We have three days to talk about semisimple rings and semisimple modules (Chapter XVII). A \textit{semisimple} $R$-module is a finite direct sum of simple modules
\[ M = S_1 \oplus \cdots \oplus S_n \]
and a \textit{semisimple ring} is a ring $R$ for which all f.g. modules are semisimple. The main reasons that I am choosing this particular topic in noncommutative algebra is for the study of representations of finite groups which is our last topic.

If $G$ is a finite group then a representation of $G$ over $\mathbb{C}$ is the same as a module over the group ring $\mathbb{C}[G]$ (also written $\mathbb{C}G$). Once we have the basic definitions it will be very easy to see that $\mathbb{C}[G]$ is a semisimple ring. This makes the representation theory of finite groups elementary.

From now on, all rings will be associative rings with $1 \neq 0$ (which may or may not be commutative) and $R$-module will usually be right $R$-modules.

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1. Simple rings and modules

We return to $R$ being a noncommutative ring. We generally take right $R$-modules $M$. Then, for any $f \in \text{End}_R(M)$, the ring of $R$-module endomorphisms of $M$, $x \in M$, $r \in R$ we have

$$f(xr) = f(x)r.$$ 

Given $f, g : M \to M$ we have

$$f(g(x)) = (fg)(x).$$

So, $M$ is a left $\text{End}_R(M)$-module. Both rules are “associativity” type rules.

The center $Z(R)$ of a ring $R$ is the set of all $z \in R$ which commutes with all other elements of $R$:

$$Z(R) := \{z \in R \mid zx = xz \ \forall x \in R\}$$

**Proposition 1.1.** $Z(R)$ is a subring of $R$. □

**Proposition 1.2.** A ring $A$ is a $K$-algebra (for $K$ a field) if and only if $A$ contains (a copy of) $K$ in its center.

**Proof.** ($\Leftarrow$) Suppose $K \subset Z(R)$. Then, for any $x, y \in R, a, b \in K$ we have:

$$(ax)(by) = abxy, \quad a(x + y) = ax + ay$$

since $b$ is central. So, $R$ is a $K$-algebra.

($\Rightarrow$) Suppose $R$ is a $K$-algebra. Then we have a ring homomorphism $\lambda : K \to R$ given by $\lambda(a) = a1_R$ where $1_R$ is the unit in $R$. Since ker $\lambda$ is an ideal in $K$, it must be zero. So, $\lambda$ is a monomorphism. We “identify” $K$ with the image of $\lambda$. Since multiplication in $R$ is $K$-bilinear

$$x(a1_R) = a(x1_R) = ax = (a1_R)x$$

So, $a1_R \in Z(R)$ for all $a \in K$. So, $\lambda(K) \subset Z(R)$. So, $R$ contains a copy of $K$ in its center. □

The basic noncommutative rings are the simple rings.

1.1. Simple rings.

**Definition 1.3.** A ring $R$ is called simple if it has no nonzero two-sided ideals.

For example, any field is simple. There is also a noncommutative version of a field:

1.1.1. Division rings.

**Definition 1.4.** A division ring is a ring $R$ in which every nonzero element has a two-sided inverse. I.e., for all $a \neq 0 \in R$ there is a $b \in R$ so that $ba = ab = 1$.

**Proposition 1.5.** $R$ is a division ring iff every nonzero element has a left inverse.

**Proof.** If every $a \neq 0 \in R$ has a left inverse $b$ (so that $ba = 1$) then $b$ also has a left inverse $c$ with $cb = 1$. But then

$$c = c(ba) = (cb)a = a$$

So, $ab = cb = 1$ making $b$ a two-sided inverse for $a$. So, $R$ a division ring. □
**Theorem 1.6.** Division rings are simple.

**Proof.** Any nonzero two-sided ideal \( I \subseteq D \) would have a nonzero element \( a \). So, \( aD \subseteq I \) would contain 1 making \( I = D \). \(

**Example 1.7.** \( \mathbb{H} = \) the ring of quaternions is a division algebra. (An algebra is a ring which contains a field in its center. The center of a ring \( R \) is the set of all \( z \in R \) so that \( zx = xz \) for all \( x \in R \). The center of \( \mathbb{H} \) is \( \mathbb{R} \). So, \( \mathbb{H} \) is an \( \mathbb{R} \)-algebra. Quaternions are:
\[
t + xi + yj + zk
\]
where \( t, x, y, z \in \mathbb{R} \). Thus \( \mathbb{H} \cong \mathbb{R}^4 \) is 4-dimensional and \( i, j, k \in \mathbb{H} \) are noncommuting elements satisfying:
\[
ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j
\]
and \( i^2 = j^2 = k^2 = -1 \).

