ENUMERATING $m$-CLUSTERS USING EXCEPTIONAL SEQUENCES

KIYOSHI IGUSA

Abstract. The number of clusters of Dynkin type has been computed case-by-case and is given by a simple product formula which has been extended to $m$-clusters. The $m$-cluster formula is easier to study since it is a polynomial in $m$ which factors as a product of linear factors. The standard case is $m = 1$. This paper is one step towards a uniform representation theoretic approach for this formula. The main result of this paper is that there is a bijection between ordered $m$-clusters and (complete) $m$-exceptional sequences, a concept that we introduce for this purpose. This holds for all hereditary artin algebras. This extends the bijection in the $m = 1$ case shown in [9]. We also show that, for type $A_n$, the linear factors of this polynomial count independent events in a random complete exceptional sequence, namely whether or not the $k$-th term in an exceptional sequence is relatively projective. The first step of the probability argument comes from [16].

For any Dynkin quiver $Q$, the number of clusters in the cluster category $[3]$ of any hereditary algebra $\Lambda$ with valued quiver $Q$ is given by the formula

$$\prod_{i=1}^{n} \frac{h + d_i}{d_i}$$

where $h$ is the Coxeter number of $Q$ and $d_i = 2, \cdots, h$ are the degrees of the principal $W$-invariant polynomials where $W$ is the Weyl group of $Q$. This has been proven case-by-case and one of the big mysteries of the subject is to explain this product formula. The formula has been generalized to $m$-clusters by Fomin and Reading [8]:

$$p_{\Lambda}(m) = \prod_{i=1}^{n} \frac{hm + d_i}{d_i}.$$ 

This generalization is easier to work with since it is a polynomial in $m$. This paper gives a new approach to this formula using exceptional sequences. Namely, we show that the set of ordered $m$-clusters is in 1-1 correspondence with the set of $m$-exceptional sequences (Definition 1.2.1). This can be viewed as a refinement of the observation that the leading coefficient of $n!p_{\Lambda}(m)$ is the number of exceptional sequences:

$$e_\Lambda = \frac{n! h^n}{\prod d_i}.$$ 

This uniform formula for $e_\Lambda$ was first observed by Chapoton [6] and was the motivation for the paper [16] which is one of the starting points of this paper. There is still no uniform proof of either of these formulas.

1. Basic definitions

We recall the definitions and basic properties of exceptional sequences from [7], [17]. Over an hereditary algebra $\Lambda$, a module $M$ is called exceptional if $M$ is rigid and Schurian where rigid
means $\text{Ext}_1(M, M) = 0$ and Schurian means $\text{End}_\Lambda(M)$ is a division algebra. An exceptional sequence of length $\ell$ for $\Lambda$ is a sequence of exceptional modules $(E_1, E_2, \cdots, E_\ell)$ so that
\[
\text{Hom}_\Lambda(E_j, E_i) = \text{Ext}_1^\Lambda(E_j, E_i) = 0 \text{ for } i < j \leq \ell.
\]
The exceptional sequence is called complete if it is of maximal length and this is well known to occur when $\ell = n$, the number of nonisomorphic simple $\Lambda$-modules. We consider exceptional sequences up to isomorphism where two sequences $(E_i)$, $(E'_i)$ are isomorphic if they have the same length and $E_i \cong E'_i$ for all $i$.

1.1. Relative projective terms. In each exceptional sequence we will define certain terms to be “relatively projective.”

For any $\Lambda$-module $M$ the (right) perpendicular category $M^\perp$ is defined to be the full subcategory of $\text{mod-}\Lambda$ of all objects $X$ so that $\text{Hom}_\Lambda(M, X) = 0 = \text{Ext}_1^\Lambda(M, X)$. The left perpendicular category $\perp M$ is defined analogously. From the six-term exact sequence for $\text{Hom}_\Lambda(M, -)$ it follows that $M^\perp$ and $\perp M$ are closed under extensions, kernels of monomorphisms and cokernels of epimorphisms. Any full subcategory of $\text{mod-}\Lambda$ having these properties is called a wide subcategory. It is well-known that the perpendicular categories $M^\perp$, $\perp M$ are finitely generated, i.e., contain a generator. Conversely, if a wide subcategory $W$ of $\text{mod-}\Lambda$ has a generator $X$ then $\perp X$ has a generator, say $Y$, and $W = Y^\perp$. Therefore, perpendicular categories are the same as finitely generated wide subcategories of $\text{mod-}\Lambda$.

A term $E_j$ in an exceptional sequence $(E_1, \cdots, E_\ell)$ will be called relatively projective if it is a projective object of the perpendicular category of the later terms, $(E_{j+1} \oplus \cdots \oplus E_\ell)^\perp$. In a complete exceptional sequence, the first term $E_1$ is always relatively projective since the perpendicular category $(E_2 \oplus \cdots \oplus E_n)^\perp$ is semi-simple. If $\Lambda$ has finite representation type then we define $f_\Lambda(x)$ to be the degree $n$ integer polynomial given by
\[
f_\Lambda(x) := \sum_{k=1}^n e_k x^k
\]
where $e_k$ is the number of complete exceptional sequences in $\text{mod-}\Lambda$ with $k$ relatively projective terms. This polynomial is suitable for computing the number of $m$-clusters in the $m$-cluster category of $\Lambda$ defined below. This polynomial also gives the number of complete exceptional sequences which, following [16], we denote by $e_A = f_\Lambda(1) = \sum e_k$.

1.2. $m$-exceptional sequences. Recall that, since $\Lambda$ is hereditary, the indecomposable objects of the bounded derived category of $\text{mod-}\Lambda$ have the form $M[k]$ where $M$ is an indecomposable $\Lambda$-module and $k \in \mathbb{Z}$. We say that $M[k]$ is exceptional if $M$ is an exceptional $\Lambda$-module. The following definition extends an idea from [9].

**Definition 1.2.1.** For any $m \geq 0$ we define an $m$-exceptional sequence of length $\ell$ for $\Lambda$ to be a sequence of exceptional objects $(E_1, \cdots, E_\ell)$ in $\mathcal{D}^b(\text{mod-}\Lambda)$ so that

1. for all $j$, $E_j = M_j[d_j]$ where $0 \leq d_j \leq m$,
2. $(M_1, \cdots, M_\ell)$ is an exceptional sequence in $\text{mod-}\Lambda$ and
3. when $d_j = m$, the module $M_j$ is relatively projective in the perpendicular category $(M_{j+1} \oplus \cdots \oplus M_\ell)^\perp$.

An $m$-exceptional sequence will be called complete if it has the maximum length $\ell = n$. For $m = 1$, an $m$-exceptional sequence is called a signed exceptional sequence [9].

It is clear from this definition that, to every exceptional sequence $(E_1, \cdots, E_\ell)$ with $k$ relatively projective terms, there correspond $(m+1)^k m^{\ell-k}$ $m$-exceptional sequences since there
are $m + 1$ possibilities for $d_j$ when $M_j$ is relatively projective and only $m$ possibilities otherwise. Therefore, for $\Lambda$ of finite type, the number of complete $m$-exceptional sequences is given by

$$g_\Lambda(m) := \sum_{k=1}^{n} e_k (m+1)^k m^{n-k} = m^n f_\Lambda \left( \frac{m+1}{m} \right).$$

We also consider the $m$-cluster category of $\Lambda$ [9]. This is defined to be the orbit category of the bounded derived category of $\text{mod-}\Lambda$ modulo the functor $\tau^{-1}[m]$:

$$C^m(\Lambda) = Db(\text{mod-}\Lambda) / \tau^{-1}[m]$$

We identify objects of $C^m(\Lambda)$ with their representative in $Db(\text{mod-}\Lambda)$ in the fundamental domain of $\tau^{-1}$ which we take to be

$$(1.1) \quad \text{mod-}\Lambda \cup \text{mod-}\Lambda[1] \cup \cdots \cup \text{mod-}\Lambda[m-1] \cup \Lambda[m]$$

Thus the indecomposable objects of $C^m(\Lambda)$ are represented by $M[j]$ for some indecomposable $\Lambda$-module $M$ where $0 \leq j \leq m$ with the additional condition that $M$ is projective when $j = m$. Two object $X[j], Y[k]$ in $C^m(\Lambda)$ are compatible if $\text{Ext}^i_{Db}(X[j], Y[k]) = \text{Ext}^i_{Db}(Y[k], X[j]) = 0$ for all $i \geq 1$ and they are not isomorphic. Equivalently, either

1. $j < k$ and $\text{Hom}_\Lambda(Y, X) = 0 = \text{Ext}^1_\Lambda(Y, X)$,
2. $j = k$ and $\text{Ext}^1_\Lambda(X, Y) = 0 = \text{Ext}^1_\Lambda(Y, X)$ and $X \ncong Y$ or
3. $j > k$ and $\text{Hom}_\Lambda(X, Y) = 0 = \text{Ext}^1_\Lambda(X, Y)$.

A maximal pairwise compatible set of exceptional objects of $C^m(\Lambda)$ has $n$ objects and is called an $m$-cluster for $\Lambda$.

The first part of the main theorem of this paper, proved in Section 3 below, is the following which generalizes the $m = 2$ case proved in the preliminary version of [9]. This new proof supersedes the one in [9].

**Theorem 1.2.2.** Let $\Lambda$ be a finite dimensional hereditary algebra over any field and $k, m \geq 0$. Then there is a 1-1 correspondence between (isomorphism classes of) ordered $k$-tuples of pairwise compatible objects in the $m$-cluster category of $\Lambda$ and (isomorphism classes of) $m$-exceptional sequences of length $k$ for $\Lambda$.

For example, when $m = 0$, there is only one 0-cluster of size $n$, namely the set of projective modules. So, there are $n!$ complete 0-exceptional sequences. These are the same as complete exceptional sequences in which every object is relatively projective. It follows that the number of maximal compatible subsets of $C^m(\Lambda)$, the $m$-clusters, is equal to

$$\frac{1}{n!} g_\Lambda(m) = \frac{m^n}{n!} f_\Lambda \left( \frac{m+1}{m} \right).$$

The case of greatest interest is when $m = 1$. The number of cluster for the algebra $\Lambda$ is

$$\frac{1}{n!} f_\Lambda (2)$$

where $f_\Lambda (2)$ is the number of signed exceptional sequences.

Some of the facts that we can see immediately from this formula are:

1. $g_\Lambda(-1) = f_\Lambda(0) = 0$ (since $E_1$ is always relatively projective)
2. $g_\Lambda(0) = e_n = n!$ (the number of ordered $m$-clusters for $m = 0$).

