
- many-body localization
- quenches
- much more…

However, usually very difficult to compute—even in free theories

Simplifies in certain theories with *many strongly-interacting* fields…

Holographic dualities

Consider a QFT with N interacting fields
for example $SU(n)$ Yang-Mills theory, $N \sim n^2$

When N is large, these fields may admit a *collective* description in terms of a small number of degrees of freedom

- classical (think of hydrodynamics)
- usually complicated

However, in certain cases, when the fields are very *strongly* interacting, it simplifies dramatically:

General relativity in $d+1$ dimensions with cosmological constant $\Lambda < 0$
(plus some matter fields)
subject to certain boundary conditions: “universe in a box”
(Maldacena ’97)

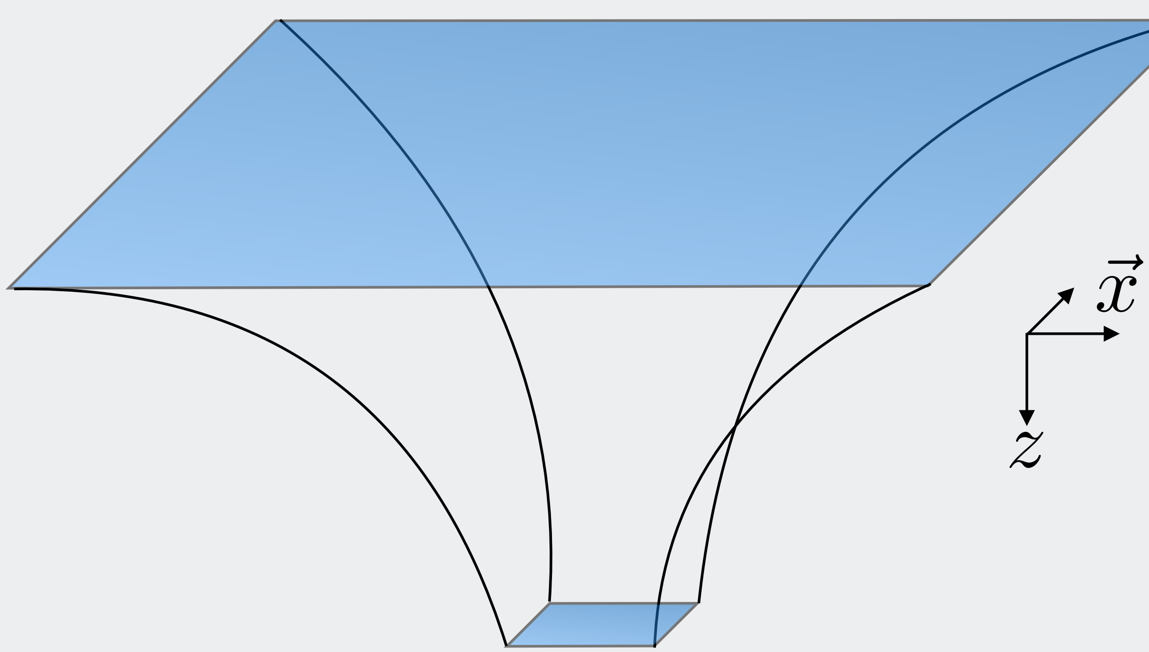
Holographic dualities

If QFT is conformal (scale-invariant), ground state is anti-de Sitter (AdS) spacetime:

AdS radius $\qquad$ d dimensions of QFT

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + dz^2 + d\vec{x}^2\right)$$

extra dimension



$z = \epsilon$ boundary of AdS, where field theory “lives”

