Appendix A

Set Theory

A.1. Sets and functions

A **set** is a collection of objets and can be denoted either by explicitly listing the elements or by giving a rule determining the members of the set. A set A is a **subset** of a set B if each element of A is an element of B and we denote this by $A \subset B$. Two sets A and B are **equal** if they contain the same elements, meaning that $A \subset B$ and $B \subset A$. The **empty set** is the set which contains no elements and is denoted by $\varnothing$. The **complement** of the set A in the set B is the elements of B that are not in A and is denoted by $B - A$. When there is no ambiguity as to the set B, the complement is denoted by A^C.

If A and B are two sets, their **union** $A \cup B$ is the collection of all elements that lie either in A or in B and the **intersection** $A \cap B$ is the collection of all elements lying both both in A and in B. If A_j is a sequence of sets indexed by $j \in J$, we can similarly define the union or intersection over all the elements in J.

If A and B are two sets, the **Cartesian product** $A \times B$ of A and B is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Similary, we can define the Cartesian product for a finite or an infinite collection of sets. For the latter, if $\{A_1, A_2, \ldots\}$ is a collection of sets indexed by $\mathbb{N}$, the **Cartesian product** $A_1 \times A_2 \times \ldots$ is the collection of all $(a_1, a_2, \ldots)$ so that $a_j \in A_j$ for each $j \in \mathbb{N}$. If A and B are sets, a **function** f from A to B is a rule that takes each element of A to one element of B. We write $f\colon A \to B$ and if f maps $a \in A$ to $b \in B$, we write $f(a) = b$. The **domain** $D = D(f)$ of the function f is A and its **range** $R = R(f)$ is the set $\{b \in B\colon$ there exists $a \in A$ so that $f(a) = b\}$. If the range of f is all of B, then f is **onto** or **surjective**. If $f(a) = f(a')$ implies that $a = a'$ for all $a, a' \in A$, then f is **one to one** or **injective**.

If $f\colon A \to B$, then the **inverse** function f^{-1} is defined for $b \in B$ by

$$f^{-1}(b) = \{a \in A : f(a) = b\}$$

and the inverse for a subset $C \subset B$ is defined by

$$f^{-1}(C) = \{a \in A : f(a) \in C\} \ .$$

Note the inverse is itself a function if and only if f is both onto and one to one.

A.2. Orderings

A **relation** R between sets A and B is a subset of the Cartesian product $A \times B$. A relation between A and itself is called a **relation on** A. A **partial ordering** on a set A is a relation $\leq$ defined on $A \times A$ so that

(1) For all $a \in A$, $a \leq a$.
(2) For all $a, b \in A$, $a \leq b$ and $b \leq a$ implies $a = b$.
(3) For all $a, b, c \in A$, $a \leq b$ and $b \leq c$ implies $a \leq c$.

When A has a partial ordering defined on it, it is called a **partially ordered** set. The set A is **totally ordered** if for all $a, b \in A$, either $a \leq b$ or $b \leq a$.

An example of a partial ordering is ordering subsets of the real numbers by inclusion: if $A, B \subset \mathbb{R}$, define $A \leq B$ if $A \subset B$. The usual ordering of the real numbers using $\leq$ is a total ordering.

If A is a partially ordered set and $B \subset A$, then the partial ordering on A **induces** an ordering on B by defining $b \leq b'$ for $b, b' \in B$ if and only if $b \leq b'$, where b and b' are considered as elements of A. If a subset B of a partially ordered set A is totally ordered by the induced ordering on B, then B is a **chain** in A. This means that when B is a chain, for all $b, b' \in B$, either $b \leq b'$ or $b' \leq b$.

Considering, again, subset of the real numbers, an example of a chain is $\{[0, a]\colon a \in \mathbb{R}, a \geq 0\}$.

If A is a partially ordered set and $B \subset A$, an element $a \in A$ is an **upper bound** for B if $b \leq a$ for all $b \in B$. Similarly, $a' \in A$ is a **lower bound** for B if $a' \leq b$ for all $b \in B$. Not every nonempty subset of a partially ordered set need have upper or lower bounds.

If A is a partially ordered set and $B \subset A$, the element $b \in A$ is the **least upper bound** for B if b is an upper bound for B and $b \leq a$ for any upper bound a of B. Similarly, b' is the **greatest lower bound** for B if b' is a lower bound and $a \leq b'$ for any lower bound a of B. Not every nonempty subset of a partially ordered set (or even a totally ordered set) that has an upper bound need have a least upper bound. For example, in the rational numbers, the set of $\{x \in \mathbb{Q} : x \leq \sqrt{2}\}$ is bounded from above but has no least upper bound in $\mathbb{Q}$.

We denote the negation of the relation $\leq$ by $\not\leq$. If A is a partially ordered set and $B \subset A$, the element $m \in B$ is **maximal** in B if $m \not\leq b$ for every $b \in B - \{m\}$. The element $m' \in B$ is **minimal** in B if $b \not\leq m'$ for all $b \in B - \{m'\}$.

We make use of the axiom of choice:

Axiom A.1. *If $A_j, j \in J$, is a family of nonempty sets, then there is a function $f\colon J \to \bigcup_{j\in J} A_j$ so that $f(j) \in A_j$ for each $j \in J$.*

We usually use the axiom of choice in the equivalent form of Zorn's Lemma:

Axiom A.2. *If A is a partially ordered set so that every chain in A has an upper bound, then A contains a maximal element.*

A.3. Equivalence relations

A relation $\sim$ on a set X is an **equivalence relation** if

(1) $x \sim x$ for all $x \in X$;

(2) If $x, y \in X$ so that $x \sim y$, then $y \sim x$;

(3) If $x, y, z \in X$ so that $x \sim y$ and $y \sim z$, then $x \sim z$.

A relation satisfying the first property is called **reflexive**, satisfying the second is called **symmetric** and satisfying the third is called **transitive**.

The standard example of an equivalence relation is ordinary $=$. A partial ordering differs from an equivalence relation only in the second property.

If $\sim$ is an equivalence relation on X, the set of all elements of X that are equivalent to a given element x is called the **equivalence class** of x. Two elements in the same equivalence class are said to be **equivalent**.

Example A.3. Let X and Y be sets and assume that $f\colon X \to Y$ is a function. Define a relation on X by saying that $x \sim y$ if $f(x) = f(y)$. This is an equivalence relation on X and the set of equivalence classes is usually denoted by $X/\sim$. We can use f to define a one to one function $\overline{f}\colon X/\sim \to Y$.

A.4. Cardinality

The set X is **finite** if there is some $n \in \mathbb{N}$ and a one to one correspondence between X and the set $\{1, 2, \ldots, n\}$. If so, we say that X has n elements and write $|X| = n$. If X is not finite, it is **infinite**.

The set X is **countably infinite** if there is a one to one correspondence between X and the natural numbers $\mathbb{N}$. The set X is **countable** if it is either finite or countably infinite and a set which is not countable is **uncountable**.

Example A.4. The integers and rational numbers are countable, but the irrationals and reals are not.

Proposition A.5. *(1) A subset of a finite set is finite.*

(2) A subset of a countable set is countable.

(3) A countable union of countable sets is countable.

(4) A finite product of countable sets is countable.

A countable product of countable sets is not necessary countable.

Appendix B

Groups and vector spaces

B.1. Groups and semigroups

If G is a set and $*$ is a binary operation on G, then G with this operation is a **group** if

(1) The operation $*$ is **associative**, meaning that $(g*h)*k = g*(h*k)$ for alll $g, h, k \in G$.

(2) There is an element $e \in G$, called the **identity** of G, such that $g * e = e * g = g$ for all $g \in G$.

(3) For each element $g \in G$ there exists $h \in G$, called the **inverse** of g, such that $g * h = h * g = e$.

If in addition to these properties, the operation $*$ is **commutative**, meaning that for $g * h = h * g$ for all $g, h \in G$, then the group is said to be **commutative**. A set G with a binary operation $*$ on it satisfying only the first two properties is called a **semigroup**.