### 1.1.2. Matrix rings.

**Definition 1.8.** For any ring \( R \) let \( \text{Mat}_n(R) \) denote the ring of \( n \times n \) matrices \((a_{ij})\) with coefficients \( a_{ij} \in R \). Addition is “coordinate-wise” \((a_{ij}) + (b_{ij}) = (c_{ij})\) where \( c_{ij} = a_{ij} + b_{ij} \). Multiplication is matrix multiplication: \((a_{ij})(b_{ij}) = (c_{ij})\) where \( c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \). A tedious and unnecessary computation will show that \( \text{Mat}_n(R) \) is a ring. (There is an easy proof which we will see later.)

**Theorem 1.9.** \( \text{Mat}_n(R) \) is isomorphic to the endomorphism ring of \( R_n \), the free right \( R \)-module on \( n \) generators.

**Proof.** Let \( e_1, \ldots, e_n \) be a basis for \( R^n \) as right module. Then every element of \( R^n \) can be written uniquely as:
\[
x = \sum_{j=1}^{n} e_j x_j, \quad x_j \in R
\]
For \( f : R^n \to R^m \) we have the matrix \( A = (a_{ij}) \) with \( a_{ij} \in R, 1 \leq i \leq m, 1 \leq j \leq n \) s.t.
\[
f(e_j) = \sum e_j a_{ij}
\]
Then
\[
f(x) = f \left( \sum_{j=1}^{n} e_j x_j \right) = \sum_{j=1}^{n} f(e_j) x_j = \sum_{i,j} e_i a_{ij} x_j = \sum_{i=1}^{m} e_i \sum_{j=1}^{n} a_{ij} x_j
\]
If \( g : R^m \to R^\ell \) is given by the matrix \( (b_{ki}) \) then
\[
g(e_i) = \sum e_k b_{ki}
\]
\[
gf(e_j) = g \left( \sum e_j a_{ij} \right) = \sum g(e_i) a_{ij} = \sum e_k b_{ki} a_{ij}
\]
Thus, the matrix of \( gf \) is the product \((b_{ki})(a_{ij})\) of the matrices of \( g, f \). It is easy to see that the matrix of a sum \( f + g \) has matrix \((a_{ij} + b_{ij})\). So, when \( n = m = \ell \) these are all square matrices and
\[
\text{End}_R(R_R^n) \cong \text{Mat}_n(R)
\]
as claimed. \( \square \)
An important special case is $n = 1$.

**Corollary 1.10.** $R \cong \text{End}(R_R)$.

One reason for using right modules is the following annoying feature of left modules.

**Corollary 1.11.** $\text{End}_R(R_R) \cong R^{op}$, the opposite ring of $R$.

**Proof.** We have an anti-isomorphism $\varphi : R \cong \text{End}_R(R_R)$ given by $\varphi(r)(x) = xr$. Then $\varphi(rs)(x) = x(rs) = \varphi(s)(xr) = \varphi(s)\varphi(r)(x)$. So, $\varphi(rs) = \varphi(s)\varphi(r)$. $\varphi$ is an isomorphism $R^{op} \cong \text{End}_R(R_R)$.

I assume that everyone knows how matrices work. However, determinant does not exist. There is no function $\det : \text{Mat}_n(R) \to R$ so that $\det(AB) = \det(A)\det(B)$.

Note that $\text{Mat}_n(R)$ is a free $R$-module with basis given by the matrices $X_{ij}$ which have a 1 in the $ij$ position and 0 everywhere else. A matrix with coefficients $a_{ij} \in R$ can be written as

$$ (a_{ij}) = \sum X_{ij}a_{ij} $$

**Theorem 1.12.** If $D$ is a division ring then $\text{Mat}_n(D)$ is simple.

**Proof.** If $J$ is a nonzero two-sided ideal in $\text{Mat}_n(D)$ then I want to show that $J = \text{Mat}_n(D)$. Let $A = (a_{ij})$ be a nonzero element of $J$. Then one of the entries is nonzero: say, $a_{ij} \neq 0$. For any $k$ we multiplying on the right by $X_{jk}$ and on the left by the matrix $X_{ki}a_{ij}^{-1}$ we see that

$$ X_{kk} = X_{ki}a_{ij}^{-1}AX_{jk} \in J $$

We get $X_{kk} \in J$ for each $k$. Adding these we get the identity matrix $I_n = \sum X_{kk} \in J$. So, $J = \text{Mat}_n(D)$ as claimed.

We will prove Wedderburn’s Theorem which implies that, up to isomorphism, there are no other examples of simple rings.

**Example 1.13.** Let $D$ be the set of $2 \times 2$ complex matrices of the form

$$ A = \begin{bmatrix} z & w \\ \overline{w} & \overline{z} \end{bmatrix} $$

where $z, w \in \mathbb{C}$. Then $D$ is closed under addition, multiplication and scalar multiplication by real numbers and contains $I_2$. So, $D$ is an $\mathbb{R}$-algebra.

