Dynamics on homogeneous spaces
and new applications to number theory

Dmitry Kleinbock

Brandeis University

Nachdiplom Lectures, ETH, Zürich

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Sasha Anisimova, Stolen Lives, Kharkiv
Recall: **Proposition 4.6.**
Let $\Gamma \subset G$ be a discrete subgroup. Then the following are equivalent:

(a) on $X = \Gamma \backslash G$ there exists a $G$-invariant probability measure $m_X$ which satisfies $m_X(g \cdot B) = m_X(B)$ $\forall B \subset X$ and $g \in G$ ($\Gamma$ is a lattice in $G$)

(b) there is a fundamental domain $F$ for $\Gamma$ in $G$ with $m_G(F) < \infty$

(c) there is a fundamental domain $F \subset G$ such that $m_G^{(r)}(F) < \infty$, and $m_G^{(r)}$ is left $\Gamma$-invariant

If any (and hence all) of these conditions hold, then $G$ is **unimodular**. That is, $m_G = m_G^{(r)}$. 
Also: using our previous work on injectivity radii we can get a general "compactness criterion" describing sequence \((x_n) \subset \Gamma \backslash G\) that go to infinity:

**Proposition 5.1.** Let \(\Gamma \subset G\) be a lattice. Then \(x_n \to \infty\) as \(n \to \infty\), meaning that for any compact \(K \subset X\) \(\exists N = N(K)\) such that \(x_n \notin K\) for \(n > N\).

\[\uparrow\]

the maximal injectivity radius \(r(x_n)\) of \(x_n\) goes to zero as \(n \to \infty\).

**Proof.** \(\uparrow\) by Lemma 4.2; \(\downarrow\) by

\[\Rightarrow m_{x_n}(X) = \infty\]

(\(\downarrow\) is not true without the finite volume assumption)
Here is an important special case:

**Definition.** A discrete subgroup $\Gamma$ of $G$ is called **co-compact** or **uniform** if the quotient space $X = \Gamma \backslash G$ is compact.

Notice that Lemma 4.2 implies that in this case $X = \pi(K)$ for some compact subset $K$ of $G$.

Hence $X$ must be of finite volume, that is, $\Gamma$ has to be a lattice.

**Example.** $\mathbb{Z}^d \subset \mathbb{R}^d$, or any lattice in an abelian (also nilpotent) group.
**Conclusion:** in order to construct non-compact homogeneous spaces we need to start with as non-commutative $G$ as possible. So let us come back to the beginning of the course, when we discussed $G = \text{PSL}_2(\mathbb{R})$. Recall:

- $G$ acts transitively on the upper-half plane $\mathbb{H}$ via
  
  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$

  and simply transitively on the unit tangent bundle $T^1\mathbb{H}$; thus can be identified with $T^1\mathbb{H}$ as a left $G$-space;

- the hyperbolic metric

  $$\langle \xi, \eta \rangle_z = \frac{\xi \cdot \eta}{(\text{Im } z)^2} \iff \|\xi\|_z = \frac{|\xi|}{\text{Im } z},$$

  is invariant by this action $\Rightarrow$ corresponds to invariant Riemannian metric on $G$. 

Classically, discrete subgroups of $G = \text{PSL}_2(\mathbb{R})$ are called **Fuchsian** groups. Here is a classical characterization in terms of their action on the hyperbolic plane.

**Definition:** Let $X$ be a locally compact metric space whith an action of a countable group $\Gamma$ by homeomorphisms. The action is said to be **properly discontinuous (PD)** if for any compact set $K \subset X$ the set

$$\{ \gamma \in \Gamma : \gamma K \cap K \neq \emptyset \}$$

is finite.

**Lemma 5.2.** An infinite subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})$ is a Fuchsian group $\iff$ its action on $\mathbb{H}$ is PD.

**Proof.** $\iff$ is clear.
Conversely, assume that $\Gamma$ is discrete, take $K = \{ w : |w| \leq R, \ \text{Im}(w) \geq \varepsilon \}$ and define

$$B := \{ g \in G : gK \cap K \neq \emptyset \}.$$ 

Need to show that $B$ is bounded in $G = \text{PSL}_2(\mathbb{R})$.

