11 Factor Sets

If K is an additive group then a general extension

$$K \rightarrowtail G \xrightarrow{\pi} Q$$

of K by a group Q is given by an action of Q on K:

$$\theta : Q \to \mathrm{Aut}(K)$$

and a "factor set"

$$f : Q \times Q \to K.$$

The factor set is supposed to measure the extent to which the extension G is not a semidirect product.

Recall that the action of Q on K is given by

$$xa = \theta(x)(a) = \ell(x) + a - \ell(x)^1$$

for all $x \in Q$ and $a \in K$ where $\ell(x)$ is any element of $\pi^{-1}(x)$. Rotman points out that $\pi^{-1}(x)$ is a coset of K in G so a choice of $\ell(x)$ for each $x \in Q$ is the same as choosing one element from each of these cosets. This is suppose to justify the terminology that the set mapping $\ell : Q \to G$ is called a *transversal* of K. We will assume that

$$\ell(1) = 0.$$

Unlike Rotman we will suppress θ from the notation and consider the action of Q on K to be a fixed part of the structure of K, i.e., we consider K to be a (left) Q*-module*.

Definition 11.1. The *factor set* $f : Q \times Q \to K$ associated to the transversal ℓ is given by

$$f(x, y) = \ell(x) + \ell(y) - \ell(xy)$$

or, equivalently,

$$\ell(x) + \ell(y) = f(x, y) + \ell(xy)$$

for all $x, y \in Q$. This is an element of $K = \ker \pi$ since $\pi(\ell(x) + \ell(y) - \ell(xy)) = xy(xy)^{-1} = 1$.

Note that ℓ is a homomorphism if and only if $f(x, y) = 0$ for all $x, y \in Q$. And G is a semidirect product if and only if K has a transversal which is a homomorphism (Theorem 10.8). Thus the factor set f measures the extent to which ℓ is not a homomorphism which in turn measures the extent to which G is not a semidirect product.

[1]Since $K \subseteq G$ the group law in both K and G is written as addition. We have to remember that K is abelian but G is not.

Lemma 11.2. *Every factor set satisfies the following* cocycle *condition for all $x, y, z \in Q$.*

$$xf(y,z) - f(xy,z) + f(x,yz) - f(x,y) = 0 \tag{1}$$

Furthermore it is normalized *in the sense that*

$$f(1,x) = 0 = f(x,1) \tag{2}$$

for all $x \in Q$.

Proof. (2) follows from the assumption that $\ell(1) = 0$. (1) follows from the following calculation.

$$\begin{aligned} f(x,y) + f(xy,z) &= \ell(x) + \ell(y) - \ell(xy) \ + \ \ell(xy) + \ell(z) - \ell(xyz) \\ xf(y,z) + f(x,yz) &= \ell(x) + \underbrace{\ell(y) + \ell(z) - \ell(yz)}_{f(y,z)} - \ell(x) + \underbrace{\ell(x) + \ell(yz) - \ell(xyz)}_{f(x,yz)} \end{aligned}$$

Both are equal to $\ell(x) + \ell(y) + \ell(z) - \ell(xyz)$. □

Conversely, given any function $f : Q \times Q \to K$ satisfying (1) and (2) [such an f is called a *normalized cocyle*], we can construct a group $G(Q, K, f)$ as follows. The underlying set is the cartesian product $K \times Q$ with multiplication given by

$$(a,x)(b,y) = (a + xb + f(x,y), xy).$$

Associativity follows from the cocycle condition:

$$\begin{aligned} ((a,x)(b,y))(c,z) &= (a + xb + f(x,y), xy)(c,z) \\ &= (a + xb + f(x,y) + xyc + f(xy,z), xyz) \\ (a,x)((b,y)(c,z)) &= (a,x)(b + yc + f(y,z), yz) \\ &= (a + xb + xyc + xf(y,z) + f(x,yz), xyz) \end{aligned}$$

The identity is $(0,1)$ since f is normalized and the inverse of (a,x) is

$$(a,x)^{-1} = (-x^{-1}a - x^{-1}f(x,x^{-1}), x^{-1}).$$

Theorem 11.3. *$G(Q,K,f)$ is an extension of K by Q having a transversal with factor set f. Conversely, any such extension is isomorphic to $G(K,Q,f)$.*

Proof. We prove the second statement first. Suppose that G is an extension of K by Q with transversal ℓ giving the factor set f. Then an isomorphism $\psi : G(Q,K,f) \cong G$ is given by

$$\psi(a,x) = a + \ell(x).$$

This is obviously a bijection and it is easy to verify that this is a homomorphism:

$$\begin{aligned} \psi(a,x) + \psi(b,y) &= a + \ell(x) + b + \ell(y) = a + xb + \ell(x) + \ell(y) \\ &= a + xb + f(x,y) + \ell(xy) = \psi((a,x)(b,y)). \end{aligned}$$

The first statement is also easy: Let $\ell(x) = (0, x)$. Then

$$\ell(x)\ell(y) = (0, x)(0, y) = (f(x, y), xy) = f(x, y)\ell(xy).$$

$\square$

We will go through four examples.