For $A \neq 0$, $\det A = |z|^2 + |w|^2 \in \mathbb{R}$. So

$$ A^{-1} = \frac{1}{\det A} \begin{bmatrix} \overline{z} & -w \\ w & \overline{z} \end{bmatrix} \in D. $$

Thus, $D$ is a division algebra. This division algebra is isomorphic to $\mathbb{H}$. An isomorphism $\varphi : \mathbb{H} \cong D$ is given by

$$ \varphi(i) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \varphi(j) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \varphi(k) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. $$
1.2. Simple modules.

**Definition 1.14.** For any ring $R$, a (left or right) $R$-module $S$ is called *simple* if $S \neq 0$ and $S$ has no proper nonzero submodules. ($M \subseteq S$ is *proper* if $M \neq S$.)

Some trivial observations without proof:

1. $S$ is simple iff 0 is a maximal proper submodule.
2. If $N \subset M$ is a maximal proper submodule then $M/N$ is simple.

**Theorem 1.15.** The right $R$-module $R$ is simple iff $R$ is a division algebra.

*Proof.* This is obvious both ways. ($\Leftarrow$) A submodule $N$ of $R$ is the same as a right ideal. If $N$ is nonzero then it has a nonzero element $a$ with inverse $b$. Then $ab = 1 \in NR$. So, $N = NR = R$. ($\Rightarrow$) Conversely, suppose that $R$ is simple and $a \neq 0 \in R$. Then $aR$ is a nonzero submodule of $R$. Therefore, $aR = R$. So, $1 = ab \in aR$ and $b$ is a right inverse for $a$. So, $R$ is a division ring. □

One of the most important theorem about simple modules is also trivial:

**Lemma 1.16** (Schur’s lemma). If $S, T$ are simple $R$-modules and $f : S \rightarrow T$ is an $R$-module homomorphism then either $f = 0$ or $f$ is an isomorphism.

*Proof.* Suppose $f \neq 0$. Then $\ker f \neq S$. Since $S$ is simple, $\ker f = 0$ and $f$ is a monomorphism. Also, the image of $f$ is a nonzero submodule of $T$. So, $f(S) = T$, i.e., $f$ is onto. So, $f$ is an isomorphism. □

**Theorem 1.17.** The endomorphism ring $\text{End}_R(S)$ of a simple module $S$ is a division ring.

*Proof.* If $S$ is simple then Schur’s lemma tells us that any nonzero element of $\text{End}_R(S)$ is an isomorphism and thus invertible. □

**Example 1.18.** Let $M = \mathbb{R}^3$ considered as a set of row vectors. Then $M$ is a right $R$-module where $R = \text{Mat}_3(\mathbb{R})$ is the ring of $3 \times 3$ real matrices.

**Claim:** $M$ is a simple right $R$-module for $R = \text{Mat}_3(\mathbb{R})$.

*Proof:* Let $N \subset M$ be a nonzero submodule. Let $v \neq 0 \in N$. Then, for any vector $w \in M = \mathbb{R}^3$, there is a linear transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ sending $v$ to $w$. Thus $w = vA$ for some $A \in R = \text{Mat}_3(\mathbb{R})$. So, $vR = M \subset N$. So, $N = M$ is the only nonzero submodule. So, $M$ is simple.

Theorem 1.17 says that $\text{End}_R(M)$ is a division algebra. In fact, one can show that $\text{End}_R(M) = \mathbb{R}$.

Generalizing this example we have:

**Theorem 1.19.** Let $K$ be a field. Then $K^n$ is a simple right $\text{Mat}_n(K)$-module. □

**Corollary 1.20.** The ring $R = \text{Mat}_n(K)$, considered as a right module over itself is a direct sum of $n$ isomorphic simple modules given by the $n$ rows of a matrix. □

This is an example of a semi-simple module.
2. Semisimple modules

2.1. Finiteness conditions. We will need both the ACC and the DCC for modules. These work the same way that they do for modules over commutative rings.

2.1.1. Noetherian modules over noncommutative rings.

Definition 2.1. An $R$-module $M$ is called Noetherian if it satisfies the ascending chain condition (ACC) for submodules. This is equivalent to saying that every submodule of $M$ is finitely generated. A ring $R$ is called left/right-Noetherian if $rR$, resp $RR$ is a Noetherian module. Equivalently, $R$ has the ACC for left/right ideals.

Just as in the commutative case we have:

Theorem 2.2. (1) If $M$ is Noetherian then every submodule and quotient module of $M$ is Noetherian.

(2) $R$ is right/left-Noetherian iff every finitely generated right/left $R$-module is Noetherian.

Noetherian modules have lots of maximal submodules, by which I mean maximal proper submodules.