Example B.1. The integers $\mathbb{Z}$ with the standard operation of addition is a group, while the natural numbers $\mathbb{N} \cup \{0\}$ with addition is a semigroup. When the operation is clear from the context, such as with $\mathbb{Z}$ or $\mathbb{N}$, we omit explicit reference to the operation.

Example B.2. The interval $[0, 1]$ with addition modulo 1 is a group. The elements of any vector space with vector addition form a group.

B.2. Vector spaces

Definition B.3. A *vector space* over a field F is a set V endowed with the operations of *(vector) addition* and *(vector) multiplication* with addition satisfying associativity, commutativity, existence of an identity element and inverses (meaning that V forms an abelian group under addition), multiplication having an identity element, and such that for all $u, v \in V$ and $v, b \in F$, we have

(1) $a(bv) = (ab)v$.

(2) $a(u + v) = au + av$.

(3) $(a + b)u = au + bu$.

For example, the trivial space $\{0\}$ is a vector space and any field is a vector space.

A nontrivial example is given by considering all n-tuples of elements from a field F, where the operations of addition and multiplication are coordinate-wise. For example, $\mathbb{R}^n$ is as vector space and has the standard basis $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$.

Definition B.4. An *affine subspace* of a vector space V is a subset that is closed under affine combinations of vectors in the space.

For example, if $\{v_i\}_{i \in I}$ is a collection of vectors in V, then

$$\{\sum_{i=1}^{N} a_i v_i \colon \sum_{i=1}^{N} a_i = 1\}$$

is an affine subspace of V.

Appendix C

Topology

C.1. Topologies

A **topology** on a space X is a collection $\mathcal{T}$ of subsets of X so that

(1) $\varnothing, X \in \mathcal{T}$.

(2) The union of elements of any subcollection in $\mathcal{T}$ lies in $\mathcal{T}$.

(3) The intersection of elements of any finite subcollection in $\mathcal{T}$ lies in $\mathcal{T}$.

A **topological space** is a pair $(X, \mathcal{T})$ consisting of a set X and a specified topology $\mathcal{T}$. A subset U of X is **open** if $U \in \mathcal{T}$.

When there is no ambiguity, we say that X itself is a topological space and omit explicitly mentioning the topology $\mathcal{T}$.

A topological space can be specified by giving the space X along with the open sets on this space. By definition, $\varnothing$ and X are both open sets.

The intersection of finitely many open sets is an open set, but this does not necessarily hold for the intersection of infinitely many open sets. A countable intersection of open sets is called a G_δ *set*.

One can always define some topology on any set:

Example C.1. If X is any set, the **trivial topology** on X is the collection of open sets consisting only of $\varnothing$ and X. The collection of all subsets of X is a topology, called the **discrete topology**.

Example C.2. Let $X = \{a, b, c\}$. Then $\big\{X, \varnothing, \{a\}\big\}$ and $\big\{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}\big\}$ are two examples of topologies on X.

Example C.3. If X is a set, we can define a topology $\mathcal{T}$ on X consisting of the collection of all subsets U of X so that $X - U$ is either finite or is all of X. More generally, we could take the collection of all subsets U of X so that $X - U$ is either countable or is all of X.

If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X, $\mathcal{T}$ is **finer** than $\mathcal{T}'$ if $\mathcal{T} \supset \mathcal{T}'$. We also say that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$. Two topologies are not necessarily comparable.

The discrete topology is finer than all other topologies and the trivial topology is coarser than all other topologies.

If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of open sets, called the **basis elements**, so that

(1) For each $x \in X$, there is some basis element $B \in \mathcal{B}$ with $x \in B$.
(2) If $x \in B_1 \cap B_2$ for basis elements $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ with $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ is a basis for a topology on X, then the **topology generated by** $\mathcal{B}$ has for its open sets all subsets U of X such that for each $x \in U$, there exists $B \in \mathcal{B}$ so that $x \in B$ and $B \subset U$. In particular, this means that all basis elements are open sets. More generally, a topology is the union of elements of a basis:

Proposition C.4. *If X is a set and $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X, then $\mathcal{T}$ is the collection of unions of elements of $\mathcal{B}$.*

Proof. The elements of $\mathcal{B}$ are by definition elements of $\mathcal{T}$. Since $\mathcal{T}$ is a topology, their union also lies in $\mathcal{T}$. Conversely, given a set $U \in \mathcal{T}$, for each $x \in U$, there exists a set $B_x \in \mathcal{B}$ with $x \in B_x$ and $B_x \subset U$. Therefore $U = \bigcup_{x \in U} B_x$ and so U is a union of elements of $\mathcal{B}$. □

Example C.5. For any set X, a basis for the discrete topology is given by the collection of all one point sets.

Example C.6. Consider the collection of all open intervals (a, b), where $a, b \in \mathbb{R}$ and $a < b$. The topology generated by this collection is called the **standard topology** on $\mathbb{R}$.

Example C.7. The collection of the interiors of all rectangles in the plane is a basis. One can define many different bases that generate the same topology as is generated by this basis. Another such basis is the interiors of all circles in the plane. See Exercise C.21.

It is useful to have a way to find a basis for a topology:

Proposition C.8. *Assume that X is a topological and that $\mathcal{C}$ is a collection of open sets of X such that for every open set U of X and every $x \in U$,* *there exists some $C \in \mathcal{C}$ so that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. We first show that $\mathcal{C}$ is a basis. For any $x \in X$, there is a basis element containing x: since X itself is an open set, we can pick any $C \in \mathcal{C}$ with $x \in C$. Now assume that $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$. Since C_1, C_2 are open, so is $C_1 \cap C_2$ and so there exists $C_3 \in \mathcal{C}$ so that $x \in C_3 \subset C_1 \cap C_2$. Therefore $\mathcal{C}$ is a basis.

Let $\mathcal{T}$ denote the topology of X and $\mathcal{T}'$ the topology on X generated by $\mathcal{C}$. We claim that they generate the same topology. Each element of $\mathcal{C}$ is an element of $\mathcal{T}$ and so arbitrary unions of elements of $\mathcal{C}$ are also in $\mathcal{T}$ since $\mathcal{T}$ is a topology. Therefore $\mathcal{T}$ is finer than $\mathcal{T}'$. Conversely, if U is an element of $\mathcal{T}'$ and $x \in U$, then since $\mathcal{C}$ generates $\mathcal{T}'$, there is an element $C \in \mathcal{C}$ with $x \in C \subset U$. By hypothesis, there is an element C' in the basis for $\mathcal{T}$ so that $x \in C' \subset C$. Thus $x \in C' \subset U$ and so $U \in \mathcal{T}$. □

C.2. Generating topological spaces

If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generated by the basis $\mathcal{B}$, where the open sets of $\mathcal{B}$ are all sets of the form $U \times V$ with U an open set in X and V an open set in Y (Exercise C.22).

If X is a topological space with topology $\mathcal{T}$ and $Y \subset X$, then the **subspace topology** or **relative topology** $\mathcal{T}_Y$ defined on Y is given by

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\} .$$

(See Exercise C.23.) We call Y a **subspace** of X and note that its open sets are the intersections of open sets of X with Y.

A subset may be open in Y without being an open set in X. In particular, this occurs when Y is not open in X.

Assume that X is a topological space with topology $\mathcal{T}$ and $X = \bigcup_{A \in \mathcal{A}} A$ is a partition of X into a union of disjoint nonempty open sets. The map $\pi \colon X \to \mathcal{A}$ taking each $x \in X$ to the element of $\mathcal{A}$ containing it is called the **quotient map**. The topology consisting of all subsets A of $\mathcal{A}$ so that $\pi^{-1}(A)$ is open in X is called the **quotient topology**.

C.3. Closed sets

A subset A of a topological space X is **closed** if $X - A$ is open.

The **interior** of a subset A of a topological space is the union of all open sets contained in A and is denoted by A^o. The **closure** of A is the intersection of all closed sets containing A and is denoted by $\overline{A}$. The **boundary** of A is $\overline{A} - A^o$ and is denoted ∂A.

If A is open, then $A = A^o$. If A is closed, then $A = \overline{A}$. The boundary $\partial A = \varnothing$ if and only if A is both open and closed.