Example 11.4. Let $G = \mathbb{Z}/p^2 = \{0, 1, 2, \cdots, p^2 - 1\}$ considered as an extension

$$K = \langle p \rangle \rightarrowtail G \xrightarrow{\pi} Q = \{1, g, g^2, \cdots, g^{p-1}\}$$

where π sends i to g^i. Then a transversal ℓ is given by

$$\ell(g^i) = i$$

for $i = 0, 1, \cdots, p - 1$. Then

$$\ell(g^i) + \ell(g^j) = i + j = \begin{cases} \ell(g^{i+j}) & \text{if } i + j < p, \\ p + \ell(g^{i+j}) & \text{if } i + j \geq p. \end{cases}$$

Consequently,

$$f(g^i, g^j) = \begin{cases} 0 & \text{if } i + j < p, \\ p & \text{if } i + j \geq p. \end{cases}$$

Example 11.5. Let G be the group of upper triangular matrices of the form:

$$\begin{pmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix}$$

where $i, j, k \in \mathbb{Z}$. This can be written as $c^k b^j a^i$ where

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The generators a, b, c are related by:

$$[a, c] = 1, \quad [b, c] = 1, \quad [a, b] = c \quad \Rightarrow \quad a^i b^j = c^{ij} b^j a^i$$

Let $K = \mathbb{Z} \cong \langle c \rangle$ with $k \in K$ corresponding to $c^k \in G$. Then $Q = \mathbb{Z} \times \mathbb{Z}$ with quotient map $\pi : G \to Q$ given by $\pi(c^k b^j z^i) = (i, j)$. The action of Q on K is trivial since c is central in G.

There is an obvious transversal given by $\ell(i, j) = b^j a^i$. Then

$$\ell(i, j)\ell(p, q) = b^j a^i b^q a^p = b^j c^{iq} b^q a^i a^p = c^{iq} b^{j+q} a^{i+p} = c^{iq} \ell(i + p, j + q)$$

so

$$f((i, p), (j, q)) = iq$$

Example 11.6. HW05.01: Let $Q = \mathbb{R}^n$ and $K = \mathbb{R}$ (considered as additive groups) with the trivial action of Q on K. Then show that any bilinear mapping

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

is a factor set.

Example 11.7. Let $G = D_8 = \langle s, t | s^4, t^2, stst \rangle$ considered as an extension of $K = \langle s^2, t \rangle = \{1, s^2, t, s^2t\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ by $Q = \{1, g\} \cong \mathbb{Z}/2$. Then the action of $g \in Q$ on K is given by:

$$gs^2 = ss^2s^{-1} = s^2, \quad gt = sts^{-1} = s^2t \quad \Rightarrow \quad g(s^2t) = t \quad \text{since } g^2 = 1$$

Take the transversal ℓ given by $\ell(g) = s$ [and $\ell(1) = 1$]. Then the associated factor set is given by

$$f(g, g) = s^2$$

(and $f(1, x) = f(x, 1) = 0$). At first glance it may appear that there are three other possibilities for $f(g, g) \in K$.

Lemma 11.8. *$f(g, g) = s^2$ or 1 for any normalized cocycle $f : Q \times Q \to K$.*

Proof. Taking $x = y = z = g$ in the cocycle condition (1) we get:

$$gf(g, g) - f(1, g) + f(g, 1) - f(g, g) = 0$$

Thus $gf(g, g) = f(g, g)$ which implies that $f(g, g)$ is s^2 or 1. □

This means that there are only two possible extensions of K by Q. However, if we take the transversal $\ell'(g) = st$ we get the factor set $f'(g, g) = 1$ [since ℓ' is a homomorphism]. Consequently,

Theorem 11.9. *$D_8 \cong D_4 \rtimes \mathbb{Z}/2$ is the only extension of D_4 by $\mathbb{Z}/2$ [with the above action of $\mathbb{Z}/2$ on D_4].*