Well, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B$, then for some $z \in K$

$$\frac{|az + b|}{|cz + d|} \leq R \quad \text{and} \quad \text{Im}(g \cdot z) = \frac{\text{Im} z}{|cz + d|^2} > \varepsilon.$$

Thus

$$|cz + d|^2 \leq \frac{\text{Im} z}{\varepsilon} \leq \frac{R}{\varepsilon} \quad \left\{ \text{(Exercise)} \right\}$$

and

$$|az + b|^2 \leq R^2 |cz + d|^2 \leq \frac{R^3}{\varepsilon}$$

$\implies$ the coefficients of all matrices in $B$ are uniformly bounded. $\square$
Recall: the Uniformization Theorem implies that every Riemann surface $M$ is conformally equivalent to a quotient $\Gamma \backslash \tilde{M}$, where $\tilde{M}$ is the universal cover of $M$, and $\Gamma$ is a discrete subgroup of $\text{Isom}^+(\tilde{M})$ (the group of orientation-preserving isometries). When the genus of $M$ is at least 2, $\tilde{M} = \mathbb{H}$ and $\text{Isom}^+(\tilde{M}) = \text{PSL}_2(\mathbb{R})$.

Thus (compact and non-compact) Riemann surfaces of higher genus and finite area

\[ \downarrow \]

(uniform and non-uniform) lattices in $\text{PSL}_2(\mathbb{R})$. 

\[ \mathbb{H}^2 \]

\[ \mathbb{H} \]
In other words, if we start with a Riemann surface $M$, we are getting an example of a discrete subgroup $\Gamma = \pi_1(M)$ together with a way to construct a fundamental domain (by cutting along some closed loops and straightening out the surface).

But what if we are given $\Gamma \subset \text{PSL}_2(\mathbb{R})$? how to construct/visualize a fundamental domain for $\Gamma$?
Recall: a fundamental domain $F \subset G$ for $\Gamma$ is defined by the condition

$$G = \bigcup_{\gamma \in \Gamma} \gamma F \quad \text{(disjoint union)}.$$

which sometimes is not convenient.

Let us replace it with the following

**Definition.** A fundamental region $D$ for $\Gamma$ is an open subset of $G$ such that

- $D \cap \gamma D = \emptyset$ for all $\gamma \notin e$;
- $G = \bigcup_{\gamma \in \Gamma} \gamma \overline{D}$.
**Definition.** Let $\Gamma$ be an infinite Fuchsian group, and let $p \in \mathbb{H}$ be a point not fixed by any element of $\Gamma$ other than the identity. The set

$$D = D_p := \{ z \in \mathbb{H} : d(z, p) < d(z, \gamma p) \text{ for all } \gamma \in \Gamma \setminus \{e\} \}$$

(the intersection of hyperbolic half-planes

$$H_\gamma := \{ z \in \mathbb{H} : d(z, p) < d(z, \gamma p) \}$$

over all $\gamma \in \Gamma \setminus \{e\}$) is called a **Dirichlet region** for $\Gamma$. 

[Diagram of hyperbolic half-planes and Dirichlet region with points $\gamma_1 p$, $\gamma_2 p$, and $\gamma_3 p$]
Lemma 5.3. Any Dirichlet region $D$ for an infinite Fuchsian $\Gamma$ is a fundamental region for the action of $\Gamma$ on $\mathbb{H}$.

The boundary of $D$ is made up of geodesic segments contained in geodesics defined by

$$L_\gamma := \{z \in \mathbb{H} : d(z, p) = d(z, \gamma p)\}$$

for $\gamma \in \Gamma \setminus \{e\}$. 
Proof. Let $D = D_p$ be a Dirichlet region.

Note: $D$ is open because the action of $\Gamma$ is PD.

Fix $z \in \mathbb{H}$. Since the $\Gamma$-orbit of $z$ is discrete, there exists $w \in \Gamma z$ such that

$$d(w, p) \leq d(\gamma w, p) = d(w, \gamma^{-1} p)$$

for all $\gamma \in \Gamma$.

