THE ORIGIN OF PICTURES:
NEAR-RINGS AND ALGEBRAIC K-THEORY

KIYOSHI IGUSA

Abstract. These are lecture notes for a talk I gave at the Algebra Seminar at the University of Connecticut on March 21, 2018. I explain the origin of “pictures” which played a large part in many of my earlier papers in topology including my PhD thesis. Since details on the combinatorics of near-rings are already published in my 1982 paper (in which they are called “quasi-rings”), I presented only the ideas and an open question about near-rings which has an important application in topology.

Introduction

Gordana Todorov, Jerzy Weyman, Kent Orr and I are working on a book about “pictures” which have gained renewed attention because they are equivalent to “scattering diagrams”. This talk is about very old work that I did to introduce pictures and their relation to the cohomology of $GL(n, \mathbb{Z})$. In particular, I will discuss the following pictures (from our book!) and their relationship to $H^3(GL(n, \mathbb{Z}), \mathbb{Z}/2)$.

The algebraic side of this story is a fun topic. The cohomology class which detects the “exotic element” of $K_3 \mathbb{Z}$ is the degree 3 class which counts the number of times (modulo 2) that commutativity of addition is used to prove that matrix multiplication is associative! This is an old result (about the obstruction to right distributivity in left near-rings) which I am happy to present in a new light.
Although I want to avoid the topology, a little bit of history is in order. The focus is the following exact sequence for any group $\pi$.

$$\pi_3A(B\pi) \xrightarrow{f} K_3\mathbb{Z}\pi \xrightarrow{\chi} H_1(\mathbb{Z}\pi, \mathbb{Z}_2\pi) \xrightarrow{g} \pi_2A(B\pi) \xrightarrow{h} K_2\mathbb{Z}\pi \to 0$$

(1) $g, h$ were defined by A. Hatcher and J. Wagoner [3].
(2) I defined $f$ in my PhD thesis [5].
(3) I showed that elements of $K_3$ of any ring can be represented by 2-dimensional “pictures” [4].
(4) $H_1(\mathbb{Z}\pi, \mathbb{Z}_2\pi)$ is $\mathbb{Z}_2\pi$ (group ring of $\pi$ over the field $\mathbb{F}_2 = \mathbb{Z}_2$ with two elements) modulo the conjugation action of $\pi$. Thus

$$H_1(\mathbb{Z}\pi, \mathbb{Z}_2\pi) = \mathbb{Z}_2\pi$$

when $\pi$ is abelian.
(5) This talk is about the mapping $\chi_\pi$.

For the trivial group $\pi = 1$ we have:

$$\pi_3A(\ast) \to K_3\mathbb{Z} \xrightarrow{\chi_1} \mathbb{Z}_2 \to \pi_2A(\ast) \to K_2\mathbb{Z} \to 0$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_{24} \to \mathbb{Z}_{48} \xrightarrow{\chi_1} \mathbb{Z}_2 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$$

- Milnor [9]: $K_2\mathbb{Z} = \mathbb{Z}_2$.
- Lee-Szczarba [8]: $K_3\mathbb{Z} = \mathbb{Z}_{48}$.
- Waldhausen-Cerf (combining results of [11], [1]): $\pi_2A(\ast) = \mathbb{Z}_2$.

Thus, we already know that $\chi_1 : K_3\mathbb{Z} \to \mathbb{Z}_2$ is surjective. In [4] I called this map the “Grassmann invariant”.

This talk has three parts:

(1) $K_3R$ is the set of deformation classes of “pictures”.
(2) $\chi_\pi$ is the obstruction to matrix associativity for “near-rings”.
(3) Morse theory on circle bundles gives picture for $K_3\mathbb{Z}$ having $\chi_1 \neq 0$ (from [7]).

1. Pictures and $H_3(G)$

**Definition 1.1.** Let $G = \langle \mathcal{X} \mid \mathcal{Y} \rangle$ be a group with generating set $\mathcal{X} = \{x, y, z, \cdots \}$ and relation set $\mathcal{Y} \subset F$ where $F$ is the free group generated by $\mathcal{X}$. Then a **picture** for $G$ is an oriented planar graph $L$ with

(1) Edges labeled with elements of $\mathcal{X}$.
(2) Vertices labeled with elements of $\mathcal{Y} \amalg \mathcal{Y}^{-1}$.
(3) Base point angle at each vertex.

so that the labels of the edges coming to a vertex, read counterclockwise starting at the basepoint angle is the relation at the vertex.

