3. Restriction and Induction

Given any ring homomorphism $S \to R$ we get a functor $R\text{-mod} \to S\text{-mod}$ by “restriction of scalars”. Given a subgroup $H$ of a group $G$, the inclusion map $\mathbb{C}H \to \mathbb{C}G$ is a ring homomorphism and therefore induces a functor

$$Res^G_H : G\text{-mod} \to H\text{-mod}.$$ 

Given a $G$-module $M$, the $H$-module $Res^G_H M$ is the same vector space $M$ where we consider only the action of the elements of $H$ on $M$. The character of this module is just the restriction of the character of $M$:

$$Res^G_H \chi(h) = \chi(h)$$

for all $h \in H$. Equivalently,

$$(3.1) \quad Res^G_H \chi_V = \chi_{Res^G_H}$$

for any $G$-module $V$.

3.1. induction. Given a representation $V$ of $H$, the induced representation $Ind^G_H V$ is the $G$-module given by

$$Ind^G_H V := \mathbb{C}G \otimes_{\mathbb{C}H} V.$$ 

For example, the regular representation of $H$ induces the regular representation of $G$:

$$Ind^G_H \mathbb{C}H := \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}H = \mathbb{C}G$$

The main theorem about restriction and induction is Frobenius reciprocity:

**Theorem 3.1** (Frobenius reciprocity). If $V$ is a representation of $H$ and $W$ is a representation of $G$ then

$$\text{Hom}_G(Ind^G_H V, W) \cong \text{Hom}_H(V, Res^G_H W)$$

This follows from:

**Theorem 3.2** (adjunction formula).

$$\text{Hom}_R(RM_S \otimes_S sV, RN) \cong \text{Hom}_S(sV, \text{Hom}_R(M_S, N))$$

And the easy formula:

$$\text{Hom}_R(R, N) \cong N$$

Letting $M = R$ and $S \subseteq R$, we get the following.

**Corollary 3.3.** If $S$ is a subring of $R$, $V$ is an $S$-module and $W$ is an $R$-module then

$$\text{Hom}_R(R \otimes_S V, W) \cong \text{Hom}_S(V, W)$$

Putting $R = \mathbb{C}G$, $S = \mathbb{C}[H]$, this gives Frobenius reciprocity.
### 3.2. induced characters.

**Definition 3.4.** Suppose that \( H \leq G \) (\( H \) is a subgroup of \( G \)) and \( \chi : H \to \mathbb{C} \) is a character (or any class function). Then the *induced character* 

\[
\text{Ind}^G_H \chi : G \to \mathbb{C}
\]

is the class function on \( G \) defined by

\[
\text{Ind}^G_H \chi(g) = \frac{1}{|H|} \sum_{h \in G} \chi(gh^{-1})
\]

where \( \chi(g) = 0 \) if \( g \notin H \).

The main theorem about the induced character is the following.

**Theorem 3.5.** If \( V \) is any representation of \( H \) then the induced character \( \text{Ind}^G_H \chi_V \) of \( V \) is the character of the induced representation \( \text{Ind}^G_H V \):

\[
\text{Ind}^G_H \chi_V = \chi_{\text{Ind}^G_H V}
\]

**3.2.1. Frobenius reciprocity for characters.** If \( f : G \to \mathbb{C} \) is any class function then the *restriction* of \( f \) to \( H \), denoted \( \text{Res}^G_H f \), is the composition of \( f \) with the inclusion map \( j : H \to G \):

\[
\text{Res}^G_H f = f \circ j : H \to \mathbb{C}
\]

**Theorem 3.6** (Frobenius reciprocity). Suppose that \( g, h \) are class functions on \( G, H \) respectively. Then

\[
\langle \text{Ind}^G_H h, g \rangle_G = \langle h, \text{Res}^G_H g \rangle_H
\]

**Proof.** Since both sides of the equation are bilinear in \( g, h \) and the irreducible characters span the vector space of all class functions, it suffices to prove these when \( g, h \) are irreducible characters:

(3.2)

\[
\langle \text{Ind}^G_H \chi_j, \chi_i \rangle_G = \langle \chi_j, \text{Res}^G_H \chi_i \rangle_H.
\]

But this follows almost immediately from abstract Frobenius reciprocity (the adjunction formula):

\[
\langle \text{Ind}^G_H \chi_j', \chi_i \rangle_G = \text{thm3.5} \langle \chi_{\text{Ind}^G_H \chi_j'}, \chi_i \rangle_G
\]

\[= \text{thm2.29} \dim \mathbb{C} \text{Hom}_G(\text{Ind}^G_H \chi_j', S_i)
\]

\[= \text{thm3.1} \dim \mathbb{C} \text{Hom}_H(S_j', \text{Res}^G_H \chi_i)
\]

\[= \text{thm2.29} \langle \chi_j', \text{Res}^G_H \chi_i \rangle_H
\]

As I pointed out in the instructions for HW 11, the version of Frobenius reciprocity given by the character formula (3.2) implies:

**Corollary 3.7.** The number of times that the simple \( G \)-module \( S_i \) appears in direct sum decomposition the induced representation \( \text{Ind}^G_H S_j' \) is equal to the number of times the number of times that \( S_j' \) appears in the direct sum decomposition of \( \text{Res}^G_H S_i \).