If all $\leq$ are $<$ then $w \in D$.

If some are $=$:

Take $w' \in [p, w]$; then

$$d(w', p) < d(w, p) = d(w, \gamma p) < d(w', \gamma p)$$

Then the geodesic segment $[p, w]$ is contained in $D$, since $p \in D$ and $d(w, p) \leq d(w, \gamma p)$ for all $\gamma \in \Gamma$.

Hence the closure of $D$ meets every $\Gamma$-orbit.
Now let us see why translates of $D$ by $\gamma \in \Gamma$ do not intersect. Let $w' = \gamma w$ for some $\gamma \in \Gamma \setminus \{e\}$. Assume that both $w$ and $w'$ lie in $D$. Then

\[
d(w, p) < d(w, \gamma^{-1} p) = d(w', p),
\]

and, similarly,

\[
d(w', p) < d(w', \gamma p) = d(w, p).
\]

This is a contradiction.
Let us test it on a specific example, taking $\Gamma = \text{PSL}_2(\mathbb{Z})$, the so-called modular group.

Clearly it is discrete, i.e. Fuchsian.

**Easy to check:**

- Let $p = yi$ where $y > 1$.
  Then $\gamma \cdot p \neq p$ for any $\gamma \in \Gamma \setminus \{e\}$.
  (thus one can use $p = yi$ to construct a Dirichlet region)

- The set
  $$D = \{ z \in \mathbb{H} : |z| > 1, \ |\text{Re}(z)| < 1/2 \}$$

  is a Dirichlet region for $\Gamma$ corresponding to $p = yi$.  

Conclusion: it is a fundamental region for the $\Gamma$-action on $\mathbb{H}$. 
Another conclusion: $\operatorname{PSL}_2(\mathbb{Z}) \backslash \operatorname{PSL}_2(\mathbb{R})$ is not compact.

Actually, why?

We have constructed a noncompact fundamental region, but does it imply that the space itself is not compact?

Example. Look at $\mathbb{Z}^2 \subset \mathbb{R}^2$.

So why doesn’t such a phenomenon happen for $D$?

- Let $\Gamma = \operatorname{PSL}_2(\mathbb{Z})$, take $p \in \mathbb{H}$, and suppose $(g_n) \subset G$ is a sequence such that $\operatorname{Im}(g_n \cdot p) \to \infty$.
  Then $\Gamma g_n \to \infty$ in $\Gamma \backslash G$

(Explanation: $r(\Gamma g_n) \to 0$.)

(Proposition 5.1)
Fact: The hyperbolic area form

\[ dA = \frac{1}{y^2} \, dx \, dy \text{ on } \mathbb{H} \]

and the hyperbolic volume form

\[ dm = \frac{1}{y^2} \, dx \, dy \, d\theta \text{ on } T^1\mathbb{H}, \]

where \( \theta \) gives the angle of the unit tangent vector at \( z = x + iy \), are both invariant under the respective actions of \( \text{PSL}_2(\mathbb{R}) \).

Consequently, \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) is a lattice in \( G = \text{PSL}_2(\mathbb{R}) \).

(Explanation:

\[ \text{Area}(D) < \int_{\sqrt{3}/2}^{\infty} \int_{-1/2}^{1/2} \frac{dx \, dy}{y^2} = \int_{\sqrt{3}/2}^{\infty} \frac{dy}{y^2} = \frac{2}{\sqrt{3}} < \infty. \]
Here is another way the space

\[ \text{PSL}_2(\mathbb{Z}) \backslash \text{PSL}_2(\mathbb{R}) \cong \text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R}) \]

shows up. Given

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = \text{SL}_2(\mathbb{R}), \]

consider the integer span of its row vectors:

\[ \Lambda = \{ k \begin{pmatrix} a \\ c \end{pmatrix} + \ell \begin{pmatrix} b \\ d \end{pmatrix} : k, \ell \in \mathbb{Z} \}. \]

It is a lattice in \( \mathbb{R}^2 \), and the covolume of this lattice (the area of the parallelogram spanned by \( v := (a \ b) \) and \( w := (c \ d) \) is equal to 1.
Say that two elements \( g, g' \in \text{SL}_2(\mathbb{R}) \) are equivalent, \( g \sim g' \), if their row vectors span the same lattice \( \Lambda \).