**Example:**
Theorem 1.2. Elements of $H_3(G)$ are given by “deformation classes” of pictures for $G$.

Example 1.3. For $G = \mathbb{Z}_3 = \langle x \mid x^3 \rangle$, the generator of $H_3(\mathbb{Z}_3) = \mathbb{Z}_3$ is:

Definition 1.4. The Steinberg group of the group ring $\mathbb{Z}\pi$ has generators $x_{ij}^u$ where $1 \leq i \neq j \leq n$ and $u \in \pi$ (with inverse written $x_{ij}^{-u}$) modulo the relations:

1. $[x_{ij}^u, x_{jk}^v] = x_{ik}^{uv}$ in the word order:
   $$x_{ij}^u x_{jk}^v x_{ij}^{-u} x_{ik}^{uv} x_{jk}^{-v}$$

2. $[x_{ij}^u, x_{kl}^v] = 1$ when $j \neq k$ and $i \neq \ell$.

Example 1.5. An example of a picture for the Steinberg group. All arcs oriented clockwise.
Theorem 1.6 (Gersten). \( H_3(\text{St}_\infty(R)) = K_3R \).

Theorem 1.7 (I-Klein). The following picture represents the generator of \( K_3\mathbb{Z} = \mathbb{Z}_{48} \).

Proof. Reducing mod 2, this picture and Example 1.3 map to the same element of \( K_3\mathbb{Z}_2 \) which is \( \mathbb{Z}_3 \) by [10]. (The homomorphism \( \mathbb{Z}_3 \to \text{St}(\mathbb{Z}_2) \) is given by sending \( x \) to \( x_{31}x_{13} \).) So, the 3-torsion part of this element is nonzero.

The near-ring associator \( \chi_1 : K_3\mathbb{Z} \to \mathbb{Z}_2 \) is nontrivial on this element. (It is the sum over values at the vertices. Only the blue circled vertex contributes 1.) Therefore, this picture also maps to the generator of the 2-torsion part of \( \mathbb{Z}_{48} \).

\[ \square \]
2. LEFT NEAR-RINGS

Definition 2.1. A left-near-ring (with unity) is structure \((R, +, \times, 0, 1)\) so that

1. \((R, +, 0)\) is a group.
2. \((R, \times, 1)\) is a monoid.
3. (left distributive) \(a(b + c) = ab + ac\) \((\Rightarrow x0 = 0)\).
4. \(0x = 0\).

In other words, \(R\) satisfies all the axioms of a ring with unity except commutativity of addition and right distributivity. In a left near-semi-ring we also drop the axiom of additive inverses.

Example: Let \(M = (M, \times, 1)\) be a monoid and let \(G(M)\) be the free group generated by \(M\). The group law will be written as addition. Thus, the inverse of \(x \in M\) is written \(-x\). \(G(M)\) is a left near-ring with multiplication defined as follows.

1. For \(a, b \in \pm M\), \(ab := (\text{sgn } a)(\text{sgn } b)|a||b|\). Elements of \(\pm M\) will be called monomials.
2. If \(a = \sum a_j, b = \sum b_k\) then the product \(ab\) is the sum of the monomials \(a_jb_k\) in lexicographic order according to the pair \((k, (\text{sgn } b)_k)\). For example,

\[
(a_1 + a_2)(b - c) = (a_1 + a_2)b + (a_1 + a_2)(-c) = a_1b + a_2b - a_2c - a_1c
\]

We use the notation:

\[ab = \sum_k \sum_j a_j b_k\]

2.1. Right associativity. In general \((a_1 + a_2)b \neq a_1b + a_2b\). These are the sums of the terms \(a_i j b_k\) in a different order:

\[
(a_1 + a_2)b = \sum_k \sum_i \sum_j a_i j b_k
\]

The (reduced) obstruction to right distributivity is the symmetric tensor:

\[
\overline{\rho} : G(M) \times G(M) \times G(M) \to S^2 \mathbb{Z}_2 M = \mathbb{Z}_2 M \otimes \mathbb{Z}_2 M
\]

\[
\overline{\rho}(a_1, a_2; b) = \sum a_i j b_k \otimes a'_i j' b_k'
\]

where the sum is over all pairs of terms which switch order from (2.1) to (2.2).