**Proposition 1.2.3.** All real roots of the polynomial $g_\Lambda(x)$ are between 0 and $-1$ including $-1$ and excluding 0.
Proof. If $m \geq 0$ then $g_\Lambda(m) > 0$ since all terms are nonnegative: $e_k(m + 1)^k m^{n-k} \geq 0$ and the $k = n$ term is $n!(m + 1)^n \geq n! > 0$. If $m < -1$ then $(-1)^n g_\Lambda(m) > 0$ since all terms are nonnegative and the $k = n$ term is positive. \qed

The second part of the main theorem, also proved in Section 3 is the following.

**Proposition 1.2.4.** The 1-1 correspondences of Theorem 1.2.2 can be chosen to be compatible with deletion of the first term in the sense that, if $(T_1, \cdots , T_k)$ corresponds to $(E_1, \cdots , E_k)$, then $(T_2, \cdots , T_k)$ corresponds to $(E_2, \cdots , E_k)$.

1.3. **Recursive formula.** We will show that the recursive formula for the number of exceptional sequences given in [16] can be modified to give a recursive formula for $f_\Lambda(x)$.

We review briefly the argument from [16] which counts the number of exceptional sequences $e_\Lambda = \sum e_k = f_\Lambda(1)$

Let $\Lambda$ be a hereditary algebra of finite type with connected valued quiver $Q$. The number of vertices of $Q$, usually denoted $n$, is the rank of $\Lambda$. For each vertex $i$ of $Q$, let $Q(i)$ be the valued quiver obtained from $Q$ by deleting the vertex $i$. Then $Q(i)$ has 1, 2 or 3 components. To count the number of exceptional sequences for $Q(i)$ we need the following lemma.

**Lemma 1.3.1.** [16] Let $A, B$ be hereditary algebras of finite type with rank $n_A, n_B$. Then there is a 1-1 correspondence between complete exceptional sequences for $A \times B$ and pairs of complete exceptional sequences for $A, B$ together with a shuffling of the two sequences. Thus

$$e_{A \times B} = \left(\frac{n_A + n_B}{n_A}\right) e_A e_B.$$

Since a term in an exceptional sequence for $A \times B$ is relatively projective if and only if it is relatively projective in $A$ or $B$ whichever it comes from, we get the following.

$$f_{A \times B}(x) = \left(\frac{n_A + n_B}{n_A}\right) f_A(x) f_B(x) \quad (1.2)$$

The recursive formula for $e_\Lambda$ is:

**Proposition 1.3.2.** [16] For $\Lambda$ of rank $n$ and Coxeter number $h$ we have:

$$e_\Lambda = \sum_{i=1}^{n} \frac{h}{2} e_{\Lambda(i)}$$

**Proof.** (Summarized from [16]) If $P_i$ and $I_j$ are in the same orbit of the Auslander-Reiten translation $\tau$ then the sum of the lengths of the $\tau$ orbits of $P_i$ and $P_j$ is $h$. Furthermore, for any indecomposable module $M$ in the union of these $\tau$ orbits, $M^\perp, P_i^\perp$ and $P_j^\perp$ are all isomorphic to $\Lambda(i)$. Since this counts each indecomposable module twice and since every complete exceptional sequence for $\Lambda$ is given by an indecomposable module $M$ preceded by a complete exceptional sequence for $M^\perp$, we get:

$$2e_\Lambda = 2 \sum_M e_{M^\perp} = h \sum_i e_{\Lambda(i)}.$$

The formula in the Proposition follows. \qed

In the above proof, in the union of the $\tau$ orbits of $P_i$ and $P_j$, the proportion of objects which are projective is exactly $\frac{2}{h}$. This gives the following.
Corollary 1.3.3. Let $\Lambda,n,h$ be as above. Then

$$f_\Lambda(x) = \left( x + \frac{h}{2} - 1 \right) \sum_{i=1}^{n} f_{\Lambda(i)}(x)$$

or, equivalently,

$$g_\Lambda(m) = \frac{hm + 2}{2} \sum_{i=1}^{n} g_{\Lambda(i)}(m).$$

2. Exceptional sequences of type $A_n$

In this section we go over the proof that the number of exceptional sequences of type $A_n$ is $(n+1)^n - 1$ while keeping track of which terms are relatively projective. Then we will see that, in a randomly chosen complete exceptional sequence $(E_n, \cdots, E_1)$, the probability that $E_k$ is relatively projective in $(E_{k-1} \oplus \cdots \oplus E_1)^{\perp}$ is equal to $\frac{k+1}{n+1}$. Furthermore, these events, for different values of $k$ are independent. This recovers the formula

$$f_{A_n}(x) = (n+1)^n - 1 \prod_{k=1}^{n} \left( (k+1)x + n - k \right)$$

for the generating function $f_{A_n}(x)$ for the number of complete exceptional sequences with any given number of relatively projective terms. The virtue of this formula is that it gives a conceptual explanation for the linear terms in the number of $m$-clusters of type $A_n$.

2.1. Subgraphs of an $h$-cycle. We begin by counting the number of subgraphs of the $h$-cycle graph $C_h$ consisting of $h = n + 1$ vertices and $h$ edges arranged in one cycle. For any $\ell \geq 0$, let $L_\ell$ denote the linear graph with $\ell$ edges and $\ell + 1$ vertices. We say that $L_\ell$ has length $\ell$. Then, $L_n$ is equal to $C_h$ minus one edge.

Consider what happens when we delete $k+1$ edges from $C_h$ where $0 \leq k \leq n$. It is clear that we will end up with $k+1$ components $L_{\lambda_i}$ with lengths adding up to $\sum \lambda_i = n - k$.

Proposition 2.1.1. For any partition $\lambda = (\lambda_0 \leq \cdots \leq \lambda_k)$ of $n - k$ into $k+1$ possibly empty parts, let $S_h(\lambda)$ be the number of subgraphs of $C_h$ isomorphic to $\bigsqcup L_{\lambda_i}$. Then

$$S_h(\lambda) = \frac{k!h}{\prod_{p} n_p!}$$

where $n_p$ is the number of parts $\lambda_i$ of size $p \geq 0$.

Proof. Consider the set of all pairs $(G,j)$ where $G$ is a subgraph of $C_h$ isomorphic to $\bigsqcup L_{\lambda_i}$ and $j$ is one of the $k+1$ edges of $C_h$ which are not in $G$. The size of this set is, evidently, $(k+1)S_h(\lambda)$. However, if we choose $j$ first, the number of possible $G$ is given by a multinomial by a standard counting argument and we get

$$(k+1)S_h(\lambda) = h\left( \begin{array}{c} k+1 \\ n_0,n_1,\cdots,n_d \end{array} \right)$$

where $d$ is the maximum value of $\lambda_i$. The formula in the Proposition follows. \qed

The numbers $S_h(\lambda)$ will be used to count exceptional sequences of length $k$ for $A_n$. 

2.2. Counting exceptional sequences. Let Λ be a hereditary algebra of type $A_n$. For any partition $\lambda = (\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_k)$ of $n - k$ into $k + 1$ possibly empty parts, let $N_h(\lambda)$ be the number of exceptional sequences $E_\ast = (E_k, \cdots, E_1)$ of length $k$ for $\Lambda$ whose right perpendicular category $(E_k \oplus \cdots \oplus E_1)^\perp$ is of type

$$A_\lambda := \prod_{\lambda_i > 0} A_{\lambda_i}.$$ 

Then we will prove the following.

**Theorem 2.2.1.**

$$N_h(\lambda) = h^{k-1} S_h(\lambda) = \frac{k! h^k}{\prod n_p!}$$

where $n_p$ is the number of parts $\lambda_i$ equal to $p$.

For a fixed $k$, the sum of all $S_h(\lambda)$ is clearly $\binom{h}{k+1}$. So, we get the immediate corollary:

**Corollary 2.2.2.** The number of length $k$ exceptional sequences of type $A_n$ is $\binom{h}{k+1}h^{k-1}$.

**Theorem 2.2.1** is proved by induction on $k$ using the following lemma.

**Lemma 2.2.3.** Given an exceptional sequence for $A_n$ of length $k-1$ with perpendicular category of type $A_{\lambda'}$, $\lambda' = (\lambda'_1, \cdots, \lambda'_{k'})$, the number of ways to add one more term (on the left end) to get an exceptional sequence of length $k$ with perpendicular category of type $A_{\lambda}$ is equal to

$$n'_{\lambda'}(c + 1)X(a, b)$$

when $\lambda = (a, b, \lambda'_1, \cdots, \lambda'_{k'})$, $\lambda'_p = c = a + b + 1$ and $0 \leq a \leq b$. Here

$$X(a, b) := \begin{cases} \frac{1}{2} & \text{if } a = b \\ 1 & \text{otherwise} \end{cases}$$

**Proof.** The added term $E_k$ in the exceptional sequence belongs to some $A_c$ and, on that sub-quiver, $E_k^\perp = A_a \times A_b$ where $a + b + 1 = c$ and $a, b \geq 0$. There are $n'_{\lambda'}$ components of the perpendicular category $\prod A_{\lambda'_i}$ of type $A_c$. Each such component has $(c + 1)X(a, b)$ objects $E$ so that $E^\perp = A_a \times A_b$. Since these are all the indecomposable objects of the perpendicular category up to isomorphism, the formula holds. \hfill \Box

**Proof of Theorem 2.2.1** The theorem states that $N_h(\lambda) = S_h(\lambda)h^{k-1}$. This is true for $k = 0$ since both sides are 1 in that case. So, suppose that $k \geq 1$. By the lemma we have:

$$N_h(\lambda_0, \cdots, \lambda_k) = \sum_{\lambda'} N_h(\lambda')n'_{\lambda'}(c + 1)X(a, b)$$

where the sum is over all $\lambda'$ obtained from $\lambda$ by “fusing” two components $a = \lambda_i$ and $b = \lambda_j$ together to form $c = a + b + 1$. However, the formula in Lemma 2.2.3 shows that $\lambda_i, \lambda_j$ should be the first occurrence of $a, b$ or the first two occurrences of $a = b$. So, we need to divide by a redundancy factor $M(a, b)$ making the correct sum equal to:

$$N_h(\lambda) = \sum_{0 \leq i < j \leq k} N_h(\lambda_i + \lambda_j + 1, \lambda_0, \cdots, \lambda_i, \cdots, \lambda_j, \cdots, \lambda_k) \frac{n'_{\lambda_i + \lambda_j + 2}X(\lambda_i, \lambda_j)}{M(\lambda_i, \lambda_j)}$$

where $c = \lambda_i + \lambda_j + 1$ and $M(\lambda_i, \lambda_j)$ is the number of times this same term appear in the sum over all $i < j$. Thus

$$M(\lambda_i, \lambda_j) = \begin{cases} \frac{1}{2} n_{\lambda_i}(n_{\lambda_i} - 1) & \text{if } \lambda_i = \lambda_j \\ n_{\lambda_i}n_{\lambda_j} & \text{otherwise} \end{cases}$$
Dividing by $X(\lambda_i, \lambda_j)$ would eliminate the $\frac{1}{2}$ in the first case. By induction on $k$ we get:

$$N_h(\lambda) = \sum_{0 \leq i < j \leq n} h^{k-1} \frac{(k-1)! n'_c(\lambda_i + \lambda_j + 2) X(\lambda_i, \lambda_j)}{\prod n'_p! M(\lambda_i, \lambda_j)}$$

$$= h^{k-1} \frac{S_h(\lambda)}{hk} \sum_{i < j} (\lambda_i + \lambda_j + 2) \left( \prod_{p \neq c} n'_p! M(\lambda_i, \lambda_j) \right)$$

$$= h^{k-1} \frac{S_h(\lambda)}{hk} \sum_{i < j} (\lambda_i + \lambda_j + 2)$$

(2.1)

since the fraction in parenthesis is equal to 1. Finally,

$$\sum_{i < j} (\lambda_i + \lambda_j + 2) = \sum_{i \neq j} \frac{\lambda_i + \lambda_j + 2}{2} = (k+1) \sum (\lambda_i + 1) - \sum (\lambda_i + 1) = k(h-k-1+k+1) = kh.$$  

So, $N_h(\lambda) = S_h(\lambda) h^{k-1}$ as claimed. □

2.3. Probability distribution of relatively projective terms. The following corollary and its proof is a generalization of the $k = 1$ case from [16]. It will be convenient to index the exceptional sequence in descending order.