Proposition 2.3. Every proper submodule of a Noetherian module is contained in a maximal submodule.

Proof. Otherwise, we would get a sequence of larger and larger submodules, contradicting the ACC. □

Note: A module which is not Noetherian might not have any maximal submodules. Corollary [2.11] gives an example.

Sometimes it is enough just to assume that $M$ is finitely generated. (We will go through Lemma [2.4] next time.)

Lemma 2.4 (finite sum lemma). If a f.g. module $M$ is a sum of submodules $M = \sum_{\alpha \in I} N_\alpha$, then there is a finite subset $J$ of the index set $I$ so that $M = \sum_{\beta \in J} N_\beta$.

Proof. To say that $M$ is the sum of the submodules $N_\alpha$ is the same as saying that the inclusion maps $N_\alpha \hookrightarrow M$ give an epimorphism

$$\bigoplus_{\alpha \in I} N_\alpha \twoheadrightarrow M$$

Suppose that $x_1, \cdots, x_n$ generate $M$ and, for each $i$, choose $y_i \in \bigoplus_{\alpha \in I} N_\alpha$ which maps onto $x_i$. Then each $y_i$ has only finitely many nonzero coordinates. Let $J \subseteq I$ be the set of all $\beta \in I$ so that some $y_i$ has a nonzero $\beta$ coordinate. Then $M = \sum_{\beta \in J} N_\beta$. □
2.1.2. Artinian modules.

Definition 2.5. An $R$-module $M$ is called Artinian if it satisfies the descending chain condition (DCC) for submodules. A ring $R$ is called left/right-Artinian if $RR$, resp $RR$, is an Artinian module.

Example 2.6. (1) If $M$ is a finite set, it is clearly Artinian (and Noetherian).

(2) Suppose $R$ is a $K$-algebra. Then any $R$-module is also a $K$-module, i.e., a vector space over $K$. If $M$ is finite dimensional as $K$-vector space, it is Artinian (and Noetherian) since any sequence of submodules has decreasing dimensions and, if $\dim N_k = \dim N_{k+1}$ then $N_k = N_{k+1}$.

Analogous to the corresponding statements for Noetherian modules and with analogous proofs we have the following.

Theorem 2.7. (1) Every submodule and quotient module of an Artinian module is Artinian.

(2) $R$ is left/right Artinian iff every f.g. $R$-module is Artinian.

(3) Every nonzero submodule of an Artinian module contains a simple submodule.

A statement which is dual to the finite sum lemma for Noetherian modules is the following.

Lemma 2.8 (finite intersection lemma). Suppose that $M$ is an Artinian module and $N = \bigcap_{a \in I} N_a$ is an intersection of submodules. Then there is a finite subset $J$ of the index set $I$ so that $N = \bigcap_{\beta \in J} N_\beta$.

Proof. Finite intersections form a descending sequence of submodules which stops when it is equal to the infinite intersection. \qed

Example 2.9. (1) $\mathbb{Z}$ is a Noetherian $\mathbb{Z}$-module but it is not Artinian. (2) Take the ring

$$\mathbb{Z}[1/p] = \{a/p^n : a \in \mathbb{Z}, n \geq 0\}$$

$\mathbb{Z} \subset \mathbb{Z}[1/p]$ is a subring (not an ideal). So the quotient $\mathbb{Z}[1/p]/\mathbb{Z}$ is a $\mathbb{Z}$-module (not a ring). This module is Artinian but not Noetherian. To see that this module is not Noetherian, take the submodule $N_n$ generated by $1/p^n + \mathbb{Z}$. This has $p^n$ elements and contains $N_{n-1}$. So,

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

is an infinite ascending sequence of submodules. So, $\mathbb{Z}[1/p]/\mathbb{Z}$ is not Noetherian.

Theorem 2.10. $\mathbb{Z}[1/p]/\mathbb{Z}$ is an Artinian $\mathbb{Z}$-module.

Proof. Every proper submodule $M \subset \mathbb{Z}[1/p]/\mathbb{Z}$ is finite! $\Rightarrow$ $M$ has DCC for submodules. Consider the possible values of $n$ in fractions $a/p^n + \mathbb{Z}$ in $M$. If the number $n$ is bounded, say $n \leq N$ then $M$ has $p^N$ elements. Conversely, suppose that $n$ in unbounded. Then $M$ contains $a/p^n$ for arbitrarily large values of $n$. But then $M$ contains $1/p^n + \mathbb{Z}$. Here is the proof: Since $a, p^n$ are relatively prime, there are integers $x, y$ so that

$$ax + p^n y = 1 \Rightarrow \frac{ax}{p^n} + y = \frac{1}{p^n}.$$
But then
\[ \frac{ax}{p^n} + \mathbb{Z} = \frac{1}{p^n} + \mathbb{Z} \in M \]
which implies that all \( \frac{a}{p^n} + \mathbb{Z} \) are in \( M \). So, \( M = \mathbb{Z}[1/p]/\mathbb{Z} \). \( \square \)

**Corollary 2.11** (of the proof). \( \mathbb{Z}[1/p]/\mathbb{Z} \) does not have a maximal submodule.

*Proof.* Any proper submodule of \( \mathbb{Z}[1/p]/\mathbb{Z} \) is contained in some \( N_n \). But \( N_n \subsetneq N_{n+1} \). \( \square \)
2.2. Definition of semisimple.