Questions:

- How to decide if \( g, g' \) are equivalent?
- How to describe equivalence classes?
**Answer:** \( g = \begin{pmatrix} v \\ w \end{pmatrix} \sim g' = \begin{pmatrix} v' \\ w' \end{pmatrix} \)

\[ \iff \quad g' = \gamma g \]

\[ v' = kv + \ell w \quad \text{and} \quad w' = mv + nw, \]

where \( k, \ell, m, n \in \mathbb{Z} \) and \( \gamma = \begin{pmatrix} k & \ell \\ m & n \end{pmatrix} \) has determinant 1.

That is, \( \gamma \in \text{SL}_2(\mathbb{Z}) \), and equivalence classes (unimodular lattices in \( \mathbb{R}^2 \)) are in one-to-one correspondence with cosets \( \Gamma g \).

In other words, \( \Gamma \backslash G \) is the space of unimodular lattices in \( \mathbb{R}^2 \), which is a \( G \)-space under the action of \( G \),

\[ g \cdot \Lambda = \Lambda g^{-1} = \{ vg^{-1} : v \in \Lambda \}. \]
**Good news:** the same approach works in a more general set-up:

- Any lattice in $\mathbb{R}^d$ has the form $\Lambda = \mathbb{Z}^d g$ for some $g \in \text{GL}_d(\mathbb{R})$.
- A fundamental domain for $\Lambda$ is given by the parallelepiped $[0,1)^d g$ which is spanned by the row vectors of $g$ and has Lebesgue measure $|\det g|$. This measure is also called the covolume $\text{cov}(\Lambda)$ of $\Lambda$.

- A lattice $\Lambda \subset \mathbb{R}^d$ is called **unimodular** if the covolume is 1. The space of all unimodular lattices in $\mathbb{R}^d$ is therefore

$$X_d := \{ \mathbb{Z}^d g : g \in G = \text{SL}_d(\mathbb{R}) \},$$

which is the orbit of $\mathbb{Z}^d$ under the right action of $G$ on the subsets of $\mathbb{R}^d$: for $B \subset \mathbb{R}^d$ and $g \in G$ this right action sends $(g, B)$ to $B g = \{ v g : v \in B \}$.

- Notice that

$$\text{Stab}_G(\mathbb{Z}^d) = \Gamma = \text{SL}_d(\mathbb{Z}),$$

so that $X_d = \Gamma \backslash G$, where $\Gamma g$ corresponds to the lattice $\mathbb{Z}^d g$. 

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HomDyn
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1. Intro
2. HSs, MPTs
3. ETs, ED, M
4. DSbgps
5. Lattices
Remark. Notice that we have several ways to build topology (metric, manifold structure) on $X_d$:

- the quotient topology, pushed by the projection $G \hookrightarrow \Gamma \backslash G$ ($\Lambda_n \to \Lambda$ if $\Lambda_n = \mathbb{Z}^d g_n$, $\Lambda = \mathbb{Z}^d g$, and $\Gamma g_n \to \Gamma g$);
- the topology coming from bases of lattices: $\Lambda_n \to \Lambda$ if we can choose a basis $v_1^{(n)}, \ldots, v_d^{(n)}$ of $\Lambda_n$ such that $v_i^{(n)} \to v_i$ for all $i$, and $(v_1, \ldots, v_d)$ is a basis of $\Lambda$;
- the topology of uniform convergence on compacta: $\Lambda_n \to \Lambda$ if for any $R > 0$ each element of $\Lambda_n \cap B(0, R)$ converges to some element of $\Lambda \cap B(0, R)$.

Claim. These topologies are the same!
To understand the space $X_d$ better, and also to see how it is relevant to number theory, we need to develop a better understanding of lattices in $\mathbb{R}^d$.