Example C.9. The interval $[0, 1] \subset \mathbb{R}$ is closed, since its complement $(-\infty, 0) \cup (1, \infty)$ is the union of two open intervals. The interval $[0, 1) \subset \mathbb{R}$ is neither open nor closed. In the discrete topology, all sets are both open and closed.

The subset $A \subset X$ is **dense** in B if every open subset of B contains a point of A. Alternately, A is dense in B if $\overline{A} = B$.

Example C.10. The rationals $\mathbb{Q}$ are dense in $\mathbb{R}$. The irrationals are dense in $\mathbb{R}$. The integers $\mathbb{Z}$ are not dense in $\mathbb{R}$.

A **neighborhood** of a point x is an open set U so that $x \in U$.

If $A \subset X$, we say that x is a **limit point** of A if every neighborhood of x has nontrivial intersection with A in some point other than x itself. The point x may or may not belong to A.

Example C.11. If $X = \mathbb{R}$ and $A = \{1/n : n \in \mathbb{N}\}$, then 0 is a limit point of A. If we let $A' = A \cup \{0\}$, then 0 is also a limit point of A'. If $B = [0, 1]$, then every point in B is limit point of B and if $B' = (0, 1)$, then every point in $[0, 1]$ is a limit point of B'.

A point $x \in X$ is said to be **isolated** if the set $\{x\}$ consisting only in the point x is a neighborhood of x.

Lemma C.12. *If X is a topological space and $A \subset X$, then A is a closed subset of X if and only if all limit points of A are elements of A.*

Proof. Assume that A is closed and suppose that there is some limit point x of A and $x \in X - A$. Since A is closed, $X - A$ is open . Thus $X - A$ is a neighborhood of x that does not contain any points of A, a contradiction. Conversely, suppose that every limit point of A lies in A. We show that the complement of A in X is open. Let $x \in X - A$. Since x is not a limit point of A, there is a neighborhood U_x of x so that $U_x \cap A = \varnothing$. Then $U = \bigcup_{x \in X - A} U_x$ is an open subset of X and so $A = X - U$ is closed in X. $\square$

C.4. Continuity

If X and Y are topological spaces, the function $f\colon X \to Y$ is **continuous** if for all open sets V of Y, the set $f^{-1}(V)$ is open in X.

Understanding when a function is continuous depends not only on the definition of the function, but also on the topologies of X and Y. To check continuity of a function, it suffices to check this condition on the basis elements of the topology on Y:

Lemma C.13. *Let X and Y be topological spaces and let $f\colon X \to Y$. Then f is continuous if and only if $f^{-1}(B)$ is open for all V in a basis for the topology on Y.*

Proof. If f is continuous, then $f^{-1}(B)$ is open for all open sets B and so in particular, for basis elements. Conversely, assume that the inverse image under f of every basis element is open. An arbitrary open set V in Y can be written as $V = \bigcup_{j\in J} B_j$, where for each j, B_j is a basis element, and so the inverse image

$$f^{-1}(V) = \bigcup_{j\in J} f^{-1}(B_j)$$

is also open. □

Example C.14. If X is a topological space, then the identity map from X to X is continuous. If X has the discrete topology on it, then every $f\colon X \to X$ is continuous.

Example C.15. If $f\colon \mathbb{R} \to \mathbb{R}$, then this abstract definition of continuity agrees with the standard $\epsilon - \delta$ definition. Given the above definition, if $x_0 \in \mathbb{R}$ and $\epsilon > 0$, then the interval $U = \big(f(x_0) - \epsilon, f(x_0) + \epsilon\big)$ is an open set in the range $\mathbb{R}$. Thus $f^{-1}(U)$ is an open set in the domain $\mathbb{R}$. Since $x_0 \in f^{-1}(U)$, we have that $(a, b) \subset f^{-1}(U)$ for some basis element (a, b) around x_0. Taking $\delta = \min(x_0 - a, b - x_0)$, then for $|x - x_0| < \delta$, the point $x \in (a, b)$. Thus $f(x) \in U$ and $|f(x) - f(x_0)| < \epsilon$. The converse implication is similar. The same argument also shows that this definition agrees with the standard $\epsilon - \delta$ definition for $f\colon \mathbb{R}^n \to \mathbb{R}^n$ for any $n \in \mathbb{N}$. This is generalized in Theorem F.4.

There are many equivalent conditions for checking continuity. We give a few.

Theorem C.16. *Let X and Y be topological spaces and assume that $f\colon X \to Y$ is a function. The following are equivalent:*

(1) f is continuous.

(2) For all subsets A of X, $f(\overline{A}) \subset \overline{f(A)}$.

(3) For all closed subsets B of Y, $f^{-1}(B)$ is closed in X.

Proof. (1) $\Longrightarrow$ (2). Let $A \subset X$ and let $x \in \overline{A}$. If U is a neighborhood of $f(x)$, then $f^{-1}(U)$ is an open set in X and $x \in f^{-1}(U)$. Thus $f^{-1}(U) \cap A \neq \varnothing$ and so $U \cap f(A) \neq \varnothing$ and we have $f(x) \in \overline{f(A)}$.

(2) $\implies$ (3). Assume that B is closed and let $A = f^{-1}(B)$. Then $f(A) \subset B$. If $x \in \overline{A}$. Then $f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B}$. Since B is closed, $\overline{B} = B$ and so $x \in f^{-1}(B) = A$. Therefore $\overline{A} \subset A$ and so A is closed.

(3) $\implies$ (1). Assume that U is open in Y. Then $Y - U$ is closed in Y and so $f^{-1}(Y-U)$ is closed in X. But $f^{-1}(U) = f^{-1}(Y - (Y-U)) = f^{-1}(Y) - f^{-1}(Y-U) = X - f^{-1}(Y-U)/$ Therefore $f^{-1}(U)$ is open. □

The next theorem gives various ways of constructing continuous functions:

Theorem C.17. *Assume that X, Y and Z are topological spaces.*

(1) A constant map $f\colon X \to Y$ is continuous.

(2) The identity map on any topological space is continuous.

(3) If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then the composition $g \circ f\colon X \to Z$ is continuous.

(4) If A is a subspace of X, then the inclusion map $j\colon A \to X$ is continuous.

(5) If A is a subspace of X and $f\colon X \to Y$ is a continuous map, then the restriction of f to A, denoted $f|_A$, is a continuous map.

(6) If $f\colon A \to X \times Y$ is given by $f(a) = (f_1(a), f_2(a))$, then f is continuous if and only if $f_1\colon A \to X$ and $f_2\colon A \to Y$ are continuous.

Proof. (1) Assume that $f(x) = y_0$ for all $x \in X$. For any open set V in Y, $f^{-1}(V)$ is X if $y_0 \in V$ and is $\varnothing$ if $y_0 \notin V$. By definition, both $\varnothing$ and X are open.

(2) The inverse of the identity map is itself, and so any open set is mapped to itself.

(3) If V is an open set in Z, then $g^{-1}(V)$ is an open set in Y and $f^{-1}(g^{-1}(V))$ is an open set in X. This set is exactly $(g \circ f)^{-1}(V)$.

(4) If V is an open set in X, then $j^{-1}(V) = V \cap A$, which is open in A by definition of the subspace topology.

(5) The restricted function is the composition of the inclusion map $j\colon A \to X$ and the map $f\colon X \to Y$ and so is itself continuous.

(6) The function f_1 is the composition of f and the projection map $\pi_1\colon X \times Y \to X$ to the first coordinate. If U is an open set in X, then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$ and so the projection is continuous. By part (3), the composition is continuous. Similarly, f_2 is the composition of two continuous maps and so is itself continuous. Conversely, assume that f_1 and f_2 are continuous and let $U \times V$ be a basis element for the topology of $X \times Y$. Then $a \in f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$, meaning that

$f_1(a) \in U$ and $f_2(a) \in V$. Thus $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$, which is the intersection of two open sets. □

Theorem C.18. *Assume that X is a topological space and that $f, g\colon X \to \mathbb{R}$ are continuous functions. Then $f+g$, $f-g$ and $f \cdot g$ are continuous and if $g(x) \neq 0$ for any $x \in X$, then f/g is continuous.*

Proof. Note that addition from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ is continuous. Define $h\colon X \to \mathbb{R} \times \mathbb{R}$ by

$$h(x) = f(x) \times g(x) \ .$$

This map is continuous by part (6) of Theorem C.17. Then $f+g$ is the composition of these two maps and so is itself continuous. Similar proofs work for the other functions. □

C.5. Topological properties

If X and Y are topological spaces and $f\colon X \to Y$ is a bijection so that both f and f^{-1} are continuous, then f is **homeomorphism**. Equivalently, f is a homeomorphism if U is open in X if and only if $f(U)$ is open in Y.