**Corollary 2.3.1.** For a random complete exceptional sequence $(E_n, E_{n-1}, \ldots, E_1)$ of type $A_n$, the probability that $E_k$ is relatively projective in $(E_{k-1} \oplus \cdots \oplus E_1)\perp$ is equal to $\frac{k+1}{h}$. Furthermore, the probability that this occurs for $k_1, k_2, \ldots, k_s$ is equal to the product $\prod \frac{k_i+1}{h}$, i.e., the probabilities for different $k$ are independent.

**Proof.** Let $B_k$ be the event that $E_k$ is relatively projective in $(E_{k-1} \oplus \cdots \oplus E_1)\perp$. Then it suffices to show the following where $P(B)$ denotes the probability of an event $B$.

1. $P(B_k) = \frac{k+1}{h}$
2. $P(B_k)$ and $(E_k \oplus \cdots \oplus E_1)\perp$ has type $A_\lambda$ is $\frac{k+1}{h} P((E_k \oplus \cdots \oplus E_1)\perp$ has type $A_\lambda$)

Proof of sufficiency: (1) is the first statement of the corollary. Since the probability distribution of relatively projective objects in $E_n, \ldots, E_{k+1}$ depends only on the type $A_\lambda$ of $(E_k \oplus \cdots \oplus E_1)\perp$, (2) implies that $B_k$ is independent of $B_{j_1} B_{j_2} \cdots B_{j_s}$ for any $j_1 > j_2 > \cdots > k$. So,

$$P(B_{j_1} B_{j_2} \cdots B_{j_s} B_k) = P(B_{j_1} B_{j_2} \cdots B_{j_s}) P(B_k) = P(B_{j_1}) P(B_{j_2}) \cdots P(B_k)$$

by induction on $n$. So, (2) implies the independence of the events $B_k$.

By summing over all $\lambda$ we see that (2) implies (1). Thus it suffices to prove (2).

To prove (2) we use the summation (2.1) in the proof of Theorem 2.2.1. We claim that, for each $i, j$, the proportion of the terms counted by the $ij$ summand having the property that $E_k$, the added term in the exceptional sequence, is relatively projective in $(E_{k-1} \oplus \cdots \oplus E_1)\perp$ is equal to $\frac{2}{c+1} = \frac{2}{\lambda_i + \lambda_j + 2}$. This is because $E_k$ is being chosen in a double $\tau$-orbit in some $A_c$, and the proportion of projective objects in each double $\tau$-orbit is $\frac{2}{c+1}$.

Thus, the total number of terms in which $E_k$ is relatively projective is equal to

$$\frac{h^{k-1} S_h(\lambda)}{hk} \sum_{i < j} 2 = N_h(\lambda) \frac{k+1}{h}$$

which implies statement (2). The corollary follows as explained above. □

We obtain the following pair of equivalent formulas the second of which is known [8].
Corollary 2.3.2. The number of \(m\)-exceptional sequences of length \(k\) for \(A_n\) is equal to

\[
\frac{1}{h} \binom{h}{k+1} \prod_{d=2}^{k+1} (mh + d)
\]

Equivalently, the number of \(k\) element sets of pairwise compatible elements of the \(m\)-cluster category for \(A_n\) is given by the generalized Narayana number:

\[
\frac{1}{(k+1)!} \binom{n}{k} \prod_{d=2}^{k+1} (mh + d)
\]

where \(h = n + 1\).

Proof. Corollaries 2.2.2 and 2.3.1 together imply that the number of \(m\)-exceptional sequences of length \(k\) for \(A_n\) is equal to

\[
\binom{h}{d} \prod_{d=2}^{k+1} \left( (m + 1) \left( \frac{d}{h} \right) + m \left( \frac{h - d}{h} \right) \right)
\]

which easily simplifies to the first expression. By the main Theorem 3.1.1 below, this is equivalent to the second formula. \(\square\)

3. Proof of main theorem

This section gives the statement and proof of the main theorem.

3.1. Statement of main theorem.

Theorem 3.1.1. There is a bijection between the set of isomorphism classes of \(m\)-exceptional sequences of length \(k\) and the set of isomorphism classes of ordered \(k\)-tuples of pairwise compatible objects in the \(m\)-cluster category for any finite dimensional hereditary algebra \(\Lambda\). Furthermore, this bijection is compatible with deletion of the first term in the sense that, if \((E_1, \cdots, E_k)\) corresponds to \((T_1, \cdots, T_k)\), then \((E_2, \cdots, E_k)\) corresponds to \((T_2, \cdots, T_k)\).

We will set up notation to give an explicit formula for this bijection. We choose one object from every isomorphism class of exceptional \(\Lambda\)-modules. Let \(E(\Lambda)\) denote the set of these objects. For set theoretic purposes we can consider this to be a subset of \(\mathbb{Z}^n\) since every exceptional module \(M\) is determined up to isomorphism by its dimension vector \(\dim M \in \mathbb{Z}^n\).

For any finitely generated wide subcategory \(W\) of \(\text{mod-}\Lambda\) (defined in section 1.1), let \(E(W) = W \cap E(\Lambda)\). Denote by \(C^m(W)\) the \(m\)-cluster category of \(W\). Objects of \(C^m(W)\) will be represented by elements of the fundamental domain of the functor \(\tau^{-1}[m]\) on the bounded derived category of \(W\), as in (1.1). Thus the exceptional objects of \(C^m(W)\) are (up to isomorphism) \(X[j]\) where \(X \in E(W)\) and \(0 \leq j \leq m\) with the additional condition that \(X\) is a relative projective object of \(W\) when \(j = m\). Let \(E^m(W)\) denote this set of objects. We call \(j\) the level of the object \(X[j]\). Let \(E^m_j(W)\) denote the set of objects of \(E^m(W)\) of level \(j\). In particular, the rank \(r \leq n\) of \(W\) is the number of elements of \(E^m(W)\). We use the abbreviation \(E^m(\Lambda) = E^m(\text{mod-}\Lambda)\). We consider the special case \(W = T^{-}\) where \(T \in E(\Lambda)\). This is a wide subcategory of rank \(n-1\). We note that \(T\) is uniquely determined by \(T^{-}\) since \(T\) is the unique exceptional object of \(\text{mod-}\Lambda\).

We use the expression exceptional pair for an exceptional sequence of length 2 in \(\text{mod-}\Lambda\). We say that two are equivalent: \((X, Y) \sim (Z, W)\) if \((X \oplus Y)^{-} = (Z \oplus W)^{-}\), i.e., if they span the same rank 2 wide subcategory \(\left((X \oplus Y)^{-}\right)\) of \(\text{mod-}\Lambda\).
3.2. The key lemma. We state the key lemma, which uses the following notation, and show how it proves the main theorem.

For any $T[k] \in \mathcal{E}^m(\Lambda)$, let $\mathcal{E}^{T[k]} \subseteq \mathcal{E}^m(\Lambda)$ denote the set of all $X[j] \in \mathcal{E}^m(\Lambda)$ which are compatible with $T[k]$. Recall that $A, B$ are compatible if $A \nRightarrow B$ and $\text{Hom}(A, B[s]) = 0 = \text{Hom}(B, A[s])$ for all $s > 0$. For $A = X[j], B = T[k]$ this is equivalent to one of the following.

1. $j < k$ and $(X, T)$ is an exceptional pair,
2. $j > k$ and $(T, X)$ is an exceptional pair or
3. $j = k$ and $T, X$ are ext-orthogonal and nonisomorphic in $\text{mod-}\Lambda$.

**Lemma 3.2.1.** For any $T[k] \in \mathcal{E}^m(\Lambda)$ there is a bijection

$$\sigma_{T[k]} : \mathcal{E}^m(T^\perp) \to \mathcal{E}^{T[k]}.$$ 

Furthermore, $A, B$ are compatible in $\mathcal{E}^m(T^\perp)$ if and only if $\sigma_{T[k]}A, \sigma_{T[k]}B$ are compatible in $\mathcal{E}^{T[k]}$.

**Proof of Theorem 3.1.1 given Lemma 3.2.1.** We show that the bijections $\sigma_{T[k]}$ give bijections $\theta_p : \{p\text{-tuples of compatible objects in } \mathcal{E}^m(\Lambda)\} \to \{m\text{-exceptional sequences of length } p \text{ for } \Lambda\}$ compatible with deletion of the first term. The bijections $\theta_p$ are defined recursively as follows. For $p = 1$ let $\theta_1$ be the identity mapping. Next, take $p \geq 2$.