**Definition 2.12.** A f.g. $R$-module $M$ is called *semisimple* if it satisfies one of the following equivalent definitions.

1. $M$ is a direct sum of finitely many simple modules.
2. $M$ is a sum of simple submodules.
3. Every submodule of $M$ is a direct summand.

First I need a trivial lemma.

**Lemma 2.13.** Suppose that $N, S$ are submodules of any module $M$ where $S$ is simple. Then $N + S$ is either equal to $N$ or to $N \oplus S$. In both cases, $N$ is a direct summand of $N + S$.

**Proof.** $N \cap S$ is a submodule of $S$. So, it is either 0 or $S$. In the first case, $N + S = N \oplus S$. In the second case, $N + S = N$. □

**Proof of equivalence of definitions.** Clearly, $(1) \Rightarrow (1') \Rightarrow (2)$. Also, by the finite sum lemma (2.4), $(2)$ implies $(2')$. Since every simple module is generated by one element (any nonzero element), $(2')$ includes the assumption that $M$ is finitely generated.

$(2') \Rightarrow (3)$ Suppose that $N \subseteq M$ and $M = \sum_{i=1}^{n} S_i$. For each $k \leq n$ let

$$N_k = N + S_1 + \cdots + S_k$$

Lemma 2.13 says that $N_k$ is a direct summand of $N_{k+1} = N_k + S_{k+1}$ for every $k$. Therefore, $N$ is a direct summand of $N_n = M$.

$(3) \Rightarrow (1)$ Since any submodule $N \subseteq M$ is a summand: $M = N \oplus K$, $N \cong M/K$ is also a quotient of $M$. Therefore, every submodule of $M$ is f.g. making $M$ Noetherian.

If $M$ is not simple then $M$ contains a maximal submodule $N_1$ and $S_1 = M/N_1$ is simple. By $(3)$, $M = S_1 \oplus N_1$. Since $N_1$ is Noetherian, if $N_1 \neq 0$ it contains a maximal submodule $N_2$. So, $S_2 = N_1/N_2$ is simple and $M/N_2 = S_1 \oplus S_2$. So $M = S_1 \oplus S_2 \oplus N_2$. Repeating this process we get

$$M = S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus N_n$$

giving an ascending sequence of submodules

$$S_1 \subset S_1 \oplus S_2 \subset \cdots \subset S_1 \oplus S_2 \oplus \cdots \oplus S_n$$

which must eventually stop. So $N_n = 0$ and $M$ is a direct sum of $n$ simple modules. □

**Example 2.14.** For any division ring $D$, the matrix ring $R = Mat_n(D)$ is a semi-simple module over itself. As a right module $R_R$ is the direct sum of the rows of $Mat_n(D)$. More precisely, let $M_i$ be the set of $n \times n$ matrices which are zero except in the $i$th row. Clearly, each $M_i$ is a right $Mat_n(D)$-module and

$$Mat_n(D) \cong M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

Each row $M_i$ is a simple $Mat_n(D)$ module since any nonzero element generates all row vectors.
2.3. Unique decomposition.

**Theorem 2.15.** The simple summands of a semisimple $R$-module are uniquely determined up to isomorphism. In other words, if

$$M = S_1 \oplus \cdots \oplus S_n = T_1 \oplus \cdots \oplus T_m$$

where $S_i, T_j$ are simple submodules of $M$, then $n = m$ and $S_i \cong T_{\sigma(i)}$ for some permutation $\sigma$ of $n$.

**Proof.** For each $j$ let

$$N_j = T_1 \oplus \cdots \oplus \widehat{T_j} \oplus \cdots \oplus T_m$$

This is a maximal submodule of $M$ since $M/N_j \cong T_j$ is simple. Also, $\bigcap N_j = 0$. So, there is some $j$ so that $S_n \not\subseteq N_j$. Since $S_n$ is simple, this implies that $S_n \cap N_j = 0$. Since $N_j$ is maximal we conclude that $M = S_n \oplus N_j$ and therefore,

$$M/S_n \cong S_1 \oplus \cdots \oplus S_{n-1} \cong N_j = T_1 \oplus \cdots \oplus \widehat{T_j} \oplus \cdots \oplus T_m$$

The theorem follows by induction on $n$. \qed

**Definition 2.16.** Define the length $\ell(M)$ of a semisimple module $M$ to be the number of simple summands in any decomposition $M = \bigoplus S_i$. Sometimes we write $\ell_R(M)$. Theorem 2.15 implies that $\ell(M)$ is well-defined.