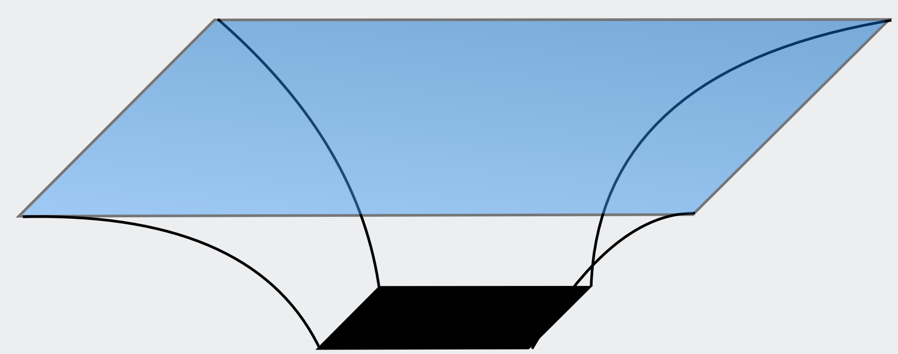
Map between boundary and bulk is non-local

If QFT is gapped (massive), space ends on wall at $z_{\max} \sim \xi$ (correlation length)

Many specific examples known in various dimensions
(mostly supersymmetric, derived from string theory)

Holographic Dualities

QFT	GR
N	AdS radius $\dfrac{R^{d-1}}{G_{\mathrm{N}}\hbar}$ Planck area
thermodynamic limit $N \to \infty$	classical limit $\hbar \to 0$
statistical fluctuations	quantum fluctuations
collective modes	gravitational waves, etc.
deconfined plasma	black hole
$S \propto N$	$S = \dfrac{\mathrm{area(horizon)}}{4G_{\mathrm{N}}\hbar}$



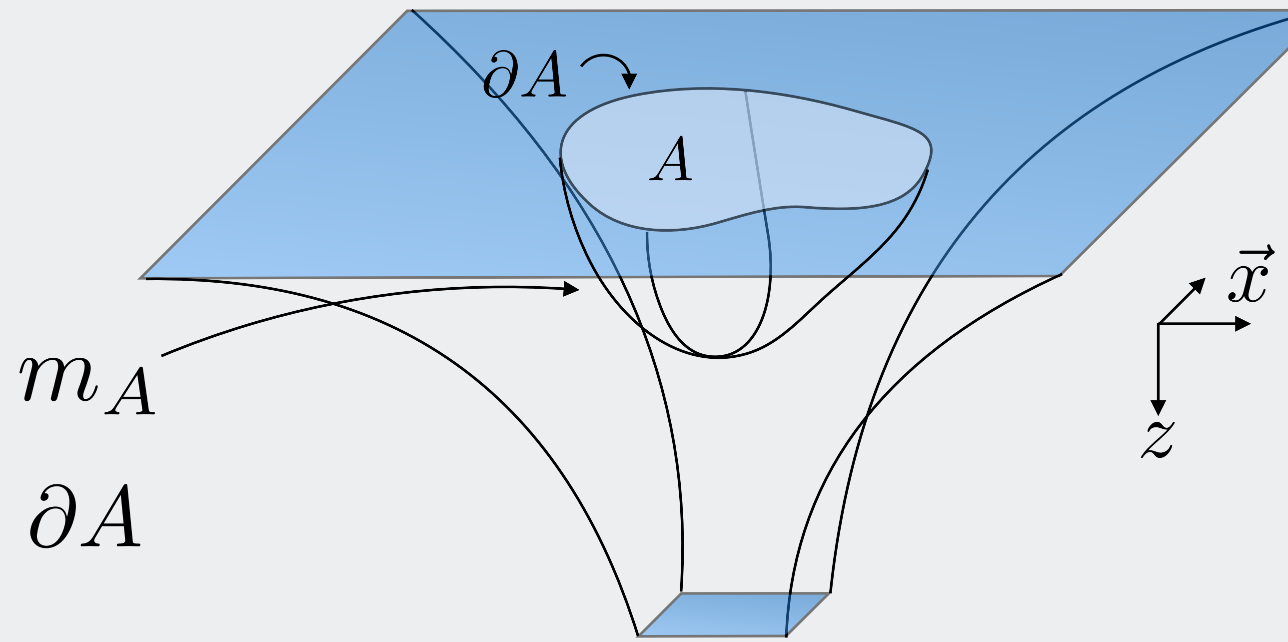
Holographic dualities are useful for computing *many* things in strongly interacting QFTs

Let’s talk about entanglement entropies...