**Lemma 5.4 (Minkowski’s Convex Body Lemma).** Let $\Lambda \subset \mathbb{R}^d$ be a lattice of covolume $V$, and let $D$ be a convex centrally symmetric subset of $\mathbb{R}^d$ with $\text{vol}(D) > 2^d V$. Then $D \cap \Lambda \neq \{0\}$.

The same conclusion holds when $\text{vol}(D) = 2^d V$ and $D$ is compact.
Proof. Since $D$ is convex and centrally symmetric, it coincides with the set

$$\frac{1}{2}D - \frac{1}{2}D = \{x - y : x, y \in \frac{1}{2}D\}.$$  

Suppose that $D \cap \Lambda = \{0\}$. Then it is not possible to find $x, y \in \frac{1}{2}D$ with $x - y \in \Lambda \setminus \{0\}$.

This amounts to saying that the natural projection $\pi$ from $\mathbb{R}^d$ onto $\mathbb{R}^d/\Lambda$ is injective when restricted to $\frac{1}{2}D$. Hence

$$\frac{1}{2^d} \text{vol}(D) = \text{vol}\left(\frac{1}{2}D\right) \leq \text{vol}\left(\pi\left(\frac{1}{2}D\right)\right) \leq \text{vol}(\mathbb{R}^d/\Lambda) = \text{cov}(\Lambda),$$

a contradiction.

The last claim follows from a compactness argument. □
Corollary 5.5 (Minkowski’s 1st theorem).
Fix any norm $\| \cdot \|$ on $\mathbb{R}^d$.
If $\Lambda \subset \mathbb{R}^d$ is a lattice of covolume $V$, then there exists a non-zero vector
in $\Lambda$ of length $\leq 2 \left( \frac{V}{c_d} \right)^{1/d}$, where $c_d$ is the volume of the unit ball
with respect to $\| \cdot \|$.

Proof. With $D = B(0, r)$,

$$\text{vol}(D) = c_d r^d = 2^d V \iff r = 2 \left( \frac{V}{c_d} \right)^{1/d}. \quad \square$$

Note that in general the inequality in Corollary 5.5 is not sharp, but it is sharp if $\| \cdot \|$ is the supremum norm ($c_d = 2^d$).
Now let us use the existence of short vectors in a lattice to define an important family of subsets of $X_d$: for any $r > 0$ let

$$Q_r := \{ \Lambda \in X_d : \Lambda \cap B(0, r) = \{0\} \}.$$ 

Clearly it depends on the choice of the norm on $\mathbb{R}^d$ (but not significantly). Corollary 5.5 implies that $Q_r = \emptyset$ if $r > 2 (\frac{1}{d}/c_d)^{1/d}$.

When $\| \cdot \|$ is the supremum norm, we have

- $Q_r = \emptyset$ if $r > 1$;
- $Q_1 \neq \emptyset$;
- $Q_r$ has non-empty interior if $r < 1$. 

(actually $V = 1$)
Example. Let $d = 2$. What is $Q_1$? what is $Q_r$ when $r$ is small?

(Nothing else)
It turns out that the picture we saw in the case $d = 2$ is valid for $d > 2$ as well. Namely, we have

**Theorem 5.6 (Mahler’s Compactness Criterion).**

$Q \subset X_d$ is relatively compact $\iff Q \subset Q_\varepsilon$ for some $\varepsilon > 0$.

The direction “$\implies$” is clear: $\Lambda$ has a very small vector

$$\downarrow$$

its injectivity radius must be very small $\implies$ it is far away.

Equivalently, if there exist $v_n \in \Lambda_n$ with $\|v_n\| \to 0$, then $(\Lambda_n)$ cannot have a convergent subsequence. In particular, $X_d$ is not compact.

So it remains to prove that $Q_\varepsilon$ is itself compact.
To prove it, we need to upgrade a proof for the case $d = 2$ to an inductive construction.