A homeomorphism gives a correspondence between the open sets of one space and the open sets of another space. Any property of a topological space that only depends on the open sets (and thus only on the topology) is thus preserved under homeomorphisms. Such a property is called **topological property** of X.

Example C.19. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x+1$. Then f is a homeomorphism and its inverse is $g(y) = \frac{1}{3}(y-1)$.

Example C.20. The function $f\colon X \to Y$ can be continuous and bijective without being a homeomorphism. Consider the circle $S^1 = \{(x,y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2\}$ and let $f\colon [0,1) \to S^1$ be defined by $f(x) = \big(\cos(2\pi t), \sin(2\pi t)\big)$. Then f is bijective and continuous, but f^{-1} is not continuous.

Exercises

Exercise C.21. Show that each of the collections in Example C.7 is a basis and that these bases generate the same topology.

Exercise C.22. Show that if X and Y are topological spaces, then the collection of sets of the form $U \times V$ with U open in X and V open in Y is a basis.

Exercise C.23. Show that $\mathcal{T}_Y$ is a topology.

Exercise C.24. Show that a topology can be defined by using closed sets instead of open sets, by showing that

(1) $\varnothing, X$ are closed sets.

(2) Intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Appendix D

Connected Spaces

The topological space X is **disconnected** if there exist nonempty open sets U, V so that $U \cup V = X$ and $U \cap V = \varnothing$. The sets U, V are called a **disconnecting set**. If X is not disconnected, then X is **connected**. An equivalent formulation is:

Proposition D.1. *The space X is connected if and only if the only sets that are both open and closed in X are $\varnothing$ and X.*

Proof. If X is not connected, then there exist $U, V \subset X$ so that both U and V are open and closed, since $V = X - U$ and $U = X - V$. Conversely, if U is a nonempty subset of X that is both open and closed, then $U, X - U$ disconnects X. □

As the definition of connected depends on open and closed sets, one needs to know the topology of X in order to determine whether or not the space is connected. This implies that if X is connected, then so is any space homeomorphic to X, meaning that connectivity is a topological property.

Clearly any one point subset is connected. A space X is said to be **totally disconnected** if the only connected subsets in X are the one point subsets.

Example D.2. If X contains at least two points and has the discrete topology on it, then X is disconnected and moreover, is totally disconnected. If X has the trivial topology on it, then X is connected, since the only nonempty subset of X is X.

Example D.3. The rationals are not connected. Given any two rationals p and q, choose an irrational a with $p < a < q$. Then $(-\infty, a), (a, \infty)$ is a disconnecting set.

Example D.4. The interval $[0, 1]$ is connected. If not, we can write $[0, 1] = U \cup V$, where U and V are nonempty disjoint open subsets. Pick $u \in U$ and $v \in V$ and without loss, assume that $u < v$. Let $A = \{a \in \mathbb{R} : [u, a] \subset U\}$. Then A is bounded from above by 1 and so has a least upper bound a_0 with $0 < a_0 < 1$. Therefore a_0 lies either in U or in V. If $a_0 \in U$, then since U is open, there exists $\epsilon > 0$ so that $(a_0 - \epsilon, a_0 + \epsilon) \subset U$, meaning in particular that $a_0 + \epsilon/2 \in U$. This contradicts a_0 being an upper bound and so instead we must have $a_0 \in V$. Again there exists $\epsilon > 0$ so that $(a_0 - \epsilon, a_0 + \epsilon) \subset V$. Then $a_0 - \epsilon/2$ is an upper bound for A, contradicting the choice of a_0 as a least upper bound. Thus there is no disconnecting set for $[0, 1]$ and so it is connected.

Proposition D.5. *If $X = U \cup V$ with U, V nonempty disjoint open subsets of X, then if $A \subset X$ is connected, then either $A \subset U$ or $A \subset V$.*

Proof. Assume that $A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing$. Then $A \cap U$ and $A \cap V$ are nonempty disjoint subsets that are open in A. However, $A = (A \cap U) \cup (A \cap V)$, a contradiction of A being connected. Therefore either $A \cap U = \varnothing$ or $A \cap V = \varnothing$, meaning that $A \subset V$ in the former case and $A \subset U$ in the latter case. □

Proposition D.6. *If $A \subset X$ is connected and $A \subset B \subset \overline{A}$, then B is also connected.*

Proof. Assume that A is connected and $A \subset B \subset \overline{A}$ but that B is not connected. Then $B = U \cup V$ where U, V are nonempty open disjoint subsets of X. By Proposition D.5, since $A \subset B$, either $A \subset U$ or $A \subset V$. If $A \subset U$, then $\overline{A} \subset \overline{U}$ and so $B \cap V = \varnothing$, since $\overline{U} \cap V = \varnothing$, contradicting the fact that $V \subset B$ is not empty. □

Proposition D.7. *If $X_j, j \in J$, are connected sets so that $\bigcap_{j \in J} X_j \neq \varnothing$, then $X = \bigcup_{j \in J} X_j$ is connected.*

Proof. Assume that $X = U \cup V$, with U, V nonempty disjoint open subsets of X. Then by Proposition D.5, for each $j \in J$, either $X_j \subset U$ or $X_j \subset V$. If $A_j \subset U$ for some $j \in J$, then since $\bigcap_{j \in J} X_j \neq \varnothing$, each A_i has nonempty intersection with U and so lies entirely within U. Therefore $X = \bigcup_{j \in J} A_j \subset U$ and $V = \varnothing$, meaning that X is connected. The same holds if $A_j \subset V$ for some $j \in J$. □

Proposition D.8. *The image of a connected space under a continuous map is connected.*

Proof. Assume that X is connected and that $f\colon X \to Y$ is continuous. The restriction of f to its range $f(X)$ is also continuous and so it suffices to assume that f is surjective and $Y = f(X)$. We show that Y is connected. If not, then there exist disjoint nonempty open sets U and V so that $Y = U \cup V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty (since f is surjective), open (since f is continuous), disjoint (since U and V are) and their union is all of X, contradicting the fact that X is connected. □

Using these theorems, we can construct other examples of connected spaces from spaces already known to be connected. For example, combining Proposition D.8 and Example D.4, we have that any interval in the real line is connected.

Appendix E

Compact Spaces

E.1. Hausdorff spaces

Intuitively, we would like finite sets, as they are in $\mathbb{R}$, to be closed sets. However, it is easy to define topologies without this property. For example, if $X = \{a, b, c\}$ and $\mathcal{T} = \big\{\varnothing, X, \{b\}\big\}$, then the set $\{b\}$ is not closed, since its complement $\{a, b\}$ is not open. We usually add an extra hypothesis on the spaces in order to satisfy this geometric intuition.

A topological space X is a **Hausdorff space** if for all $x, y \in X$ with $x \neq y$, there exist neighborhoods U and V of x, y, respectively, so that $U \cap V = \varnothing$.

Theorem E.1. *A finite set in a Hausdorff space is closed.*

Proof. Consider a one point set $\{x\}$. For any $y \neq x$, there exist open sets U and V so that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. Since $V \cap \{x\} = \varnothing$, y is not in the closure of the set $\{x\}$. Therefore the closure of the set $\{x\}$ is the set itself, meaning that it is closed. Since a finite set is the union of finitely many one point sets, a finite set of points is closed. □

Theorem E.2. *If X is a Hausdorff space and $A \subset X$, then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. Assume that x is a limit point of A, but that some neighborhood U of x intersects A only in finitely many points. Then U also intersects $A - \{x\}$ in finitely many points $x_1, x_2, \ldots, x_n$. By Theorem E.1 the set $\{x_1, x_2, \ldots, x_n\}$ is closed and so $X - \{x_1, x_2, \ldots, x_n\}$ is open. But then

$U \cap (X - \{x_1, x_2, \ldots, x_n\})$ is a neighborhood of x that does not intersect $A - \{x\}$, a contradiction of x being a limit point of A.