Let $(T_1, \ldots, T_p)$ be an ordered $p$-tuple of compatible objects in $\mathcal{E}^m(\Lambda)$ and let $T_p = T[k]$ be the last object. Then, $T_i$ for $i < p$ are compatible elements of $\mathcal{E}^{T[k]}$. So, by the lemma, $\sigma_{T[k]}^{-1}(T_i)$ are compatible objects of $\mathcal{E}^m(T^\perp)$. So, $\sigma_{T[k]}^{-1}(T_1, \ldots, T_{p-1}) = (X_1, \ldots, X_{p-1})$ is an ordered $(p - 1)$-tuple of compatible objects in the wide subcategory $T^\perp$ of $\text{mod-}\Lambda$. Since $T^\perp$ is isomorphic to $\text{mod-}\Lambda'$ for some hereditary algebra $\Lambda'$, by induction on $p$, we have the bijection $\theta_{p-1} : \{(p - 1)\text{-tuples of compatible objects in } \mathcal{E}^m(T^\perp)\} \to \{m\text{-exc. seq. of length } p - 1 \text{ for } T^\perp\}$ which is given by $\theta_{p-2}$ when first terms are deleted. Let

$$\theta_{p-1}(X_1, \ldots, X_{p-1}) = (E_1, \ldots, E_{p-1})$$

be the corresponding $m$-exceptional sequence in $T^\perp$. Then we define $\theta_p$ by

$$\theta_p(T_1, \ldots, T_p) = (E_1, \ldots, E_{p-1}, T[k]).$$

Since $\theta_{p-2}(T_2, \ldots, T_{p-1}) = (X_2, \ldots, X_{p-1})$, it follows that $\theta_{p-1}(T_2, \ldots, T_p) = (E_2, \ldots, E_p)$ as required.

Conversely, let $(E_1, \ldots, E_p)$ be an $m$-exceptional sequence for $\Lambda$. Let $E_p = T[k]$. Then $(E_1, \ldots, E_{p-1})$ is an $m$-exceptional sequence in $T^\perp$ and

$$\theta_{p-1}^{-1}(E_1, \ldots, E_{p-1}) = (X_1, \ldots, X_{p-1})$$

is a $(p - 1)$-tuple of compatible objects in $T^\perp$. Applying $\sigma_{T[k]}$ gives $\sigma_{T[k]}(X_1, \ldots, X_{p-1}) = (T_1, \ldots, T_{p-1})$ a $(p - 1)$-tuple of compatible objects in $\mathcal{C}^m(\Lambda)$. So, $\theta_{p-1}^{-1}(E_1, \ldots, E_p) = (T_1, \ldots, T_p)$. By construction the maps $\theta_p, \theta_{p-1}$ are inverse to each other. Since the maps $\theta_p$ are compatible with deletion of the first term, so are the maps $\theta_{p-1}$. Thus the key lemma implies the main theorem. □
3.3. Outline of proof of key lemma. We will construct the bijection
\[ \sigma_{T[k]} : \mathcal{E}^m(T^\perp) \to \mathcal{E}^T[k] \]
by decomposing each set into a disjoint union of subsets and constructing bijections between corresponding subsets as shown schematically as follows.

\[ \begin{align*}
\mathcal{E}^m(T^\perp) &= \mathcal{E}_0 \cup \cdots \cup \mathcal{E}_{k-1} \cup \mathcal{B}_k \cup \mathcal{A}_{k+1} \cup \mathcal{B}_{k+1} \cup \cdots \cup \mathcal{A}_m \cup \mathcal{B}_m \\
\mathcal{E}^T[k] &= \mathcal{E}_0' \cup \cdots \cup \mathcal{E}_{k-1}' \cup \mathcal{B}_k' \cup \mathcal{A}_k' \cup \mathcal{B}_{k+1}' \cup \cdots \cup \mathcal{A}_{m-1}' \cup \mathcal{B}_m'
\end{align*} \]

Here \( \mathcal{E}_j = \mathcal{E}_j^m(T^\perp) \) and \( \mathcal{E}_j' = \mathcal{E}_j^T[k] \) denote the subsets of \( \mathcal{E}^m(T^\perp) \) and \( \mathcal{E}^T[k] \), resp., of objects of level \( j \). It follows from the definitions that, for \( j < k \), \( \mathcal{E}_j = \mathcal{E}_j' \). So, we take the identity mapping

\[ \sigma_{T[k]}|_{\mathcal{E}_j} = \text{id} : \mathcal{E}_j \overset{\sim}{\to} \mathcal{E}_j', \quad 0 \leq j < k. \]

With the exception of the bijection \( \mathcal{B}_k \to \mathcal{B}_k' \), all other bijections in Chart 3.1 are given by mutation of exceptional sequences \( (X, T) \leftrightarrow (T, Y) \). This gives a bijection \( \gamma : \mathcal{E}(T^\perp) \cong \mathcal{E}(T^\perp) \) where the formula for \( Y = \gamma(X) \) is given by examining four cases. We pick out one case of this bijection. Let \( \mathcal{A} \subseteq \mathcal{E}(T^\perp) \) and \( \mathcal{A}' \subseteq \mathcal{E}(T^\perp) \) be given by

\[ \mathcal{A} = \{ X \in \mathcal{E}(T^\perp) : \exists \text{ mono } X \hookrightarrow T^s \} \]
\[ \mathcal{A}' = \{ Y \in \mathcal{E}(T^\perp) : \exists \text{ epi } T^s \twoheadrightarrow Y \}. \]

The bijection \( \gamma : \mathcal{E}(T^\perp) \cong \mathcal{E}(T^\perp) \) sends \( \mathcal{A} \) onto \( \mathcal{A}' \) and gives a bijection \( \alpha : \mathcal{A} \cong \mathcal{A}' \). Corresponding objects \( X \in \mathcal{A} \leftrightarrow Y \in \mathcal{A}' \) are related by the canonical short exact sequence

\[ 0 \to X \overset{f}{\to} T^s \overset{g}{\to} Y \to 0 \]

where \( f, g \) are minimal left/right \( T \)-approximations of \( X, Y \) respectively. Let \( \mathcal{B} = \mathcal{E}(T^\perp) - \mathcal{A} \) and \( \mathcal{B}' = \mathcal{E}(T^\perp) - \mathcal{A}' \) be the union of the other cases. Thus, \( \gamma \) induces a bijection \( \beta : \mathcal{B} \cong \mathcal{B}' \).

For \( j > k \) we define \( \sigma_{T[k]}(X[j]) \) by

\[ \sigma_{T[k]}(X[j]) = \begin{cases} 
\alpha(X)[j-1] & \text{if } X \in \mathcal{A} \\
\beta(X)[j] & \text{otherwise}
\end{cases} \]

We will show that this gives a bijection

\[ \sigma_{T[k]}|_{\mathcal{A}_j} = \alpha_j : \mathcal{A}_j \to \mathcal{A}'_{j-1} \]

between \( \mathcal{A}_j = \{ X[j] : X \in \mathcal{A} \} \) and \( \mathcal{A}'_{j-1} = \{ Y[j-1] : Y \in \mathcal{A}' \} \) for \( k < j \leq m \). We will verify that \( \mathcal{A}_j \subseteq \mathcal{E}_j^m(T^\perp) \) for all \( k < j \leq m \) and \( \mathcal{A}'_j \subseteq \mathcal{E}_j^T[k] \) for all \( k \leq j < m \).

We will show that, for \( k < j \leq m \), Formula 3.2 also gives a bijection

\[ \sigma_{T[k]}|_{\mathcal{B}_j} = \beta_j : \mathcal{B}_j \to \mathcal{B}'_j \]

where \( \mathcal{B}_j = \mathcal{E}_j - \mathcal{A}_j \) and \( \mathcal{B}'_j = \mathcal{E}_j' - \mathcal{A}'_j \).

For \( j = k \leq m \) let \( \mathcal{B}_k = \mathcal{E}_k \) and \( \mathcal{B}'_k = \mathcal{E}_k' - \mathcal{A}'_k \). The bijection

\[ \sigma_{T[k]}|_{\mathcal{B}_k} = \beta_k : \mathcal{B}_k \to \mathcal{B}'_k \]

is constructed by a special argument which can be summarized by saying that \( \beta_k(X[k]) \) is either \( X[k] \) or \( \gamma(X)[k] \), whichever lies in the set \( \mathcal{B}'_k \). A similar description summarizes the bijection \( \sigma_{T[k]} \) on all objects of \( \mathcal{E}^m(T^\perp) \) (See Proposition 3.5.4).

This completes the description of each object and each morphism in Chart 3.1. The final step is to show that \( \sigma_{T[k]} \) preserves compatibility. I.e., \( X[j], X'[j'] \) are compatibility in \( \mathcal{E}^m(T^\perp) \)
if and only if $\sigma_{T[k]}(X[j]), \sigma_{T[k]}(X'[j'])$ are compatibility in $\mathcal{E}^m \Lambda$. This will follow from the fact that $X, X', T$, appropriately order, form an exceptional sequence.

3.4. The bijection $\alpha_j : A_j \to A'_{j-1}$. Recall from the previous section that $A$ is the set of all $X \in \mathcal{E}(T^\perp)$ so that there is a monomorphism $X \hookrightarrow T^s$. We need the following observation.

**Remark 3.4.1.** Any homomorphism $X \hookrightarrow T^s$ induces an epimorphism $\text{Ext}^1_\Lambda(T^s, Z) \to \text{Ext}^1_\Lambda(X, Z)$ for any $Z$ since $\Lambda$ is hereditary. In particular, $\text{Ext}^1_\Lambda(X, Z) = 0$ for $Z \in T^\perp$. For $X \in T^\perp$ this implies that $X$ is a relatively projective object of $T^\perp$.

The standard bijection $\gamma : \mathcal{E}(T^\perp) \cong \mathcal{E}(\perp T)$ sends any $X \in A$ to $\gamma(X) = Y$, the cokernel of the minimal $T$-approximation of $X$:

$$0 \to X \overset{f}{\to} T^s \to Y \to 0$$

We call this bijection $\alpha : A \to A' = \gamma(A)$. Recall (from the previous section) that, for $k < j \leq m$, $A_j$ denotes the set of all $X[j]$ where $X \in A$ and, for $k \leq j < m$, $A'_j$ denotes the set of all $Y[j]$ where $Y \in A'$.

**Remark 3.4.2.** By Remark 3.4.1, any $X \in A$ is relatively projective in $T^\perp$ and, by the dual argument, any $Y \in A'$ is relatively injective in $\perp T$.

**Lemma 3.4.3.** $A_j \subseteq \mathcal{E}^m_j(T^\perp)$ for all $k < j \leq m$ and $A'_j \subseteq \mathcal{E}^m_j(T^\perp)$ for all $k \leq j < m$.

**Proof.** Since $(X, T)$ is exceptional, $X[j] \in \mathcal{E}^m_j(T^\perp)$ for $j < m$. For $j = m$ we have, by Remark 3.4.1, that $X[m] \in \mathcal{E}^m_m(T^\perp)$. Thus $A_j \subseteq \mathcal{E}^m_j(T^\perp)$ for all $k < j \leq m$.