**Example 2.17.** For $R = \text{Mat}_n(D)$,

$$\ell(R_R) = n$$

since $M = R_R$ is a direct sum of its $n$ rows which are simple right $R$-modules. As $D$-module we have

$$\ell(R_D) = n^2$$

since $R_D \cong D^{n^2}$.

**Corollary 2.18.** Submodules and quotient modules of semisimple modules are semisimple. And $\ell(N) + \ell(M/N) = \ell(M)$ for any submodule $N$ of a semisimple module $M$.

**Proof.** By (3), $M = N \oplus K$ for some submodule $K \cong M/N$. Each a quotient of $M$ and therefore a sum of finitely many simple modules. So, they are both semisimple. Decomposing $N, K$ into direct sums of $n, m$ simple modules, we get a decomposition of $M = N \oplus K$ into $n + m$ simple modules. So,

$$\ell(M) = n + m = \ell(N) + \ell(M/N)$$

\qed

**Corollary 2.19.** Semisimple modules are both Noetherian and Artinian.

**Proof.** For any increasing or decreasing sequence of submodules the lengths increase or decrease. \qed
2.4. **Jacobson radical.** An Artinian module $M$ is semisimple iff its Jacobson radical is zero.

**Definition 2.20.** The *Jacobson radical* $rM$ (or $JM$) of any $R$-module $M$ is defined to be the intersection of all maximal (proper) submodules of $M$. If $M$ has no maximal submodules then $rM = M$. For example, the radical of $M = \mathbb{Z}[1/p]/\mathbb{Z}$ is $rM = M$.

**Example 2.21.**

1. $r\mathbb{Z} = 2\mathbb{Z} \cap 3\mathbb{Z} \cap 5\mathbb{Z} \cap \cdots = 0$ since there is no positive integer $n$ which is divisible by every prime $p$.

2. For $M = \mathbb{R}^2$ as $\mathbb{R}$-module, the maximal submodules of $M$ are the linear subspaces. The intersection of these is 0. So, $rM = 0$.

**Proposition 2.22** (Naturality of $rM$). If $f : M \to N$ is a homomorphism of $R$-modules, then $f(rM) \subseteq rN$.

**Proof.** For any maximal submodule $L_\alpha \subseteq N$, $f^{-1}(L_\alpha)$ is either equal to $M$ or to a maximal submodule of $M$. Therefore,

$$f^{-1} \bigcap L_\alpha = \bigcap f^{-1}(L_\alpha)$$

contains $rM$ which is what we wanted to show. \qed

**Theorem 2.23.** Suppose that $M$ is an Artinian $R$-module. Then

1. $M/rM$ is semisimple.
2. $M$ is semisimple iff $rM = 0$.

**Example 2.21** (2): $M$ is semisimple. In (1), $\mathbb{Z}$ is not semisimple since $\mathbb{Z}$ is not Artinian.

**Proof.** By the finite intersection lemma (2.8), the Jacobson radical of $M$ is a finite intersection of maximal submodules: $rM = \bigcap N_i$. Thus we have an exact sequence:

$$0 \to rM \to M \to \bigoplus M/N_i$$

Since each $N_i$ is maximal, $M/N_i$ is simple. So, $M/rM$ is also semisimple, being isomorphic to a submodule of $\bigoplus M/N_i$.

This shows that $rM = 0$ implies $M$ is semisimple. Conversely, suppose $M = \bigoplus S_i$. Then the kernel of each projection $M \to S_i$ is a maximal submodule $N_i$ and $rM \subseteq \bigcap N_i = 0$. \qed
3. Semisimple rings

3.1. Jacobson radical of a ring. The theorem is that an Artinian ring is semisimple if and only if its Jacobson radical is zero. But we need to define the terms.

**Definition 3.1.** The Jacobson radical $rR$ of a ring $R$ is defined to be the intersection of all maximal right ideals. (This is the same as the Jacobson radical of the module $R_R$. Rings always have maximal ideals by Zorn’s Lemma. So, $rR 
eq R$.)

As a special case of the previous theorem we have the following.

**Corollary 3.2.** If $R$ is a right Artinian ring then $R_R$ is semisimple iff $rR = 0$.

**Definition 3.3.** A ring $R$ is called semisimple if every f.g. $R$-module is semisimple.

**Remark 3.4.** All f.g. $R$-modules are semisimple iff $R_R$ is semisimple.

**Proof:** Every f.g. module is a quotient of $R^n_R$ for some $n$ and every quotient of a semisimple module is semisimple.