**Definition.** Let $\Lambda$ be a lattice in $\mathbb{R}^d$. We define the $k$th *successive minimum* of $\Lambda$ by

$$
\lambda_k(\Lambda) := \min \left\{ r \left| \begin{aligned}
\Lambda & \text{ contains } k \text{ linearly independent } \\
& \text{ vectors of norm } \leq r
\end{aligned} \right. \right\}.
$$

Clearly $\lambda_1(\Lambda) \geq r \iff \Lambda \in Q_r$. 
**Theorem 5.7 (Minkowski’s 2nd Theorem + Reduction).**

Let \( \Lambda \) be a lattice in \( \mathbb{R}^d \). Then

(i) \( \lambda_1(\Lambda) \cdots \lambda_d(\Lambda) \asymp \text{cov}(\Lambda) \);

(ii) there is a \( \mathbb{Z} \)-basis \( \{v_1, \ldots, v_d\} \subset \Lambda \) of \( \Lambda \) such that

\[
\|v_1\| = \lambda_1(\Lambda), \ldots, \|v_d\| = \lambda_d(\Lambda).
\]

This basis is "almost orthogonal".

Here \( \asymp \) (and later \( \gg \)) means that the ratio is bounded from both sides (or from one side) by a constant dependent only on \( d \).

**Remark.** An informal meaning of this theorem is that \( \Lambda \) possesses a nice (reduced = almost orthogonal) basis, and the technique to get such a basis from an arbitrary one is referred to as reduction theory.
Proof. By induction on $d$: 

\[ \lambda_1 \] 

\[ (d = 1) \]

\[ V_{\mathbf{x} + \mathbf{\lambda}} \]

\[ \mathbb{R}^k \]
**Proof of Theorem 5.6.**

Let \( \{\Lambda_n\} \subset Q_\varepsilon \) be any sequence. Then, by Theorem 5.7, \( \Lambda_n \) has a basis \( v_1^{(n)}, \ldots, v_d^{(n)} \) with

\[
\varepsilon \leq \lambda_1(\Lambda) = \|v_1^{(n)}\| \ll \cdots \ll \|v_d^{(n)}\|
\]

and

\[
\|v_1^{(n)}\| \cdots \|v_d^{(n)}\| \asymp 1,
\]

which implies that

\[
\varepsilon \leq \|v_i^{(n)}\| \leq \varepsilon^{-(d-1)} \quad \text{for } i = 1, \ldots, d
\]

\( \implies \) can choose a convergent subsequence.
Another use of reduction theory is to prove

**Theorem 5.8.** $SL_d(\mathbb{Z})$ is a lattice in $SL_d(\mathbb{R})$.

For this we need to understand something about the structure of $G = SL_d(\mathbb{R})$ and the way the Haar measure on $G$ is built.

**Theorem 5.9 (Iwasawa Decomposition).**

$U \times A \times K \rightarrow G = UAK$ is a 1-1 correspondence, where

- $U = \{\text{lower-triangular unipotent matrices}\}$;
- $A = \{\text{diagonal matrices with positive entries}\}$;
- $K = SO(d)$.

**Proof:** Gram-Schmidt orthogonalization.
How does it affect the measure?
Here is a general principle for building measures on products of groups.

**Lemma 5.10.** Let $G$ be a $\sigma$-compact unimodular group, and let $V, W \subset G$ be closed subgroups such that

- $V \cap W = \{e\}$;
- the product set $VW$ contains a neighborhood of $e \in G$.

Let $\phi : V \times W \to G$ be the product map

$$\phi(g, h) = gh \in VW \subset G.$$ 

Then the Haar measure $m_G$ restricted to $VW$ is proportional to the pushforward $\phi_* \left( m_V \times m_W^{(r)} \right)$, where $m_V$ is a left Haar measure on $V$ and $m_W^{(r)}$ is a right Haar measure on $W$. 
Proof. We will actually use $\Phi : V \times W \rightarrow G$ defined by

$$\Phi(g, h) = \phi(g, h^{-1}) = gh^{-1}.$$

Note that for any $g, g_0 \in V$ and $h, h_0 \in W$ we can write

$$\Phi((g, h)(g_0, h_0)) = \Phi(gg_0, hh_0) = gg_0(hh_0)^{-1}$$
$$= gg_0 h_0^{-1} h^{-1} = g \Phi(g_0, h_0) h^{-1}.$$  