For $k < j < m$, the condition of $(T, Y)$ being exceptional is equivalent to $Y[j] \in \mathcal{E}^m_j(T^\perp)$. For $j = k$, the additional condition that $\text{Hom}_\Lambda(T, Y) \neq 0$ implies that $\text{Ext}^1_\Lambda(T, Y) = 0$. Therefore $Y, T$ are ext-orthogonal modules or, equivalently, $Y[k] \in \mathcal{E}^m_k(T^\perp)$. So, for all $k \leq j < m$, $A'_j \subseteq \mathcal{E}^m_j(T^\perp)$. □

The bijection $\alpha_j : A_j \to A'_{j-1}$ is given by sending $X[j]$ to $Y[j-1]$ where $Y = \alpha(X)$.

3.5. The bijection $\beta_j : B_j \to B'_j$. We recall the notation from the outline: $B = \mathcal{E}(T^\perp) - A$ and $B' = \mathcal{E}(\perp T) - A'$. The bijection $\gamma : \mathcal{E}(T^\perp) \cong \mathcal{E}(\perp T)$ will be denoted with the letters $X, Y$. Thus $Y = \gamma(X)$. There are four cases, but one of them is $A \cong A'$.

**Remark 3.5.1.** The bijection $\beta : B \cong B'$ falls into three cases.

<table>
<thead>
<tr>
<th>$X \in B$</th>
<th>$X = Y$</th>
<th>$T, Y$ are hom-ext orthogonal</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(b)</td>
<td>(c)</td>
</tr>
<tr>
<td>$\text{Ext}^1_\Lambda(X, T) \neq 0$</td>
<td>$T^s \hookrightarrow Y \twoheadrightarrow X$</td>
<td>$\exists f : X \to T^s$</td>
</tr>
<tr>
<td>$\exists g : T^s \to Y$</td>
<td>$\text{Ext}^1_\Lambda(T, Y) \neq 0$</td>
<td></td>
</tr>
</tbody>
</table>

**Lemma 3.5.2.** The bijection $\beta : B \to B'$, sending $X$ to $Y$, has the property that $Y$ is projective in $\text{mod-} \Lambda$ if and only if $X$ is a relatively projective object in $T^\perp$.

**Proof.** (from [9]) In all three cases, we have an isomorphism

$$\text{Ext}^1_\Lambda(X, Z) \cong \text{Ext}^1_\Lambda(Y, Z)$$

for all $Z \in T^\perp$. This is trivial when $X = Y$ and follows from the six term Hom-Ext series in the other cases. If $Y$ is projective these are zero and thus $X$ is relatively projective in $T^\perp$. 
Conversely, suppose that $X$ is relatively projective in $T^\perp$ and $Y$ is not projective. Then we will obtain a contradiction. Let $Z \in \text{mod-}\Lambda$ be minimal so that $\text{Ext}_A^1(Y, Z) \neq 0$. Since $\text{Ext}_A^1(X, -) \cong \text{Ext}_A^1(Y, -)$ on $T^\perp$ we have $Z \notin T^\perp$. Therefore, either

1. $\text{Hom}_A(T, Z) \neq 0$ or
2. $\text{Ext}_A^1(T, Z) \neq 0$ (and $\text{Hom}_A(T, Z) = 0$).

In Case (1) there is a right exact sequence $T \to Z \to W \to 0$ where $W$ is smaller than $Z$. Since $Y \in T^\perp$, we have $\text{Ext}_A^1(Y, Z) \cong \text{Ext}_A^1(Y, W) \neq 0$ contradicting the minimality of $Z$.

In Case (2), we consider the universal extension $Z \hookrightarrow E \to T^s$ given by choosing a basis for $\text{Ext}_A^1(T, Z)$ over the division algebra $\text{End}_A(T)$. This gives the six term exact sequence

$$\text{Hom}_A(T, Z) \to \text{Hom}_A(T, E) \to \text{Hom}_A(T, T^s) \xrightarrow{\sim} \text{Ext}_A^1(T, Z) \to \text{Ext}_A^1(T, E) \to \text{Ext}_A^1(T, T^s)$$

where the middle map is an isomorphism by construction and the two end terms are zero: $\text{Hom}_A(T, Z) = 0$ by assumption and $\text{Ext}_A^1(T, T^s) = 0$ since $T$ is exceptional. Therefore, the other two terms are zero and $E \in T^\perp$. But this implies $\text{Ext}_A^1(Y, E) \cong \text{Ext}_A^1(X, E) = 0$. We have another exact sequence

$$\text{Hom}_A(Y, T^s) \to \text{Ext}_A^1(Y, Z) \to \text{Ext}_A^1(Y, E)$$

which gives $\text{Ext}_A^1(Y, Z) = 0$ since $Y \in T^\perp$. This contradiction completes the proof.  

We recall our notation. For all $k < j < m$, $B_j$ denotes the set of all $X[j]$ where $X \in B$ and $B'_j$ denotes the set of all $Y[j]$ where $Y \in B'$. Then $B_j \subseteq E_j$ and $B'_j \subseteq E'_j$ and we have a bijection $\beta_j : B_j \cong B'_j$ given by $\beta_j(X[j]) = (\beta(X))[j]$.

For $j = m > k$, $B_m = E_m - A_m$. This is the set of all $X[m]$ where $X \in B$ and $X$ is also relatively projective in $T^\perp$. We also have our notation: $B'_m = E'_m$. This is the set of all $Y[m]$ where $Y \in B'$ and $Y$ is also a projective module. By Lemma 35.2, the bijection $\beta : B \cong B'$ gives a bijection $\beta_m : B_m \cong B'_m$ by $\beta_m(X[m]) = (\beta(X))[m]$.

For $j = k \leq m$, $B_k = E_k$ including objects $X[k]$ where $X \in A$. $B'_k = E'_k - A'_k$ is the set of all objects $Z[k]$ where $Z, T$ are ext-orthogonal and there does not exist an epimorphism $T^s \to Z$ since such objects $Z$ lie in $A'$. Note that $A'_m$ is empty since there is no epimorphism $T^s \to Z$ when $Z$ is projective.

**Lemma 35.3.** There is a bijection $\beta_k : B_k \cong B'_k$ given by

$$\beta_k(X[k]) = \begin{cases} X[k] & \text{if } \text{Ext}_A^1(X, T) = 0 \\ (\beta(X))[k] & \text{otherwise} \end{cases}$$

with inverse $\beta_k^{-1}$ given by

$$\beta_k^{-1}(Y[k]) = \begin{cases} Y[k] & \text{if } \text{Hom}_A(T, Y) = 0 \\ (\beta^{-1}(Y))[k] & \text{otherwise} \end{cases}$$

**Proof.** When $\text{Hom}_A(T, Y) \neq 0$ we know that $Y \in T^\perp$. So, the right $T$ approximation of $Y$ is either a monomorphism $g : T^s \hookrightarrow Y$ or an epimorphism $T^s \twoheadrightarrow Y$ and the second case is excluded since $Y \in A'$ in that case. Then $\beta^{-1}(Y) = X$ is the cokernel of $g$ and $\beta(X) = Y$. 

Lemma 3.6.2. Let $X, T$ be an exceptional pair in $\mathcal{E}(T^\perp) \cong \mathcal{E}^T[k]$ if the corresponding triple in the following sequence is exceptional:

\[
\begin{align*}
(a) & \quad X, T \text{ are hom-ext orthogonal } & X = Y, T_Y, Y \text{ are hom orthogonal} \\
(b) & \quad \text{Ext}_A^1(X, T) \neq 0 & T^* \hookrightarrow Y \twoheadrightarrow X & \exists g : T^* \hookrightarrow Y \\
(c) & \quad \exists f : X \twoheadrightarrow T^* & X = Y & \exists h : Y \hookrightarrow T^* \\
(d) & \quad \exists h : X \hookrightarrow T^* & X = Y & \exists h : Y \hookrightarrow T^*
\end{align*}
\]

The special case $k = m$ needs further checking. In this case $T$ is projective and Case (c) will not occur. In Case (d), $X$ is projective since it is a submodule of a projective module. So, we can take $Y = X$. Cases (a) and (b) are the same as in Remark 3.5.1. So, Lemma 3.5.2 tells us that $X$ is relatively projective in $T^\perp$ if and only if $Y$ is a projective module. Therefore, the above formula gives a bijection $\mathcal{B}_m \rightarrow \mathcal{B}'_m$ when $k = m$.

Combining all of these cases, we obtain the desired bijection as summarized by the following.

Proposition 3.5.4. The bijection $\sigma = \sigma_{T[k]} : \mathcal{E}^m(T^\perp) \cong \mathcal{E}^T[k]$ is given by

1. $\sigma(X[i]) = X[i]$ if $X[i] \in \mathcal{E}^T[k]$.
2. If $X[i] \notin \mathcal{E}^T[k]$ then $\sigma(X[i]) = Y[j]$ where $Y = \gamma(X) \in \mathcal{E}(T^\perp)$ and $j = i$ or $i - 1$ so that $(-1)^i \dim X$ and $(-1)^j \dim Y$ are congruent modulo $\dim T$.

3.6. Compatibility. We will prove that $\sigma = \sigma_{T[k]} : \mathcal{E}^m(T^\perp) \cong \mathcal{E}^T[k]$ preserves compatibility:

Proposition 3.6.1. $X[i], X'[i']$ are compatible objects of $\mathcal{E}^m(T^\perp)$ if and only if $Y[j] = \sigma(X[i])$ and $Y'[j'] = \sigma(X'[i'])$ are compatible in $\mathcal{E}^T[k]$.

We need the following lemmas.

Lemma 3.6.2. Let $X, X' \in \mathcal{E}(T^\perp)$ and consider the pairs:

\[(X, X'), \quad (X, \gamma(X')), \quad (\gamma(X), \gamma(X'))\]

If any of these is an exceptional pair, the other two are exceptional pairs.

Proof. A pair is exceptional if the corresponding triple in the following sequence is exceptional:

\[(X, X', T), \quad (X, T, \gamma(X')), \quad (T, \gamma(X), \gamma(X')).\]

But any one of these gives the other two by mutation of exceptional sequences. □

Lemma 3.6.3. Let $(X, X')$ be an exceptional pair in $\mathcal{E}(T^\perp) = A \cup B$ and let $Y = \gamma(X), Y' = \gamma(X') \in \mathcal{E}(\perp T) = A' \cup B'$. Let $Y'' \in \mathcal{E}(T^\perp), Y'' \in \mathcal{E}(\perp T)$ be given by mutation of exceptional sequences:

\[(X, X', T) \sim (X', X'', T) \sim (X', T, Y'') \sim (T, Y', Y'') \sim (T, Y, Y')\]

Then there are unique rational numbers $a, b$ and signs $\varepsilon, \varepsilon' = \pm 1$ so that

\[
\begin{align*}
\dim X + a \dim X' + b \dim X'' &= 0 \\
\dim Y + \varepsilon a \dim Y' + \varepsilon' b \dim Y'' &= 0
\end{align*}
\]

Furthermore

1. $\varepsilon = 1$ if and only if $X, X'$ both lie in $A$ or they both lie in $B$.
2. $\text{Ext}_A^1(X, X') \neq 0$ if and only if $a > 0$
3. $\text{Ext}_A^1(Y, Y') \neq 0$ if and only if $\varepsilon a > 0$
Proof. The existence and uniqueness of $a, b \in \mathbb{Q}$ follows from the fact that $\dim X', \dim X''$ are linearly independent vectors in $\mathbb{Q}^n$ and $\dim X$ is in their span.