**Corollary 3.5.** A ring is semisimple iff it is right-Artinian (and Noetherian) and its Jacobson radical is zero.

**Proof.** ($\Leftarrow$) This follows from the previous corollary and remark.

($\Rightarrow$) If $R$ is a semisimple ring, $R_R$ is a semisimple module which implies that it is Artinian and Noetherian and its radical is zero.

**Theorem 3.6.** The Jacobson radical of $R$ is a two-sided ideal.

**Proof.** $rR$ is clearly a right ideal. So, let $a \in R$. Then left multiplication by $a$ is an $R$-module homomorphism $\rho(a) : R_R \to R_R$. By naturality of $r$ (Prop. 2.22) this implies that $a(rR) \subseteq rR$. So, $rR$ is also a left ideal.

**Corollary 3.7.** Simple Artinian rings are semisimple.

**Corollary 3.8.** Division rings and matrix rings over division rings are semisimple.

**Proof.** Division rings are clearly Artinian. So, they are semisimple. The matrix ring $Mat_n(D)$ is finitely generated as a $D$-module and therefore is Artinian (since all right ideals are also $D$-submodules). So, it is also semisimple by the previous corollary.
3.2. Wedderburn structure theorem. Here is the last theorem of Part C.

**Theorem 3.9 (Wedderburn structure theorem).** A ring $R$ is semisimple if and only if it is a finite product of matrix rings over division rings:

$$R \cong \prod Mat_{n_i}(D_i)$$

To show that these products are semisimple we need the following lemma.

**Lemma 3.10.** A f.g. module over the product of two rings $R \times S$ is a direct sum $M_R \oplus N_S$ of a f.g. $R$-module $M_R$ and a f.g. $S$-module $N_S$.

By induction on $n$: a f.g. module over a product of rings $R_1 \times R_2 \times \cdots \times R_n$ is a sum of modules $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ where $M_i$ is a f.g. $R_i$-module.

**Proof.** The unit $1 = (1, 1)$ of the ring $R \times S$ can be written as a sum:

$$1 = (1, 0) + (0, 1) = e_1 + e_2$$

where $e_i$ are central, orthogonal idempotents. (*Central* means $e_i x = xe_i$ for all $x$, *orthogonal* means $e_i e_2 = 0$ and *idempotent* means $e_i^2 = e_i$.) If $M$ is any $R \times S$ module then any $x \in M$ can be written uniquely as $x = x_1 + xe_1 + xe_2$. Thus

$$M = Me_1 \oplus Me_2.$$ 

Since $e_1 S = Se_1 = 0$, the right action of $S$ on $Me_1$ is zero. So, the action of $R \times S$ on $Me_1$ factors through $(R \times S)/S \cong R$ and the action of $R \times S$ on $Me_2$ factors through $S$. If $M$ is f.g. then $Me_1, Me_2$ are finitely generated modules over $R, S$ respectively. □

**Example 3.11.** A module over the product $\mathbb{Z} \times \mathbb{R}$ is $A \oplus B$ where $A$ is an additive group and $B$ is a real vector space. A module over $(\mathbb{Z} \times \mathbb{R}) \times \mathbb{C}$ is $(A \oplus B) \oplus C$ where $A \oplus B$ is a $\mathbb{Z} \times \mathbb{R}$-module (i.e., $A$ is an additive group and $B$ is a real vector space) and $C$ is a complex vector space.

**Lemma 3.12.** If $R, S$ are semisimple rings then their product $R \times S$ is semisimple.

**Proof.** By the previous lemma, any f.g. $R \times S$-module $M$ is a direct sum of a f.g. $R$-module $A$ and a f.g. $S$-module $B$. Since $R, S$ are semisimple, $A$ and $B$ are finite direct sums of simples modules. So, $M = A \oplus B$ is also a finite direct sum of simple modules. □

We know, by Corollary 3.8, that $Mat_{n_i}(D_i)$ are semisimple. Lemma 3.12 proves that any finite product $\prod Mat_{n_i}(D_i)$ is semisimple.

### 3.2.1. endomorphisms.

Suppose that $M$ is a semisimple right $R$-module. Then we want to show that the endomorphism ring $\text{End}_R(M_R)$ is one of the rings in the Wedderburn structure theorem. This will prove the structure theorem because of the observation (Corollary 1.10) that $\text{End}_R(R_R) \cong R$.

In the decomposition $M = \bigoplus S_i$, some of the simples $S_i$ may be isomorphic to each other. We use the notation $nS_i$ to denote a direct sum of $n$ copies of the simple module $S_i$. Then we can write:

$$M \cong \bigoplus_{i=1}^{m} n_i S_i$$
where the $S_i$ are nonisomorphic.