Conversely, if any neighborhood of x contains infinitely many points of A, then the neighborhood must intersect A in some point other than x and so x is a limit point of A. □

Theorem E.3. *A subspace of a Hausdorff space is Hausdorff. The product of countably many Hausdorff spaces is Hausdorff.*

Proof. Assume that X is Hausdorff and Y is a subspace of X. Let $x, y \in Y$ be distinct points. There exist open sets U and V in X with $x \in U$ and $y \in V$ so that $U \cap V = \varnothing$. Then $x \in U \cap Y$ and $y \in V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \varnothing$ and so Y is Hausdorff.

For the second statement, assume that $X_n, n \in \mathbb{N}$, is a countable collection of Hausdorff spaces and let $Y = \prod_{n\in\mathbb{N}} X_n$. Let x, y be distinct points in Y and write $x = \{x_n : n \in \mathbb{N}\}$ and $y = \{y_n : n \in \mathbb{N}\}$. Since x and y are distinct, for at least one $j \in \mathbb{N}$, $x_j \neq y_j$. Therefore, we can choose open sets U_j and V_j in X_j so that $x_j \in U_j$ and $y_j \in V_j$ and $U_j \cap V_j = \varnothing$. Let $U = \prod_{n\in\mathbb{N}} W_n$, where $W_n = X_n$ for all $n \neq j$ and $W_j = U_j$ and let $V = \prod_{n\in\mathbb{N}} W'_n$, where $W'_n = X_n$ for all $n \neq j$ and $W'_j = V_j$. Then U is a neighborhood of x and V is a neighborhood of y and since points in U and V do not agree on the j-th coordinate, $U \cap V = \varnothing$. □

Although many of the theorems we prove do not actually need that the space be Hausdorff, we usually include it as an assumption in order to simplify the presentation.

E.2. Compact Spaces

A collection of subsets of a space X **covers** X if the union of all elements in the collection equals X. It is an **open covering** if each element in the collection is open. The space X is **compact** if every open covering of X contains a finite collection that covers X. This finite collection is called a **subcovering**.

A set is **perfect** if it is compact and has no isolated points.

Clearly any finite set is compact, as any open covering of this set is itself finite. In general, there is no simple check if a set is compact.

Example E.4. The real line $\mathbb{R}$ is not compact. The collection of intervals $\{(n, n+5) : n \in \mathbb{Z}\}$ covers $\mathbb{R}$ but has no finite subcovering.

Example E.5. The open interval $(0, 1)$ is not compact. Let $U_n = (1/n, 1 - 1/n)$. Then the collection $\{U_n : n \in \mathbb{N}\}$ is an open cover of $(0, 1)$ but does not contain any finite subcovering.

Example E.6. The closed interval $[0, 1]$ is compact. Assume that $\{U_n\}$ is an open cover of $[0, 1]$. Let

$$C = \{x \in [0,1] : \text{finitely many of the } U_n \text{cover } [0,x)\} .$$

Then $C \neq \varnothing$ since some open set covers 0. Furthermore, 1 is an upper bound for C. Therefore C has a least upper bound c. If $c = 1$, then we have finitely many sets in the collection cover $[0, 1)$ and by adding in a set that covers 1, we have a finite covering for $[0, 1]$. So assume that $0 \leq c < 1$. If $c \in C$, assume that $U_{j_1}, U_{j_2}, \ldots, U_{j_k}$ is a covering for $[0, c)$. Choose some set $U_{k'}$ with $c \in U_{k'}$. Then $\{U_{j_1}, U_{j_2}, \ldots, U_{j_k}, U_{k'}\}$ is an open cover of $[0, c)$, contradicting the fact that c is an upper bound for C. If $c \notin C$, there is some U_k so that $c \in U_k$ and no finite collection of the U_n cover $[0, c) - U_k$, contradicting the choice of c as the least upper bound for C. Therefore $c = 1$ and so $[0, 1]$ is compact.

The proof given in this example is general and can be used to show that any closed interval in the real line is compact.

We now prove some simple facts about compact spaces.

Theorem E.7. *A subspace Y of a space X is compact if and only if every open covering of Y by sets open in X contains a finite subcollection covering Y.*

Proof. Assume that Y is compact and let $\mathcal{U} = \{U_j\}_{j \in J}$ be a covering of Y by sets open in X. Then

$$\{U_j \cap Y : j \in J\}$$

is a covering of Y by sets open in Y. Therefore some finite subcollection $\{U_{j_1} \cap Y, U_{j_2} \cap Y, \ldots, U_{j_n} \cap Y\}$ covers Y and so $\{U_{j_1}, U_{j_2}, \ldots, U_{j_n}\}$ is a finite subcollection that covers Y.

Conversely, if $\mathcal{U} = \{U_j\}_{j \in J}$ is a covering of Y by open sets in Y, then for each $j \in J$, we can choose a set V_j that is open in X so that $U_j = V_j \cap Y$. Then $\mathcal{V} = \{V_j\}_{j \in J}$ is a covering of Y by sets open in X. By assumption, some finite subcollection of $\mathcal{U}$ covers Y and the corresponding subcollection of $\mathcal{V}$ covers Y. □

Theorem E.8. *A closed subset of a compact space is compact.*

Proof. Assume that Y is a closed subset of the compact space X and let $\mathcal{U}$ be a covering of Y by sets that are open in X. Then $\mathcal{U} \cup X - Y$ is an open covering of X and so some finite subcollection covers X. Then this same subcollection (omitting $X - Y$ if it is included) is a finite covering for Y. □

Theorem E.9. *A compact subset of a Hausdorff space is closed.*

Proof. Assume that Y is a compact subset of the Hausdorff space X. We show that $X - Y$ is open. Let $x_0 \in X - Y$. Since X is Hausdorff, for each $y \in Y$, we can choose neighborhoods U_y of x_0 and V_y of y so that $U_y \cap V_y = \varnothing$. Then $\mathcal{V} = \{V_y : y \in Y\}$ is an open covering of Y and so a finite subcollection $V_{y_1}, V_{y_2}, \ldots, V_{y_k}$ covers all of Y. Set $V = V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_k}$ and $U = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_k}$. Then V is an open set containing Y and is disjoint from the open set U. Therefore there is a neighborhood of x_0 that is disjoint from Y. Taking the union over all x of such neighborhoods, we have that $X - Y$ is open. □

Corollary E.10. *A subset of a compact Hausdorff space is compact if and only if it is closed.*

Theorem E.11. *If $f : X \to Y$ is continuous and onto and X is compact, then Y is compact.*

Proof. Assume that $\mathcal{U}$ is an open covering of $f(X)$. Since f is continuous,

$$\{f^{-1}(U) : U \in \mathcal{U}\}$$

is an open covering of X and so a finite subcollection $f^{-1}(U_1), f^{-1}(U_2), \ldots,$ $f^{-1}(U_k)$ covers all of X. Then $U_1, U_2, \ldots, U_k$ covers all of Y. □

Corollary E.12. *Assume that X is compact and Y is Hausdorff. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.*

Proof. It remains to show that f^{-1} is continuous and so it suffices to show that images of closed sets in X are closed in Y. Assume that A is a closed set in X. Then A is compact by Theorem E.8. By Theorem E.11, $f(A)$ is compact. By Theorem E.9, $f(A)$ is closed in Y. □

Theorem E.13. *If X and Y are compact, then $X \times Y$ is compact.*

Proof. Assume that $\mathcal{U}$ is an open cover of $X \times Y$. For each $x_0 \in X$, the slice $x_0 \times Y$ is compact (it is homeomorphic to Y) and so finitely many elements $U_1, U_2, \ldots, U_k$ of $\mathcal{U}$ cover it. Let $A = U_1 \cup U_2 \cup \ldots \cup U_k$. Then A is an open set that contains the slice $x_0 \times Y$.
Claim: A contains $B \times Y$ for some open set B containing x_0.