The equation for $\gamma$ gives us that $\dim Y$ is congruent to $\delta \dim X$ modulo $\dim T$ where $\delta = -1$ for $X \in \mathcal{A}$ and $\delta = 1$ for $X \in \mathcal{B}$. This gives:

$$\delta \dim Y + \delta' a \dim Y' + \delta'' b \dim Y'' = c \dim T$$

for some $c \in \mathbb{Q}$. However, $\dim Y', \dim Y'', \dim T$ are linearly independent since $(Y', Y'', T)$ form an exceptional sequence. And $\dim Y, \dim Y', \dim Y''$ are linearly dependent. So, $c = 0$.

Statement (1) is now clear: $\delta, \delta'$ are both negative when $X, X' \in \mathcal{A}$ and they are both positive if $X, X' \in \mathcal{B}$. So, $\varepsilon = \delta'/\delta = 1$ in these cases and $\varepsilon = -1$ in the other cases.

If $\operatorname{Ext}_1^\Lambda (X, X') \neq 0$ then $a = 1 > 0$. Otherwise, either $\operatorname{Hom}_\Lambda (X, X') \neq 0$, in which case $a < 0$ or both are zero in which case $a = 0$. This proves (2) and (3) is similar. □

Proof of Proposition 3.6.1. Throughout this proof we will use the notation $Y[j] = \sigma(X[i]), Y'[j'] = \sigma(X'[i'])$. By reversing the order if necessary we may assume that $i \leq i'$ and $j \leq j'$.

There are four cases.

Case 1: $i < i', j < j'$.

When $i < i'$, $X[i], X'[i']$ are compatible if and only if $(X, X')$ is an exceptional sequence. By the formula for $\sigma$, $(Y, Y')$ is equal to either $(X, X'), (X, \gamma(X'))$ or $(\gamma(X), \gamma(X'))$. Whichever it is, by Lemma 3.6.2 $(Y, Y')$ is exceptional if and only if $(X, X')$ is exceptional. So, compatibility of $X[i], X'[i']$ is equivalent to compatibility of $Y[j], Y'[j']$.

Case 2: $i < i', j = j'$.

In this case $i' = j + 1$, $X[i] \in \mathcal{B}_i, X'[i'] \in \mathcal{A}_{j+1}, Y'[j'] \in \mathcal{A}_j$, there is a short exact sequence

$$X' \hookrightarrow T^s \rightarrow Y'$$

giving the correspondence $Y' = \alpha(X')$, and $Y$ is equal to $X$ or $\gamma(X)$. The case $Y = X \neq \gamma(X)$ occurs only when $i = k$, $\operatorname{Hom}_\Lambda (X, T) \neq 0$ and $\gamma(X)$ is either the kernel or cokernel of the universal map $X \rightarrow T^s$.

Suppose that $X[i], X'[i']$ are compatible. Then $(X, X')$ is an exceptional pair. So, $(Y, Y')$ which is either $(X, Y')$ or $(\gamma(X), Y')$ is an exceptional pair by Lemma 3.6.2. Then

1. $\operatorname{Ext}_1^\Lambda (\gamma(X), Y') = 0$ since $Y'$ is a relatively injective object of $\downarrow T$.
2. $\operatorname{Ext}_1^\Lambda (T, Y') = 0$ since $\operatorname{Hom}_\Lambda (T, Y') \neq 0$ and $(T, Y')$ is an exceptional pair.

But $X, \gamma(X)$ and $T^s$ form an exact sequence in which $X$ is not the last item. Therefore, by right exactness of $\operatorname{Ext}_1^\Lambda (-, Y')$, we conclude that $\operatorname{Hom}_\Lambda (X, Y') = 0$. Therefore, $Y, Y'$ are ext-orthogonal regardless of whether $Y = X$ or $Y = \gamma(X)$. So, $Y[j], Y'[j]$ are compatible.

Conversely, suppose that $Y[j], Y'[j]$ are compatible. Then we claim that $X[i], X'[i']$ are compatible, or equivalently $(X, X')$ is an exceptional pair. Since $X'$ is relatively projective in $\uparrow T$, we have $\operatorname{Ext}_1^\Lambda (X', X) = 0$. So, it suffices to show $\operatorname{Hom}_\Lambda (X', X) = 0$. But $X \in T^\perp$. So, $\operatorname{Hom}_\Lambda (X', X) \cong \operatorname{Ext}_1^\Lambda (Y', X) \cong \operatorname{Ext}_1^\Lambda (Y', Y) = 0$

where the second isomorphism follows from the fact that $Y' \in \downarrow T$ and $X, Y, T^s$ form an exact sequence in which $X, Y$ are adjacent terms. (See Remark 3.5.1.)

Case 3: $i = i', j < j'$.

Then $j = i - 1$, we have an exact sequence $X \hookrightarrow T^s \rightarrow Y$ giving the bijection $\alpha(X) = Y$ and $Y' = \beta(X') \in \downarrow T$. If $Y[i-1], Y'[i]$ are compatible, then $(Y, Y')$ form an exceptional pair
and thus \((X, X')\) also forms an exceptional pair. This implies that \(X, X'\) are ext-orthogonal since \(X\) is relatively projective in \(T^\perp\). Thus \(X[i], X'[i]\) are compatible.

Conversely suppose that \(X[i], X'[i]\) are compatible. Since \(Y' \in \perp T\) we have, as in Case 2,
\[
\text{Hom}_A(Y', Y) \cong \text{Ext}^1_A(Y', X) \cong \text{Ext}^1_A(X', X) = 0
\]
and \(\text{Ext}^1_A(Y', Y) = 0\). Therefore \(Y[i-1], Y'[i]\) are compatible.

Case 4: \(i = i', j = j'\).

In this case we need to prove that \(X, X'\) are ext-orthogonal if and only if \(Y, Y'\) are ext-orthogonal. The case \(i < k\) being trivial, we assume that \(i = i' \geq k\).

Suppose that \(i > k\). Then \(Y = \gamma(X), Y' = \gamma(X')\). If \(X, X'\) are ext-orthogonal they form an exceptional pair in some order, say \((X, X')\), and so does \((Y, Y')\). In the linear relation in Lemma 3.6.3 we are in the case when \(\varepsilon = 1\) since \(X, X\) are either both in \(\mathcal{A}\) or both in \(\mathcal{B}\). Therefore, \(\text{Ext}^1_A(X, X') = 0\) if and only if \(\text{Ext}^1_A(Y, Y') = 0\). So, \(Y, Y'\) are ext-orthogonal. The converse works in the same way.

Finally, suppose that \(i = k\). Then we also have \(j = k\). The bijection \(\beta_k : \mathcal{B}_k \cong \mathcal{B}'_k\) is given by \(\beta\) and the argument in the previous paragraph applies except in Cases (c),(d) in the chart in the proof of Lemma 3.5.3. Also, \(\beta_k\) is the identity except in Case (b). So, the only case we need to check is when one of the objects, say \(X[k]\) is in Case (b) and the other, \(X'[k]\) is in Case (c) or (d). In other words, \(Y = \beta(X)\) is given by the exact sequence
\[
T^s \hookrightarrow Y \twoheadrightarrow X
\]
and \(Y' = X' \in T^\perp\) has the property that \(\text{Ext}^1_A(X', T) = 0\) and \(\text{Hom}_A(X', T) \neq 0\). The exact sequence tells us that
\[
\text{Ext}^1_A(X, X') \cong \text{Ext}^1_A(Y, X') = \text{Ext}^1_A(Y, Y').
\]
Also, \(\text{Ext}^1_A(Y', T) = 0\) implies that
\[
\text{Ext}^1_A(Y', Y) \cong \text{Ext}^1_A(Y', X) = \text{Ext}^1_A(X', X).
\]
Therefore, \(X[k], X'[k]\) are compatible if and only if \(Y[k], Y'[k]\) are compatible. □

4. More properties of \(m\)-exceptional sequences

We show how \(m\)-exceptional sequences are related to the bijection between the set of (isomorphism classes of) \(m\)-cluster tilting objects in the \(m\)-cluster category of \(\text{mod-} \Lambda\) and (isomorphism classes of) \(m\)-\(\text{Hom}\leq 0\)-configurations from \([4]\) which in the sequel we refer to as “\(m\)-configurations”. We use this to derive the expected tropical duality formula (Corollary 4.2.2) relating the dimension vectors of these objects. Then we obtain a convenient reformulation of the mutation formula for \(\delta\)-vectors (Theorem 4.3.3) using a sign convention which matches \(c\)-vectors and dimension vectors of components of \(m\)-configurations. Finally, we give an example of this mutation formula. These formulas were first obtained in type \(A_n\) in \([11]\).

4.1. Relation to Buan-Reiten-Thomas \([4]\). First, the definitions. As always, \(\Lambda\) is a finite dimensional hereditary algebra over a field \(K\) with \(n\) simple objects \(S_i\).

**Definition 4.1.1.** \([4]\) An \(m\)-configuration in the bounded derived category \(D^b(\text{mod-} \Lambda)\) is defined to be an object \(M\) of this derived category with \(n\) (nonisomorphic) components \(M_i[k_i]\) satisfying the following conditions.

1. \(0 \leq k_1 \leq m\)
2. \(\text{Hom}_{D^b}(M_i[k_i], M_j[k_j - s]) = 0\) for all \(i \neq j\) and \(s \geq 0\).
(3) $M_1, \ldots, M_n$ form a complete exceptional sequence in some order.

In [4] a bijection is constructed between the set of (isomorphism classes of) $m$-configurations and $m$-cluster tilting objects for any hereditary algebra $\Lambda$. This statement and proof will be reexamined in Theorem 4.1.3 below.

This bijection has been extended to a bijection between simple minded collections and two term silting objects in the category of bounded projective complexes [13], [14]. However, the relation with exceptional sequences is special to the hereditary case.

We need the following definitions and sign conventions to make Definition 4.1.1, when expressed in terms of the dimension vectors of $M_j[k_j]$, agree (when $m=1$) with the characterization of $c$-vectors given in [18].