By Schur’s lemma, there are no homomorphisms from $S_i$ to $S_j$ for $i \neq j$. Therefore,

$$\text{End}_R(M) \cong \prod_{i=1}^{m} \text{End}_R(n_iS_i)$$

So, it suffices to show the following lemma.

**Lemma 3.13.** If $S$ is a simple right $R$-module then

$$\text{End}_R(nS) \cong \text{Mat}_n(D)$$

where $D = \text{End}_R(S_R)$.

**Proof.** An isomorphism $\varphi : \text{End}_R(nS) \to \text{Mat}_n(D)$ is given as follows. For any $f : nS \to nS$ let $\varphi(f) \in \text{Mat}_n(D)$ be the matrix with $ij$-coordinate (in $D = \text{End}_R(S)$) given by the composition

$$p_i \circ f \circ t_j : S \xrightarrow{t_j} nS \xrightarrow{f} nS \xrightarrow{p_i} S$$

where $t_j : S \to nS$ is the inclusion of the $j$th summand and $p_i : nS \to S$ is the projection to the $i$th summand.

This is a homomorphism since

$$\varphi(f \circ g) = (p_i \circ f \circ g \circ t_j) = (p_i \circ f \circ \sum t_k \circ p_k \circ g \circ t_j)$$

$$= (p_i \circ f \circ t_k)(p_k \circ g \circ t_j) = \varphi(f)\varphi(g)$$

This uses the equation $\sum t_k \circ p_k = \text{id}$ from the definition of direct sum.

To show that $\varphi$ is an isomorphism, we give the inverse: $\varphi^{-1}(f_{ij}) = \sum t_i \circ f_{ij} \circ p_j$. \qed

**Summary of proof of Wedderburn Theorem 3.9.**

$\text{Mat}_n(D)$ simple $\Rightarrow \text{Mat}_n(D)$ semisimple $\Rightarrow \prod \text{Mat}_n(D_i) \text{ semisimple}$

**Lemma 3.12**

Conversely, if $R$ is semisimple. So, $R_R = \bigoplus n_iS_i$ and

$$R \cong \text{End}_R(R_R) \cong \prod \text{End}(n_iS_i) \cong \prod \text{Mat}_n(D_i)$$

**Lemma 3.13**

where $D_i = \text{End}_R(S_i)$ is a division ring by Schur’s Lemma. \qed
3.2.2. algebraically closed fields. We want to talk about algebras $A$ over an algebraically closed field $K$. (Recall that this means $K \subset Z(A)$.)

It is easy to see that the center of any division ring is a field since, if $a \in D$ is central, so is $1/a$.

Since the center of $A$ acts on all $A$-modules, every module over a $K$-algebra will be a vector space over $K$. If a $K$-algebra $A$ is finite dimensional as a vector space over $K$, then it is clearly Artinian.

**Theorem 3.14.** The only finite dimensional division algebra over an algebraically closed field $K$ is $K$ itself.

**Proof.** Suppose that $D$ is an $n$ dimensional division algebra over an algebraically closed field $K$. Take any $a \in D$. Left multiplication by $a$ gives a $K$-linear endomorphism of the $n$-dimensional vector space $D$:

$$\mu(a) : D \cong K^n \rightarrow D \cong K^n$$

Since $K$ is algebraically closed, it contains all of the eigenvalues (roots of the characteristic polynomial) of this endomorphism. Let $\lambda \in K$ be one of these eigenvalues. Then left multiplication by $a - \lambda$ is singular, i.e., there is an $x \neq 0 \in D$ (the eigenvector) so that $(a - \lambda)x = 0$. But $D$ is a division algebra. So, this implies that $a = \lambda \in K$. Since $a \in D$ was arbitrary, this implies that $D = K$. □

**Example 3.15.** The division ring $\mathbb{H}$ or quaternions is a finite dimensional division algebra over $\mathbb{R}$ but not over $\mathbb{C}$ since $\mathbb{C} \subset \mathbb{H}$ is not central.

**Corollary 3.16.** The only finite dimensional semisimple algebras over $\mathbb{C}$ are finite products of matrix algebras:

$$R = \prod Mat_{n_i}(\mathbb{C}).$$

**Remark 3.17.**

1. The dimension of $R$ is the sum of squares: $\dim R = \sum n_i^2$.
2. $R$ is commutative if and only if all $n_i = 1$.
3. For example, if $R$ is noncommutative of dimension 6 then $R \cong \mathbb{C} \times \mathbb{C} \times Mat_2(\mathbb{C})$.

In Part D we will see that the group ring $\mathbb{C}[G]$ of any finite group is a semi-simple $\mathbb{C}$-algebra with dimension $|G|$, the order of $G$. This implies, e.g., that

$$\mathbb{C}[S_3] \cong \mathbb{C} \times \mathbb{C} \times Mat_2(\mathbb{C}).$$