For each point $w \in x_0 \times Y$, choose open subsets B_w in X and V_w in Y so that $B_w \times V_w \subset A$. Taking the union over all w in the slice, we have an open cover of $x_0 \times Y$ and by compactness a finite subcollection $B_{w_1} \times V_{w_1} \cup \ldots \cup B_{w_k} \times V_{w_k}$ covers all of $x_0 \times Y$. Set $B = B_{w_1} \cap B_{w_2} \cap \ldots \cap B_{w_k}$. Then

$$B \times Y \subset \bigcup_{j=1}^{k} (B_{w_j} \times V_{w_j}) \subset A$$

and the claim is proven.

By compactness, $B \times Y$ is covered by finitely many elements $U_1, U_2, \ldots, U_m$ of $\mathcal{U}$. Therefore, for each $x \in X$, we can choose an open set W_x of x so that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{U}$. By the compactness of X, there is a finite subcollection $W_1, W_2, \ldots, W_n$ that covers X. Furthermore, we have that $W_1 \times Y, W_2 \times Y, \ldots, W_n \times Y$ covers all of $X \times Y$. As each of these sets can be covered with finitely many elements of $\mathcal{U}$, we have that $X \times Y$ can be covered by finitely many elements of $\mathcal{U}$. □

This is a special case of a more general theorem known as Tychonoff's Theorem: the product of countably many nonempty spaces is compact if and only if each of the spaces is compact.

E.3. Finite intersection property

A collection $\mathcal{A}$ of subsets of X satisfies the **finite intersection property** if for every finite collection $\{A_1, A_2, \ldots, A_n\}$ of $\mathcal{A}$, the intersection $A_1 \cap A_2 \cap \ldots \cap A_n \neq \varnothing$.

Theorem E.14. *A topological space X is a compact if and only if for every collection $\mathcal{A}$ of closed sets satisfying the finite intersection condition, $\bigcap_{A \in \mathcal{A}} A \neq \varnothing$.*

Proof. Consider the collection $\mathcal{U} = \{X - A : A \in \mathcal{A}\}$. This is a collection of open sets since $\mathcal{A}$ consists of closed sets. The collection $\mathcal{A}$ covers X if and only if $\bigcap_{U \in \mathcal{U}} U = \varnothing$. A finite collection $\{A_1, A_2, \ldots, A_n\}$ of $\mathcal{A}$ covers X if and only if $U_1 \cap U_2 \cap \ldots \cap U_n = \varnothing$, where $U_j = X - A_j$, for $j = 1, 2, \ldots, n$. The space X being compact is equivalent to showing that for any collection $\mathcal{U}$ of open sets so that no finite subcollection covers all of X, then the whole collection $\mathcal{U}$ does not cover X, and so by applying this to the complementary set $\mathcal{U}$, we have the statement of the theorem. □

Appendix F

Metric Spaces

A **metric** on a set X is a function $d\colon X \times X \to \mathbb{R}$ so that

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

(2) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(3) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

The function d is called the **metric** or **distance** function on X. The third property is known as the **triangle inequality**.

F.1. Metric Topology

For $\epsilon > 0$, the **ϵ-ball around** x is the set of points with distance ϵ of x and is denoted by $B(x; \epsilon)$. Thus

$$B(x; \epsilon) = \{y : d(x, y) < \epsilon\} \ .$$

In case of ambiguity as to which distance function is meant, we write $B_d(x; \epsilon)$.

If d is a metric on X, then the collection of $B(x; \epsilon)$ for all $x \in X$ and all $\epsilon > 0$ is a basis for a topology on X called the **metric topology** (Exercise F.21).

Example F.1. Given a set X, define a metric d on X by

$$\begin{aligned} d(x, y) &= 1 \quad if\ x \neq y \\ d(x, y) &= 0 \quad if\ x = y\ . \end{aligned}$$

The topology induced by d is the discrete topology.

Example F.2. The standard metric on $\mathbb{R}$ is given by $d(x, y) = |x - y|$. It induces the standard topology on $\mathbb{R}$.

Example F.3. On Euclidean space $\mathbb{R}^n$, define the **norm** $||x||$ of $x = (x_1, x_2, \ldots, x_n)$ by $||x|| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$. The **Euclidean metric** d is defined by

$$d(x, y) = ||x - y|| = \left((x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2\right)^{1/2} .$$

The square metric ρ is defined by

$$\rho(x, y) = \max\left\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\right\} .$$

For $n = 1$, these two metrics agree, but the differ for $n \geq 2$. However, the topologies induced by both metrics agree with the product topology (exercise F.22).

A topological space X is said to be **metrizable** if there exists a metric d on the set X that induces the topology of X. A **metric space** is a pair (X, d), where X is a nonempty metrizable space and d is a metric that induces the topology of X.

Two metrics on a space X are **equivalent** if they generate the same topologies (see Exercise F.23).

A metrizable topological space X is Hausdorff: given $x, y \in X$ with $x \neq y$, taking balls around x and y of size $\epsilon = d(x, y)/2$, we have separated the two points.

If A is a subspace of the metric space (X, d), then the restriction of d to A generates a metric topology on A. The product of two metric spaces is a metric space.

If X is a metric space with metric d, a subset A of X is said to be **bounded** if there exists some M so that $d(x, y) \leq M$ for all $x, y \in A$. For a bounded set A, the **diameter** of A is defined to be

$$\operatorname{diam} A = \operatorname{lub}\{d(x, y) : x, y \in A\} .$$

Boundedness is not a topological property, as it depends on the metric used to give the topology.

F.2. Continuous functions in metric spaces

In a metric space, there are many other useful equivalent conditions for a function to be continuous. The first is the equivalence with the ϵ-δ definition of continuity from calculus.

Theorem F.4. *Let X and Y be metrizable spaces with metrics d_X and d_Y, respectively. Then $f : X \to Y$ is continuous if and only if for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ so that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \epsilon$.*

Proof. Assume that f is continuous and let $x \in X$ and $\epsilon > 0$. Consider the ball $B = B(f(x);\epsilon)$ around $f(x)$ of radius ϵ. Then $f^{-1}(B)$ is an open set and contains the point x. Therefore there exists some $\delta > 0$ so that it contains the ball around x of radius δ. Taking any y in this ball, we have that $f(y)$ lies in B. Conversely, assume that U is an open set in Y and let $x \in f^{-1}(U)$. Then $f(x) \in U$ and so there exists some ϵ-ball $B(f(x);\epsilon)$ contained in U. By assumption, there exists $\delta > 0$ so that $f(B(x;\delta)) \subset B(f(x);\epsilon)$, meaning that $B(x;\delta)$ is contained in $f^{-1}(U)$ and so $f^{-1}(U)$ is open. □

A sequence of points $x_1, x_2, \dots$ in a metrizable space X is said to **converge** to $x \in X$, if for any neighborhood U of x, there exists N so that $x_n \in U$ for all $n \geq N$. We write $x_n \to x$.

In a Hausdorff space, the limit of a convergent sequence is unique: if x_n converges to x and $y \neq x$, then by choosing disjoint neighborhoods U_x of x and U_y of y, since U_x contains x_n for all sufficiently large n, U_y can not.

Theorem F.5. *Let X be a metrizable space. Then $f\colon X \to Y$ is continuous if and only if for any convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$.*

Proof. Assume that f is continuous and that $x_n \to x$. Let U be a neighborhood of $f(x)$. Then $f^{-1}(U)$ is a neighborhood of x and so there exists N so that $x_n \in f^{-1}(U)$ for all $n \geq N$. This means that $f(x_n) \in U$ for all $n \geq N$.