**Definition 4.1.2.** The $(m)$-slope vector of $M[k]$ is defined by $\text{slope}(M[k]) := \dim M t^m k \in \mathbb{Z}[t]^n$. We define the dimension vector of $M[k]$ to be $\dim(M[k]) := (-1)^m k \dim M$. This is the slope vector evaluated at $t = -1$. We define the $\tilde{c}$-vectors of the $m$-cluster $T$ to be the slope vectors of the components $X_j[\ell_j]$ of the corresponding $m$-configuration:

$$\tilde{c}_j = \text{slope}(X_j[\ell_j]) = \dim X_j t^m \ell_j \in \mathbb{Z}[t]^n$$

The $c$-vectors of the $m$-cluster tilting object $T$ are $c_j = \tilde{c}_j|_{t=-1} = (-1)^m \ell_j \dim X_j \in \mathbb{Z}^n$. We also use round brackets to indicate slope:

$$X(s) := X[m-s].$$

The following is a reformulation of the main theorem of [4] in terms of $m$-exceptional sequences:

**Theorem 4.1.3.** [4] Let $T = (T_1[k_1], T_2[k_2], \ldots, T_n[k_n])$ be an ordered $m$-cluster so that $(T_n, T_{n-1}, \ldots, T_1)$ is a complete exceptional sequence. Then the $m$-exceptional sequence corresponding to $T$ is equal to the set of components of the $m$-configuration corresponding to the $m$-cluster $T$.

**Remark 4.1.4.** Note that there may be more than one way to arrange the objects $T_j[k_j]$ so that $(T_n, \ldots, T_1)$ is a complete exceptional sequence. However, any two arrangements will differ by a sequence of transpositions of consecutive hom-ext-orthogonal objects. The statement of the theorem includes the statement that, if two consecutive $T_i$ commute in this way, the corresponding components $E_j$ of the $m$-configuration also commute (are hom-ext orthogonal). For example, one could arrange the terms $T_j[k_j]$ so that $k_1 \geq k_2 \geq \cdots \geq k_n$. However, assuming this would needlessly complicate the proof.

We need to go through the proof (the same as in [4]) since this proof leads us to the “tropical duality formula” Corollary 4.2.2 below.

**Proof.** We follow the method in [4] which also appears in [2] Prop 3.3 and [1]. Namely, we apply to the exceptional sequence $(T_n, T_{n-1}, \ldots, T_1)$ the Garside braid move given by taking the objects on the left one at a time and moving them over the objects on the right which haven’t moved yet. The result is $(E_1, \ldots, E_n)$ where, by construction, $E_n = T_n$. The slopes are determined by a simple formula. The statement of the theorem is that this is a special case of the bijection between $m$-cluster tilting objects and $m$-exceptional sequences.

We recall that the correspondence $\theta_n : T \mapsto E$ is given by induction on $n$ by

$$\theta_n(T_1[k_1], \ldots, T_n[k_n]) = (\theta_{n-1}(X_1[\ell_1], \ldots, X_{n-1}[\ell_{n-1}], T_n[k_n]))$$

where $X_j[\ell_j] = \sigma_{T_n[k_n]}^{-1}(T_j[k_j])$. Recall from Proposition 3.5.4 that $X_j[\ell_j]$ is uniquely determined by the property that $(T_n, T_j) \sim (X_j, T_n)$ and the slope of $X_j$ is equal to or one less than the slope
of \( T_j \). In other words, \( \theta_n \) is the same as the bijection given by action of the Garside braid on the exceptional sequence \((T_n, \cdots, T_1)\). We note that, by the properties of the braid group action on exceptional sequences, \((X_{n-1}, \cdots, X_1)\) is an exceptional sequence and, by the properties of the bijection \( \sigma_{T_n[k_n]} \), \( X_i \) are compatible (mutually ext-orthogonal). Furthermore, \( X_i, X_{i+1} \) are hom-ext-orthogonal if and only if \( T_i, T_{i+1} \) are hom-ext-orthogonal. Therefore, \( X_i[\ell_i] \) satisfy the induction hypotheses. So, the process can be iterated and the proof is complete.

4.2. Tropical duality. From the proof of Theorem 4.1.3 we obtain a version of the tropical duality formula for \( m \)-clusters. (Corollary 4.2.2 below.)

Lemma 4.2.1. In the correspondence \( E = \theta(T) \), the slope of each component \( E_j \) is either equal to or one less than the slope of \( T_j \).

Proof. Let \( X_j \) be as in the proof of Theorem 4.1.3. By Remark 3.4.1 if the slope of \( X_j \) differs from the slope of \( T_j \) then \( X_j \) is relatively projective in the smaller category \( T_n^\perp \) and slope \( X_j = \text{slope} T_j - 1 \). Also, \( T_j \) cannot be projective since there is a non-split short exact sequence \( X_j \to T_n^\perp \to T_j \). Thus, in the next step, when \( X_j \) is replaced by \( X_j' \in (T_n \oplus T_{n-1})^\perp \), \( X_j' \) must have the same slope as \( X_j \) if \( X_j \) is projective. By Lemma 3.5.2 once an object becomes relatively projective, it will remain relatively projective through the later steps. Thus, in the sequence \( T_j \to X_j \to X_j' \to \cdots \to E_j \), the slope cannot change more than once.

Corollary 4.2.2. The dimension vectors of components of an \( m \)-cluster tilting object \( T = \bigoplus T_i[k_i] \) and of the dual \( m \)-configuration \( X = \bigoplus X_j[\ell_j] \) are related by the formula:

\[
V^t EC = D
\]

where \( V \) has columns \( \dim T_i[k_i] = (-1)^{m-k_i} \dim T_i \), \( C \) has columns \( c_j = (-1)^{m-\ell_j} \dim X_j \), \( D \) is the diagonal matrix with diagonal entries \( f_k = \dim_k \text{End}(T_k) \) and \( E \) is the Euler matrix with entries \( e_{ij} = \dim_k \text{Hom}(S_i, S_j) - \dim_k \text{Ext}^1(S_i, S_j) \). Furthermore, the slope of each \( T_j \) is either equal to or one more than the slope of \( \tilde{c}_j \), whichever gives the correct sign for the pairing \( \langle \dim T_j[k_j], c_j \rangle = +f_j \).

Remark 4.2.3. The formula \( V^t EC = D \) is equivalent to the tropical duality formula \( G^t DC = D \) from \cite{15} where \( G \) is the matrix of \( g \)-vectors (times \((-1)^m \)) to match the sign shift of \( C \). In \cite{12} we shift the sign of \( C \) but not of \( G \) so that \( G^t DC = (-1)^m D \). The equations are equivalent since \( V^t = G^t D E^{-1} \). This follows from the well-known fact that the rows of \( DE^{-1} \) are the dimension vectors of the indecomposable projective \( \Lambda \)-modules and the \( k \)-th \( g \)-vector gives the projective presentation of the \( k \)-th component \( T_k \) of the \( m \)-cluster tilting object \( T \).

Proof. This follows from the proof of Theorem 4.1.3 where we recall that \( \theta(T) = (E) = (E_1, \cdots, E_n) \) and \( X_k \) are given by the braid move \( (T_n, T_k) \sim (X_k, T_n) \).

Claim 1: \( \langle \dim T_i, \dim E_j \rangle = 0 \) if \( i < j \) where \( \langle x, y \rangle := x^tEy \).

Proof: Since \( \dim E_j \) is a linear combination of \( \dim X_k \) for \( k \geq j \) which are linear combinations of \( \dim T_k \) by induction, \( \langle \dim T_i, \dim E_j \rangle \) is a linear combination of \( \langle \dim T_i, \dim T_k \rangle \) for \( k \geq j > i \). But these are all zero since \( (T_n, \cdots, T_1) \) is an exceptional sequence.

Claim 2: \( \langle \dim T_i, \dim E_j \rangle = 0 \) if \( i > j \).

Proof: By induction on \( n - i \). First, let \( i = n \). Then, for \( j \leq k < n \), \( X_k \in T_n^\perp \) for \( X_k[\ell_k] = \sigma_{T_n[k_n]}^{-1}(T_j[k_j]) \). So, \( \langle \dim T_i, \dim X_k \rangle = 0 \). But \( \dim E_j \) is a linear combination of \( \dim X_k \) for \( k \leq j < n \). So, \( \langle \dim T_i, \dim E_j \rangle = 0 \).

Next, suppose \( j < i < n \). Then, by induction we have \( \langle \dim T_k, \dim E_j \rangle = 0 \) for all \( i < k \leq n \) and \( \langle \dim X_i, \dim E_j \rangle = 0 \) since \( (n-1) - i < n - i \). But the braid relation \( (T_n, T_i) \sim (X_i, T_n) \)
implies that, either (a) $X_i = T_i$, or (b) there is a short exact sequence $X_i \hookrightarrow T_n^s \twoheadrightarrow T_i$ or $T_i \hookrightarrow X_i \twoheadrightarrow T_n^s$. In any case we have
\begin{equation}
\dim T_i = \delta \dim X_i + s \dim T_n
\end{equation}
for $\delta = \pm 1$ and some integer $s$. So,
\begin{equation}
\langle \dim T_i, \dim E_j \rangle = \delta \langle \dim X_i, \dim E_j \rangle + s \langle \dim T_n, \dim E_j \rangle = 0.
\end{equation}

Claim 3: For $i = j$, $\langle \dim T_i, \dim E_i \rangle = \pm f_i$ where $f_i = \dim_K \text{End}(T_i)$.

Proof. By iteration the formula (4.1) we see that $\dim T_i$ is equal to $\delta \dim E_i$ plus a linear combination of $\dim T_k$ for $i < k \leq n$. Since $\langle \dim T_i, \dim T_k \rangle = 0$ for all such $k$ we get
\begin{equation}
f_i = \langle \dim T_i, \dim T_i \rangle = \delta \langle \dim T_i, \dim E_i \rangle
\end{equation}
as claimed.

Finally, we note that the sign $\delta = \pm 1$ in (4.1) is equal to $-1$ only when $T_i, X_i$ are related by a short exact sequence of the form $X_i \hookrightarrow T_n^s \twoheadrightarrow T_i$. In this case the slope of $X_i$ is one less than the slope of $T_i$ and this can happen only once for each $i$. Therefore, either $\langle \dim T_i, \dim E_i \rangle = f_i$ and $T_i, E_i$ have the same slope or $\langle \dim T_i, \dim E_i \rangle = -f_i$ and the slope of $E_i$ is one less than the slope of $T_i$. The proof of the duality formula is complete. \qed

4.3. **Mutation formula for $\bar{c}$-vectors.** For each $m$-configuration $X$ we construct a sequence of overlapping wide subcategories $H_s(X)$ of $\text{mod-}\Lambda$ which we call “horizontal” or “vertical” subcategories depending on the parity of $s$. Then we show that each mutation of $X$ can be construed to take place in the cluster category of one of these subcategories.