Holographic Entanglement Entropy

Ryu-Takayanagi '06:

$$S_A = \frac{\text{area}(m_A)}{4G_{\text{N}}\hbar}$$

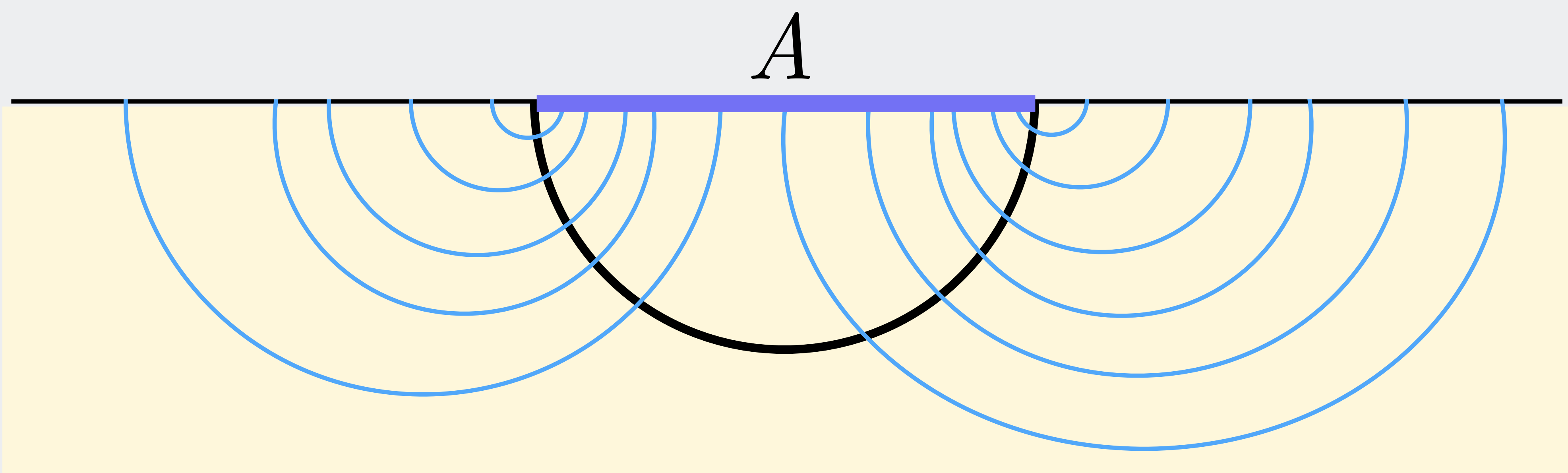


$m_A =$ minimal surface anchored to ∂A

hangs down in order to minimize area

MH-Freedman '16: $S_A =$ max # "bit threads" connecting A to rest of boundary

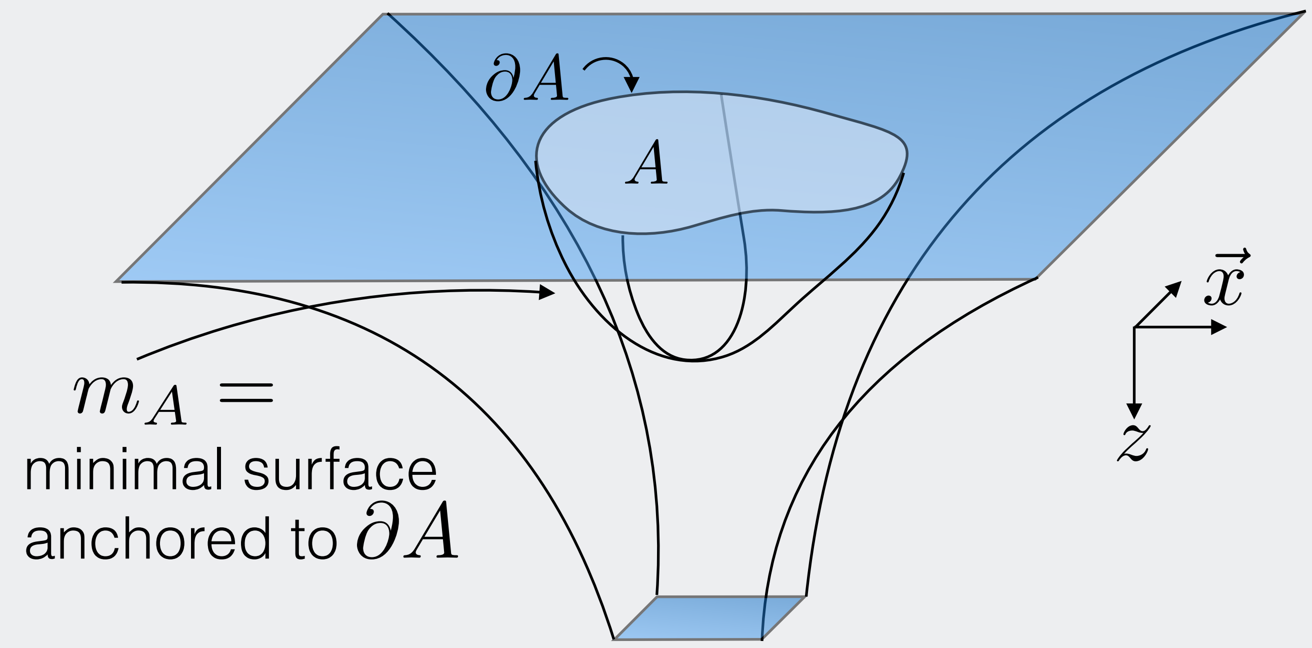