Conversely, let A be a subset of X. By Theorem C.16, it suffices to show that $f(\overline{A}) \subset \overline{f(A)}$. Assume that $x \in \overline{A}$. For a positive integer n, let $B(x;1/n)$ be the ball of radius $1/n$ around the point x. For each n, choose $x_n \in B(x;1/n) \cap A$. If U is a neighborhood of x, then it contains an ϵ ball centered at x. By taking N so that $1/N < \epsilon$, we have that U contains x_n for all $n \geq N$. Therefore if $x \in \overline{A}$, there is a sequence of points x_n in X that converges to x. By hypothesis, this implies that $f(x_n)$ converges to $f(x)$ and so $f(x_n) \in f(A)$. Therefore $f(x) \in \overline{f(A)}$ and $f(\overline{A}) \subset \overline{f(A)}$. □

F.3. Uniform Convergence

If $f_n\colon X \to Y$ is a sequence of functions from a set X to a metric space Y with metric d, then the sequence $\{f_n\}$ **converges uniformly** to the function $f\colon X \to Y$ if for all $\epsilon > 0$, there exists an integer N so that $d(f_n(x), f(x)) < \epsilon$ for all $n \geq N$ and all $x \in X$.

Note that uniform convergence depends on the metric on Y and not just on the topology of Y.

Theorem F.6. *If $f_n \colon X \to Y$ is a sequence of continuous functions from a topological space X to a metric space Y, then if $\{f_n\}$ converges uniformly to f, then f is continuous.*

Proof. Assume that U is an open set in Y and let $x_0 \in f^{-1}(U)$. Set $y_0 = f(x_0)$. Choose ϵ so that $B(y_0; \epsilon) \subset U$. By uniform convergence, there exists N so that for all $n \geq N$ and all $x \in X$, $d(f_n(x), f(x)) < \epsilon/4$. By continuity of f_N, choose a neighborhood V of x_0 so that f_N maps V into the $\epsilon/2$ ball in Y centered at $f_N(x_0)$. Then

$$\begin{aligned} d(f(x), f(x_0)) \leq \\ d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) < \\ \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon \ , \end{aligned} \tag{F.1}$$

where the first and third estimates follow by the choice of N and the second by the choice of V. Thus f takes V into $B(y_0; \epsilon)$ and so into U. □

F.4. Complete Metric Spaces

If (X, d) is a metric space, a sequence of points $\{x_n\}$ is a **Cauchy sequence** if for all $\epsilon > 0$, there exists N so that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Any convergent sequence is a Cauchy sequence. However the converse does not always hold. The metric space (X, d) is **complete** if every Cauchy sequence in X is convergent.

Example F.7. The rational numbers with Euclidean distance is not a complete metric. For example, the sequence $3, 3.1, 3.14, 3.141, 3.1415, \ldots$ is Cauchy, but does not converge in $\mathbb{Q}$, only in $\mathbb{R}$.

Theorem F.8. *A metric space (X, d) is complete if and only if every Cauchy sequence has a convergent subsequence*

Proof. By definition, if X is complete then every Cauchy sequence converges. Conversely, assume that $\{x_n\}$ is a Cauchy sequence and that $\{x_{n_k}\}$ converges to a point x. We show that the entire sequence $\{x_n\}$ converges. Given $\epsilon > 0$, since the sequence is Cauchy, there exists N so that $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N$. Since the sequence converges to x, there exists $j \geq N$ so that $d(x_{n_j}, x) < \epsilon/2$. Then for all $n \geq N$,

$$d(x_n, x) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x) < \epsilon \ .$$

□

Corollary F.9. *Every compact metric space is complete.*

Example F.10. Euclidean space $\mathbb{R}^k$ is complete with the usual metric d. Assume that $\{x_n\}$ is a Cauchy sequence. Pick N so that $d(x_n, x_m) \leq 1$ for all $n, m \geq N$. Then

$$M = \max\{d(x_1, 0), d(x_2, 0), \ldots, d(x_{N-1}, 0), d(x_N, 0) + 1\}$$

is an upper bound for $d(x_n, 0)$ for all n. Therefore the set of points in this sequence is a bounded subset of $\mathbb{R}^k$ and so all the points in this sequence lie in the cube $[-M, M]^k$. The cube is compact and so the sequence has a convergence subsequence and so by the previous lemma, the space is complete.

The converse of Corollary F.9 is false: a complete metric space might not be compact without some extra condition.

A metric space (X, d) is **totally bounded** if for all $\epsilon > 0$, there exists a finite covering of X by balls of radius ϵ. Totally bounded implies bounded, but the converse is false.

Example F.11. $\mathbb{R}$ with the usual metric is neither bounded nor totally bounded. However, $\mathbb{R}$ with the metric $\rho(x, y) = \min\{|x - y|, 1\}$ is bounded and not totally bounded. The subset $(-1, 1) \subset \mathbb{R}$ with $d(x, y) = |x - y|$ is totally bounded, but is not complete. The closed subset $[-1, 1] \subset \mathbb{R}$ is totally bounded and complete.

Theorem F.12. *A metric space (X, d) is compact if and only if it complete and totally bounded.*

Proof. If X is a compact metric space, then X is complete by Corollary F.9. It is totally bounded since any open covering has a finite subcovering.

Conversely, let $\{U_j\}$ be an open cover of X. Since X is totally bounded, X can be written has the union of finitely many sets of diameter bounded by 1. If there were no finite subcover of $\{U_j\}$ which covers all of X, then there would exist a set $A_1 \subset X$ with $\operatorname{diam} A_1 \leq 1$ so that no finite subcollection of $\{U_j\}$ covers A_1. Inductively, we can construct subset $A_n \subset X$ so that the $\operatorname{diam} A_n \leq 1/n$, $A_{n+1} \subset A_n$ for all n and no finite subcollection of $\{U_j\}$ covers A_n. Such a construction is possible since each A_n is totally bounded, each can be decomposed into a finite number of sets of diameter bounded by $1/(n+1)$ and we take one of these to be A_{n+1}.

Choose $x_n \in A_n$. If $m \leq n$ and $m \leq p$, then $x_n, x_p \in A_m$ and so $d(x_n, x_p) \leq 1/m$. Therefore the sequence $\{x_n\}$ is Cauchy and so $x_n \to x \in X$. There exists some k so that $x \in U_k$ and since U_k is open, there exists $\epsilon > 0$ so that $B(x; \epsilon) \subset U_k$. For some $m \in \mathbb{N}$, $x_n \in B(x; \epsilon/2)$ for all $n \geq m$. Taking $n \geq m$ so that also $n > 2/\epsilon$, we have that $A_n \subset B(x; \epsilon) \subset U_j$, a contradiction that A_n is not covered by any finite subcollection. $\square$

F.5. Sequential and Limit Point Compactness

We prove two other conditions that are equivalent to compactness in a metric space. These equivalences do not hold without some extra assumption, such as metrizability.

If A is a subset of a topological space X, a point $x \in X$ is a **limit point** of A if it belongs to the closure of $A - \{x\}$. This means that every neighborhood of x intersects A in some point other than the point x.

The space X is **limit point compact** if every infinite subset of X has a limit point. The space X is **sequentially compact** if every sequence has a convergent subsequence.

Lemma F.13. *If the space X is compact, then it is limit point compact.*

Proof. Let A be a subset of X and assume that A has no limit point. We show that A is finite. Since A has no limit point, it contains all of its limit points and so is closed. Since X is compact, A is also compact. Since no $a \in A$ is a limit point, for each a we can pick an open set U_a containing a so that U_a does not intersect $A - \{a\}$. Then A is covered by these open sets and by compactness, a finite subcovering covers A. Since each U_a contains only one point of A, we have that A is finite. $\square$

Lemma F.14. *Assume that $\mathcal{U}$ is an open covering for a metric space (X, d). If X is sequentially compact, then there exists some $\delta > 0$ so that every subset of X with diameter less than δ is contained in some element of $\mathcal{U}$.*

The number δ in this lemma is called the **Lebesgue number** of the covering $\mathcal{U}$.

Proof. If not, then for any $n \in \mathbb{N}$, there is some point x_n so that the ball $B(x_n; 1/n)$ is not contained in any set $U \subset \mathcal{U}$. If the collection of points $\{x_n\}$ is finite, then set x_0 to be some point that appears infinitely often in the list. If $\{x_n\}$ is infinite, let x_0 be a limit point, which exists by Lemma F.13. In both cases, x_0 is a limit point of the sequence $\{x_n\}$.