Now compose with the intersection pairing \(\varepsilon = \langle \cdot, \cdot \rangle : \mathbb{Z}_2 M \otimes \mathbb{Z}_2 M \to \mathbb{Z}_2 M\) (Consider \(\mathbb{Z}_2 M\) as the set of all finite subsets of \(M\) and take intersection.) Then:

\[
\varepsilon \overline{\rho}(a_1, a_2; b) = \sum \langle a_i j b_k, a'_i j' b_k' \rangle \in \mathbb{Z}_2 M
\]

is the set of elements of \(M\) which are commuted with themselves an odd number of times.
2.2. Matrix multiplication. Let $M_n(G(M))$ be the set of $n \times n$ matrices with entries in $G(M)$. If $A, B, C \in M_n(G(M))$ we compare $A(BC)$ and $(AB)C$. Then entries of $A(BC)$ are:

$$A(BC)_{pq} = \sum_k \sum_i A_{pi} B_{ik} C_{kq}$$

For $(AB)C$ we have:

$$(AB)C_{pq} = \sum_k (A_{p1} B_{1k} + A_{p2} B_{2k} + \cdots + A_{pn} B_{nk}) B_{kq}$$

To get from one to the other, we need to distribute the product, then permute the terms. We get a (reduced) associator

$$\alpha(A, B, C)_{pq} = \sum_{i<j} \rho(A_{pi} B_{ik}, A_{pj} B_{jk}; C_{kq}) + \sum A_{pi} B_{ik} C_{kq} \tilde{\otimes} A_{pi'} B_{i'k'} C_{k'q}$$

Take trace followed by the intersection pairing to get

$$\varepsilon \circ Tr \circ \alpha(A, B, C) \in \mathbb{Z}_2 M$$

**Theorem 2.2** (I: 1982). This is a 3-cocycle representing a class in $H^3(\text{GL}_n(\mathbb{Z}M); \mathbb{Z}_2 M)$ which restricts to $\chi_M \in H^3(\text{St}_n(\mathbb{Z}M); \mathbb{Z}_2 M)$ coming from algebraic $K$-theory.

**Conjecture 2.3** (Hatcher). For $M = \pi$ any abelian group, the image of $\chi_{\pi} : K_3 \mathbb{Z} \pi \to \mathbb{Z}_2 \pi$ has order 2.

It is enough to prove this for the Klein 4-group $\pi = \mathbb{Z}_2 \times \mathbb{Z}_2$. This significance of this statement is that the cokernel of the map $\chi_{\pi} : K_3 \mathbb{Z} \pi \to \mathbb{Z}_2 \pi$ is a subgroup of $\pi_0 \text{Diff}(M \times I \text{ rel } M \times 0)$ for $\pi_1 M = \pi$.

3. Calculation

The formula for $\chi_{\pi}$ is given as follows.

(1) Label each region of the picture $L$ with an element $r \in \text{GL}_n(\mathbb{Z} \pi)$ starting with the unique unbounded region which we label with the identity matrix $I_n$ and, when we pass a wall from left to right, change the label by right multiplication by the label of the wall:

$$r \overset{u}{\uparrow} x_{ij}^u r x_{ij}^u$$

This is well-defined since the labels around each vertex multiply to $I_n$.

(2) $\chi_{\pi}(L)$ is the sum over all vertices carrying commutators of the form $[x_{ij}^u, x_{ik}^v]$ of the following element of $\mathbb{Z}_2 \pi$:

$$\sum_p r_{pi} (u s_{jp}, v s_{kp})$$

where $s = (s_{qp})$ is the inverse of $r = (r_{pq})$ and $r$ is the label of any one of the regions adjacent to the vertex.
In the example on page 4, the blue circle has $x_{12}$ coming into the lower right. However, it should be coming into the upper left and, therefore, when drawn properly, $x_{12}$ should cross $x_{13}$. So, $(i, j, k) = (1, 2, 3)$. The matrices $r, s$ are

$$
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
$$

The only value of $p$ for which $s_{2p}, s_{3p}$ are both nonzero is $p = 3$ and

$$r_{p} \langle s_{2p}, s_{3p} \rangle = 1$$

So, this vertex contributes 1 to the value of $\chi_{1}(L)$. Each of the other vertices contribute 0. So, $\chi_{1}(L) = 1$.

In Example 1.5 there are three intersections which contribute a total of 0. The introduction had another of the five pictures on which (3.1) needs to be zero in order to define a mapping

$$\chi_{\pi} : K_{3}Z_{\pi} \rightarrow Z_{2\pi}$$

REFERENCES


DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM, MA 02454
E-mail address: igusa@brandeis.edu