In the $m$-cluster category of $\text{mod-}\Lambda$, each component $T_k$ of each $m$-cluster tilting object $T$ can be replaced with one of $m$ other objects which are given by $\mu_k^+(T_k), (\mu_k^+)^2(T_k), \cdots, (\mu_k^+)^s(T_k)$ and $\mu_k^-(T_k), (\mu_k^-)^2(T_k), \cdots, (\mu_k^-)^{m-s}(T_k)$ where the formulas for the “positive” and “negative” mutation operators $\mu_k^+, \mu_k^-$ are defined using distinguished triangles [13]. We use the sign convention given by slope. We recall that there are distinguished triangles
\begin{equation}
T_k \to B \to \mu_k^-(T_k) \to T_k[1], \quad T_k[-1] \to \mu_k^+(T_k) \to B' \to T_k
\end{equation}
where $B, B'$ are left/right add $T/T_k$-approximations of $T_k$. The slope of $\mu_k^+(T_k)$ is greater than or equal to the slope of $T_k$ and analogously for $\mu_k^-(T_k)$.

The mutation formula is better behaved for the corresponding $\bar{c}$-vectors since $\mu_k^+$ always increases the slope of $\bar{c}_k$ by one and $\mu_k^-$ always decreases the slope of $\bar{c}_k$ by one. In fact, positive and negative mutation of $\bar{c}$ is given by the following formula which needs a definition first.

Let $\varepsilon \beta \in \mathbb{Z}^n$ be a real Schur root for $\Lambda$ where $\varepsilon = \pm 1$ is its sign and $\beta$ is a positive real Schur root. Then we define the vector $\psi_s(\varepsilon \beta) \in \mathbb{Z}[t]^n$ to be $t^s \beta$ if $\varepsilon = +1$ and $t^{s+1} \beta$ if $\varepsilon = -1$.

**Remark 4.3.1.** Given $X$ an $m$-configuration and consecutive integers $s < s + 1$, let $\mathcal{A}_s(X)$ denote wide subcategory of $\text{mod-}\Lambda$ spanned by modules $M_i$, $i = 1, \cdots, h_s$, so that either $M_i(s)$ or $M_i(s + 1)$ is a component of $X$ (Recall that $M_i(s) = M_i[m - s]$). Then $\mathcal{A}_s(X)$ is a rank $h_s$ wide subcategory of $\text{mod-}\Lambda$ so that $\mathcal{A}_s(X) \subseteq \mathcal{A}_t(X)^\perp$ for all $t \geq s + 2$. In fact $\mathcal{A}_s(X)$ is the intersection of the perpendicular categories
\begin{equation}
\mathcal{A}_s(X) = \bigcap_{t \geq s + 2} \mathcal{A}_t(X)^\perp \cap \bigcap_{r \leq s - 2} \mathcal{A}_r(X).
\end{equation}

Note that, in general, $\mathcal{A}_s \cap \mathcal{A}_t$ is nonzero only when $|t - s| \geq 2$. 
For $i = 1, \cdots, h_s$, let $c_i(s) = \dim M_i$ or $-\dim M_i$ depending on whether $M_i(s)$ or $M_i(s+1)$ is a component of $X$, respectively. Then Definition 4.1.1 implies that the vectors $c_i(s)$ satisfy the condition of (13) and therefore form a set of $c$-vectors for some cluster tilting object for $A_s(X)$. Let $Z(s) \cong \mathbb{Z}^{h_s}$ be the additive subgroup of $\mathbb{Z}^n$ freely generated by these $h_s$ elements. Then, for any positive real Schur root $\beta$ of $\Lambda$, $\beta \in Z(s)$ if and only if the corresponding exceptional $\Lambda$-module $M_{\beta}$ lies in the wide subcategory $A_s(X)$. The reason is the $A_s(X)$ is given by the linear condition (4.2) whose integer solution set is $Z(s)$. We define $\psi_s(\beta), \psi_s(-\beta)$ to be the objects in the bounded derived category of $\Lambda$ equal to $M_{\beta}$ at level $[m-s]$ (with slope $s$) or level $[m-s-1]$ with slope $s+1$, respectively:

$$\psi_s(\beta) = M_{\beta}(s), \psi_s(-\beta) = M_{\beta}(s+1)$$

Recall that $X = \bigoplus M_i[k_i]$ is a fixed $m$-configuration.

**Lemma 4.3.2.** Let $v_i \in Z(s) \subset \mathbb{Z}^n$, $i = 1, \cdots, h_s$ satisfying the following.

1. Each $v_i$ is a positive or negative real Schur root of $\Lambda$. Let $N_i$ or $N_i[1]$, resp., be the corresponding object of the derived category of $A_s(X)$.
2. After possibly rearranging the terms, $(N_1, \cdots, N_{h_s})$ form an exceptional sequence with negative terms (terms corresponding to negative $v_i$) coming after all positive terms.

Then, the direct sum of the $h_s$ objects $\psi_s(v_i)$ together with all components of $X$ of slope not equal to $s$ or $s+1$ is another $m$-configuration for $\Lambda$.

**Proof.** This follow directly from the definition of an $m$-configuration. \qed

Recall that the $\tilde{c}$-vectors of an $m$-cluster tilting object of $\Lambda$ are the refined dimension vectors $\tilde{c}_i = \dim M_i t^{s_i} \in \mathbb{Z}[t]^n$ for the components $M_i(s_i) = M_i[m-s_i]$ of the $m$-configuration $X$.

**Theorem 4.3.3.** Suppose that $\tilde{c}_k$ has slope $s$ (resp. $s+1$). Then the positive (resp. negative) mutation $X' = \mu_k^+(X)$ (resp. $X' = \mu_k^-(X)$) of $X$ is uniquely determined by its $\tilde{c}$-vectors $\tilde{c}_j$ which are related to the $\tilde{c}$-vectors $\tilde{c}_j$ of $X$ as follows.

1. $\tilde{c}'_k = \tilde{c}_k t$ (resp. $\tilde{c}'_k = \tilde{c}_k t^{-1}$).
2. $\tilde{c}'_j = \tilde{c}_j$ if the slope of $\tilde{c}_j$ is not equal to $s$ or $s+1$.
3. $\tilde{c}'_j = \tilde{c}_j$ if $b_{kj} \leq 0$ (resp. $b_{kj} \geq 0$).
4. When $b_{kj} > 0$ (resp. $b_{kj} < 0$) and $\tilde{c}_j$ has slope either $s$ or $s+1$ then:
   a) $\tilde{c}'_j$ has slope $s$ or $s+1$.
   b) $\tilde{c}'_j = c_j + |b_{kj}|c_k$ in $\mathbb{Z}^n$.

**Proof that this works:** It satisfies the definition of (14) which is the following.

1. components of the same slope are hom-orthogonal
2. components can be arranged in an exceptional sequence
3. components of smaller slope come before components of larger slope in the exceptional sequence.

We show that these properties hold for $X'$, the mutated $X = \bigoplus X_j$.

Suppose that $\delta = +$ and $X_k$ has slope $s$. Then only those components of $X$ of slopes $s, s+1$ are changed and the number of them, say $h$, is unchanged. Let $X_1, \cdots, X_h$ denote these components of $X$ and let $X_j = M_j(\ell_j) = M_j[m-\ell_j]$ where $M_j$ is a $\Lambda$-module and $\ell_j \in \{s, s+1\}$. Let $A_s$ be the smallest wide subcategory of $mod-\Lambda$ containing the $h$ modules $M_j$. Then $A_s \cong mod-\Lambda_s$ for some hereditary subcategory of $mod-\Lambda$ containing the $h$ modules $M_j$. Then $A_s \cong mod-\Lambda_s$ for some hereditary algebra $\Lambda_s$ of rank $h$ and the vectors $v_j = (-1)^{\ell_j-s} \dim M_j$ form the set of $c$-vectors of a cluster tilting object for $\Lambda_s$ by (18) (since we use the sign convention designed to make this step work).
The mutation formula (⋆) is equivalent, by the correspondence \( \tilde{c}_j \leftrightarrow v_j \), to the standard mutation formula for \( c \)-vectors for \( \Lambda_s \). Furthermore, all components of \( X \) of slope less than \( s \), resp. larger than \( s + 1 \), are right perpendicular, resp. left perpendicular, to all objects in \( \mathcal{A}_s = \text{mod-} \Lambda_s \). By Lemma 4.3.2 the new \( \tilde{c}_j \) vectors satisfy the required conditions.

Let \( X' \) be the mutated \( m \)-configuration. Then the dual \( T' \) of \( X' \) clearly has the property that it differs from the dual \( T \) of \( X \) only in its \( k \)-th component \( T_k \):

(a) For \( i \neq j, k \),

\[
\langle \dim T_i, c'_j \rangle = \langle \dim T_i, c_j \rangle + |b_{kj}| \langle \dim T_i, c_k \rangle = 0
\]

\[
\langle \dim T_i, c'_k \rangle = -\langle \dim T_i, c_k \rangle = 0
\]

(b) For \( i = j \),

\[
\langle \dim T_j, c'_j \rangle = \langle \dim T_j, c_j \rangle + |b_{kj}| \langle \dim T_j, c_k \rangle = f_j
\]

So, \( T_i \) satisfies the tropical equations (Corollary 4.2.2) characterizing \( T'_i \) and, thus, \( T'_i = T_i \) for all \( i \neq k \). The equations which characterize \( T'_k \) are:

\[
\langle \dim T'_k, c'_j \rangle = \langle \dim T'_k, c_j \rangle + |b_{kj}| \langle \dim T'_k, c_k \rangle = 0
\]

\[
\langle \dim T'_k, c'_k \rangle = -\langle \dim T'_k, c_k \rangle = f_k
\]

If we write \( \dim T'_k = \sum a_j \dim T_j \) then the second equation gives \( a_k = -1 \) and the first equation gives:

\[
a_j f_j = \begin{cases} |b_{kj}| f_k & \text{if the sign of } b_{kj} \text{ is } \delta \\ 0 & \text{otherwise} \end{cases}
\]

If the sign of \( b_{kj} \) is \( \delta \) then:

\[
a_j = \begin{cases} |b_{jk}| & \text{if the sign of } b_{kj} \text{ is } \delta \\ 0 & \text{otherwise} \end{cases}
\]

and, by Corollary 4.2.2 \( T'_k \) is uniquely determined by its dimension vector. \( \square \)

In the next paper \cite{12} the wide subcategories \( \mathcal{A}_s \) are used to give a visualization of \( m \)-clusters analogous to the semi-invariant pictures \cite{10} (also known as “scattering diagrams”) used in \cite{5} to derive properties of maximal green sequences.

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**References**


Department of Mathematics, Brandeis University, Waltham, MA 02454
E-mail address: igusa@brandeis.edu