Choose some U_{x_0} so that $x_0 \in U_{x_0}$. Since U_{x_0} is open, there is some δ_0 so that U_{x_0} contains a balls of size δ_0. Choose $N \in \mathbb{N}$ so that $1/N < \delta_0/2$ and $d(x_0, x_N) < \delta_0/2$. But then the ball of radius $1/N$ around x_N is entirely contained in U_{x_0}, a contradiction. $\square$

If (X, d_X) and (Y, d_Y) are compact metric spaces, function $f\colon X \to Y$ is **uniformly continuous** if for all $\epsilon > 0$, there exists $\delta > 0$ so that for any points $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2)) < \epsilon$.

Theorem F.15. *Assume that (X, d_X) and (Y, d_Y) are compact metric spaces and that $f\colon X \to Y$ is continuous. Then f is uniformly continuous.*

Proof. Fix $\epsilon > 0$. Consider the open covering of Y by balls of radius $\epsilon/2$ and let $\mathcal{U}$ be the open covering of X obtained by taking the inverse image of these balls under f. Let δ be a Lebesgue number for $\mathcal{U}$. Then if $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, the images $f(x_1), f(x_2)$ lie in some ball of radius $\epsilon/2$, which means that they lie within ϵ of each other. □

Theorem F.16. *If X is metrizable, then the following are equivalent:*

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Proof. In Lemma F.13, we showed that compactness implies limit compactness and in the proof of Lemma F.14, we showed that limit compactness implies sequential compactness. We are left with showing that sequential compactness implies compactness.

We proceed by contradiction. Let $\mathcal{U}$ be an open covering of X that has no finite subcovering and let δ be a Lebesgue number of this covering. Let x_1 be an arbitrary point in X and consider the ball $B(x_1; \delta)$. This ball is not all of X, or we would have a covering of all of X by an open set in $\mathcal{U}$ containing this ball. So we can choose a point x_2 not contained in this ball. Continuing, we can inductively find a point x_{n+1} that is not contained in the union $B(x_1; \delta) \cup B(x_2; \delta) \cup \ldots \cup B(x_n; \delta)$, since no finite subcollection of balls of radius δ covers all of X. But then this sequence has no convergent subsequence, since any ball of radius $\delta/2$ contains x_n for at most one n. □

F.6. Baire category theorem

Theorem F.17. *If X is a complete metric space, then the intersection of a countable collection of dense and open sets in X is dense in X.*

Note that a compact metric space is complete.

Proof. Assume that $\{U_j\}_{j\in\mathbb{N}}$ is a collection of dense open sets in X. Let V be any open set in X. We show that $V \cap \bigcap_{j\in\mathbb{N}} U_j \neq \varnothing$. Since U_1 is dense in X, we can choose $x_1 \in U_1 \cap V$. As both U_1 and V are open, so is there intersection and we can choose $r_1 > 0$ so that $B(x_1; r_1) \subset U_1 \cap V$. Since U_2 is dense in X, there exists $x_2 \in U_2 \cap B_1$. Since U_2 is open, we can choose a sufficiently small ball around x_2 that is contained in U_2. Picking the radius r_2 of this ball with $r_2 < \min(\frac{1}{2}r_1, r_1 - d(x_1, x_2))$, we can further assume that $\overline{B_2} \subset B_1$. Continuing inductively, we can define balls $B_n = B_n(x_n; r_n)$ with $\overline{B_n} \subset B_{n-1}$, $B_n \subset U_n$ and $r_n \to 0$.

Fix $N \in \mathbb{N}$. By constructions, for all $n, m > N$ we have $x_n \in B_N$ and $x_m \in B_N$. This means that $d(x_n, x_m) \leq 2r_N$ for all $n, m > N$. Since

$r_n \to 0$, the sequence $\{x_n\}$ is Cauchy and since X is complete, there exists $x \in X$ so that $x_n \to x$ as $n \to \infty$. Thus for all $n > N$, we have $x_n \in B_{N+1}$ and so $x \in \overline{B_{N+1}} \subset B_N \subset U_N$. Therefore $x \in V \cap \bigcap_{j\in\mathbb{N}} U_j$. Since V was an arbitrary open set in X, $\bigcap_{j\in\mathbb{N}} U_j$ is dense in X. □

A set A is **nowhere dense** if the complement of $\overline{A}$ is dense. This means that $\overline{A}$ contains no nonempty open set. As a corollary to Baire's Theorem, we have:

Corollary F.18. *In a complete metric space, no nonempty open set is a countable union of nowhere dense open sets.*

Proof. Assume that V is a nonempty open set and let $\{U_n\}_{n\in\mathbb{N}}$ be a countable collection of nowhere dense sets. Then for each n, $W_n = X - \overline{U_n}$ is a dense open subset and so by Theorem F.17, $\bigcap_{n\in\mathbb{N}} W_n$ is dense. In particular, there exists $x \in V \cap \bigcap_{n\in\mathbb{N}} W_n$. This means that $x \notin \bigcup_{n\in\mathbb{N}} U_n$ and so $V \not\subset \bigcup_{n\in\mathbb{N}} U_n$. □

A set is said to be of **first category** if it is the union of a countable collection of nowhere dense sets. In this terminology, the corollary states that in a complete metric space, no nonempty open set is of first category. A set is **residual** if it is the complement of a set of first category and it is of **second category** if it is not of first category.

F.7. Upper and lower semicontinuous functions

If f is an extended real valued function on a topological space X, it is **lower semicontinuous** if $\{x \in X : f(x) > \alpha\}$ is open for all $\alpha \in \mathbb{R}$ and is **upper semicontinuous** if $\{x \in X : f(x) < \alpha\}$ is open for all $\alpha \in \mathbb{R}$. It follows immediately from the definition that a real valued function is continuous if and only if it is both upper and lower semicontinuous. Note that f is upper semicontinuous if and only if $-f$ is lower semicontinuous.

Example F.19. A characteristic function of a set A is the function that is 1 for all $x \in A$ and 0 otherwise. Characteristic functions of open sets are lower semicontinuous, while characteristic functions of closed sets are upper semicontinuous.

Theorem F.20. *An upper (or lower) semicontinuous function on a complete metric space is continuous for a residual set of points.*

Proof. We follow the proof in Furstenberg [**27**]. We show that the points of discontinuity lie in the union of countably many nowhere dense sets.

Assume that f is upper semicontinuous. (If f is lower semicontinuous, we apply the same argument to $-f$.) For $\epsilon > 0$ and let A_ϵ be the set of points x for which there exist sequences $x'_n \to x$, $x''_n \to x$ and $f(x'_n) - f(x''_n) < \epsilon$. Then since f is upper semicontinuous, A_ϵ is closed. We must show that it has empty interior.

First assume that $f(x)$ is bounded from below. Say that x lies in the interior of A_ϵ. Since f is upper semicontinuous, if x_n is close enough to x, then $f(x'_n) \leq f(x) + \epsilon/2$ and so also $f(x''_n) < f(x) - \epsilon/2$. Also, for n sufficiently large, x''_n is also an interior point of A_ϵ and so we can repeat this process. However, we cannot continue this indefinitely if f is bounded from below. This means that it must have empty interior. If f is unbounded from below, we can replace it by $e^{f(x)}$ and use the same argument. □

Exercises

Exercise F.21. Check that the metric topology is a topology. Show that the metric topology can be defined by its open sets. Namely, show that a set U is open in the metric topology if and only if for each $y \in U$, there exists $\epsilon > 0$ so that $B(y; \epsilon) \subset U$.

Exercise F.22. Show that the Euclidean and square metrics are metrics. Show that the topologies induced by each of these metrics agrees with the product topology.

Exercise F.23. Assume that d and d' are metrics on a set X. Show that they are equivalent if and only if given any $x \in X$ and any $\epsilon > 0$, there exist $\epsilon_1, \epsilon_2 > 0$ so that $B_{d_1}(x; \epsilon) \subset B_{d_2}(x; \epsilon)$ and $B_{d_2}(x; \epsilon) \subset B_{d_1}(x; \epsilon)$.

Exercise F.24. Show that a compact metrizable space has a countable basis. (A topological space with a countable basis for its topology is said to be **second countable**.)

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