Let $\nu := (\Phi^{-1})_* m_G$. Then, for $B \subset V \times W$ and $(g, h) \in V \times W$ we have

$$\nu((g, h)B) = m_G(\Phi((g, h)B)) = m_G(g \Phi(B) h^{-1})$$
$$= m_G(\Phi(B)) = \nu(B).$$

It follows that $\nu$ is a left Haar measure on $V \times W$, and so must be proportional to $m_V \times m_W$.

But $\phi$ and $\Phi$ differ only by the inverse in the second component, and the inverse map sends $m_W$ to a measure proportional to $m_W^{(r)}$, so the lemma follows. \qed
Definition. A set of the form $\Sigma_{s,t} = U_s A_t K$ where $s, t > 0,$

$$U_s = \left\{ \begin{pmatrix} 1 \\ * & \ddots \\ * & & 1 \end{pmatrix} : |*| \leq s \right\}$$

and

$$A_t = \left\{ \text{diag}(a_1, \ldots, a_d) : \left| \frac{a_{i+1}}{a_i} \right| \geq t \text{ for } i = 1, \ldots, d - 1 \right\}$$

is called a Siegel domain.

**Proposition 5.11.** For any $t \leq \sqrt{3}/2$ and $s \geq 1/2$ the Siegel domain $\Sigma_{s,t}$ is surjective (that is, $\pi(\Sigma_{s,t}) = X_d$).

**Proof:** Apply the Gram-Schmidt orthogonalization to the reduced basis constructed in Theorem 5.7.
To complete the proof of Theorem 5.9, it remains to show the following

**Lemma 5.12.** For any $s, t > 0$, we have $m_G(\Sigma_{s,t}) < \infty$.

**Proof.** This is just a higher-dimensional analog of our computation for $d = 2$ (finiteness of hyperbolic area of the fundamental region for $\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})$). The steps are:

1. show that $G$ is simple (the commutator $[G, G]$ contains all the elementary unipotent subgroups), hence unimodular;

2. using Thereom 5.9 (Iwasawa Decomposition) and Lemma 5.10, write $m_G$ as a product of a left Haar measure on $UA$ and a (right = left) Haar measure on $K$ (the latter can be ignored for our computation);

3. a left Haar measure on the group $UA$ of lower-triangular matrices is given by $\left(\prod_{i<j} \frac{a_i}{a_j}\right) (d*) da_1 \cdots da_d$

4. the integral over the Siegel set converges $\implies$ the measure is finite. \qed
An explanation for (3): (taking \( d = 2 \) for simplicity)

Claim. With the notation \( v_{a,b} := \begin{pmatrix} a^{-1} & 0 \\ b & a \end{pmatrix} \),
a left Haar measure on \( V = \{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \} \) is given by \( \frac{dadb}{a^2} \).

Proof. Indeed,

\[
v_{\alpha,\beta} v_{a,b} = v_{a',b'}, \quad \text{where} \quad a' := \alpha a \quad \text{and} \quad b' := \beta a^{-1} + \alpha a,
\]

\[
\begin{pmatrix} \alpha^{-1} & 0 \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ b & a \end{pmatrix} = \begin{pmatrix} (\alpha a)^{-1} & 0 \\ \beta a^{-1} + \alpha b & \alpha a \end{pmatrix}
\]

so for a function \( f \) on \( V \) we can write

\[
\int_V f(v_{\alpha,\beta} v_{a,b}) \frac{dadb}{a^2} = \int_V f(v_{a',b'}) \frac{da'db'}{(a')^2}.
\]
**Summary:** looking from far away, the space $X_2$ is just a ray $\{a_t i : t \geq 0\}$.

Similarly, if we ignore compact parts, the space $X_d$ can be thought of as the cone $A^{\sqrt{3}/2}$, with the measure exponentially thinning at infinity.