Each bit thread has cross section of 4 Planck areas
Represents entangled pair of qubits between A and complement



Holographic Entanglement Entropy

Ryu-Takayanagi '06:

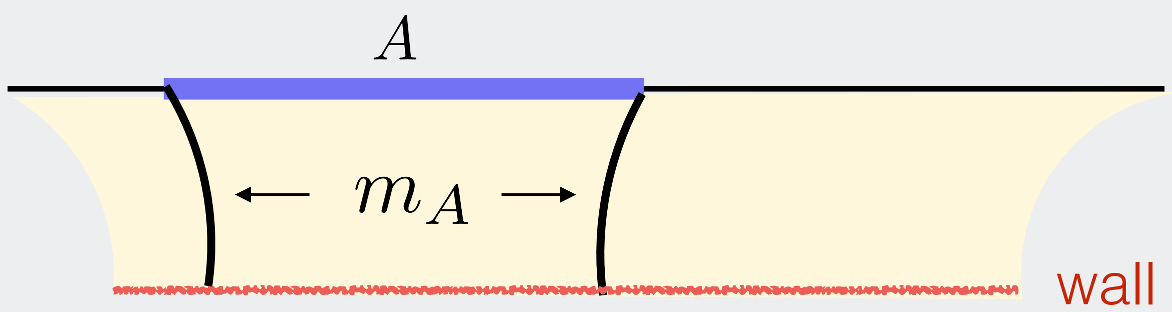
$$S_A = \frac{\mathrm{area}(m_A)}{4 G_{\mathrm{N}} \hbar}$$



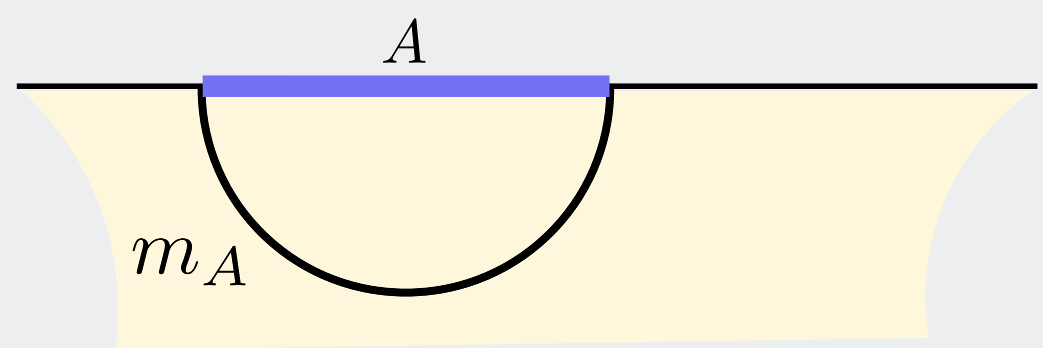
Geometrizes entanglement:

- Area-law UV divergence due to infinite area of m_A near boundary
- Gapped theory:
  minimal surface extends to wall
  (Klebanov, Kutasov, Murugan '07)

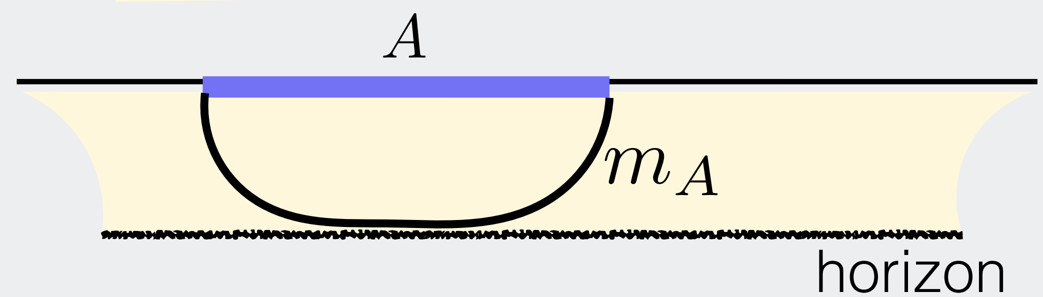
$$S_A = \mathrm{area}(\partial A)\left(\frac{1}{\epsilon^{d-1}} - \frac{1}{\xi^{d-1}}\right) + \cdots$$



- Conformal theory in $d = 1$: $S_A = \frac{c}{3} \ln \frac{L}{\epsilon}$



- Finite temperature:
  minimal surface hugs horizon
  => extensive entropy $s(T)\mathrm{volume}(A)$



Democratizes Bekenstein-Hawking: not about horizons!

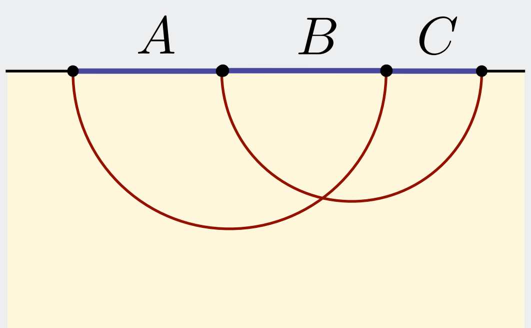
Holographic Entanglement Entropy

Quantum information theory is built into *classical spacetime geometry*

Example: Strong subadditivity

$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

$S_{AB} + S_{BC} =$



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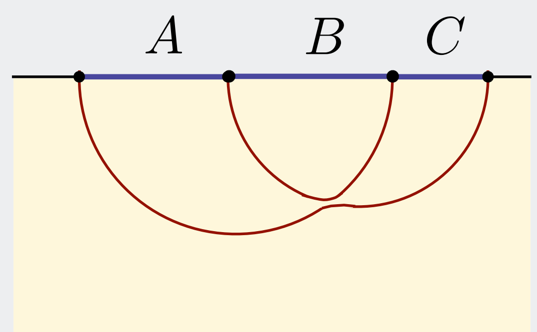
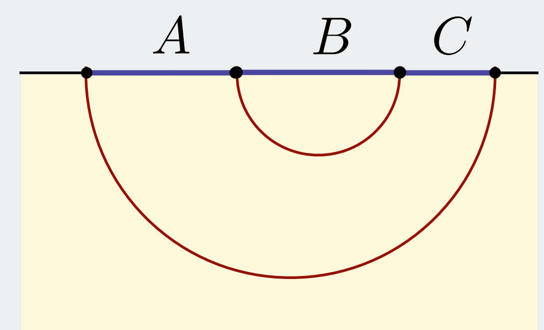


$\geq$

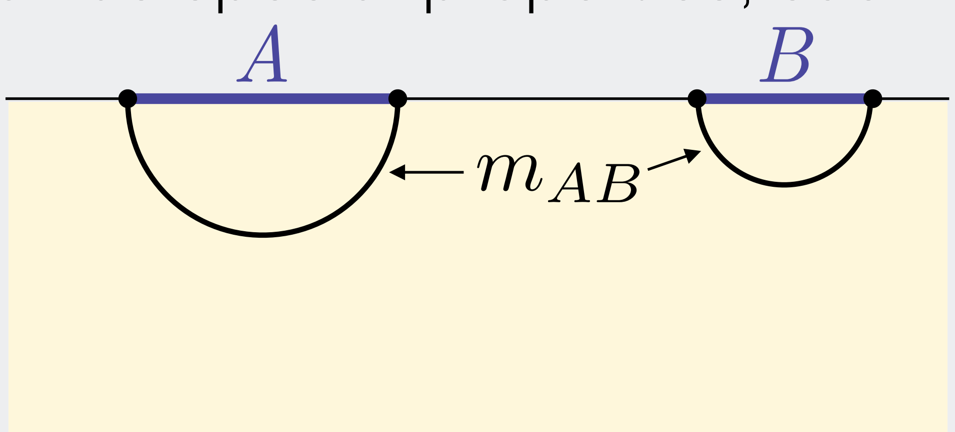


$= S_{ABC} + S_B$

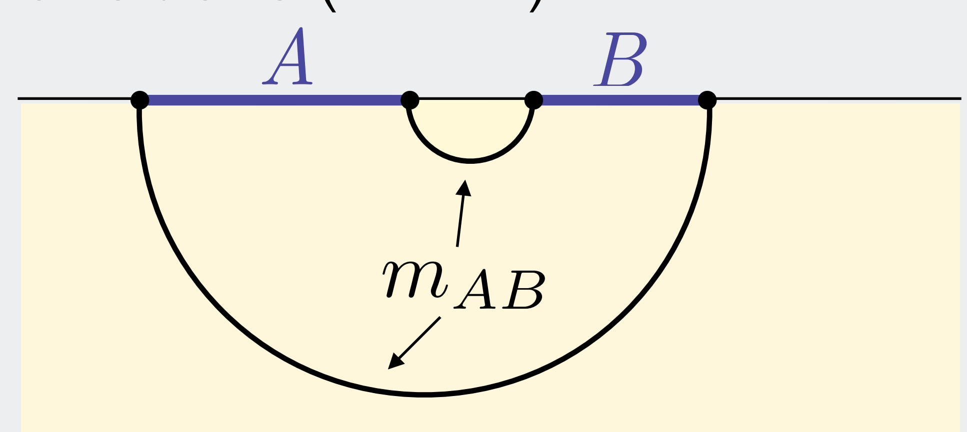
(MH-Takayanagi '07)

In fact, RT formula obeys *all* general properties of entanglement entropies
(Hayden-MH-Maloney '11; MH '13)

Also has special properties, such as phase transitions (MH '10)



$$S_{AB} = S_A + S_B \quad \Rightarrow \quad I_{A:B} = 0$$



$$S_{AB} < S_A + S_B \quad \Rightarrow \quad I_{A:B} > 0$$

Holographic Entanglement Entropy

Many other developments:

- Many discoveries about entanglement in field theories using holography
- Time dependence (Hubeny-Rangamani-Takayanagi '07)
- Derivation of RT formula (MH '10, Lewkowycz-Maldacena '13)
- Einstein equation from RT (Lashkari-McDermott-Van Raamsdonk '13)
- Quantum ($1/N$) corrections (Faulkner-Lewkowycz-Maldacena '13)
- Special properties obeyed by holographic entropies (Bao et al '15)
- Tensor networks for modelling holography (Swingle '08)
- Bit threads & entanglement structures (Cui et al '18)
- Subregion duality (MH-Hubeny-Lawrence-Rangamani '14)
- Holography as quantum error-correcting code (Almheiri-Dong-Harlow '14)
- Other quantitites: relative entropies, entanglement of purification, ...
- ...

The big one:

- Does space emerge from entanglement? From bit threads?