Dynamics on homogeneous spaces
and new applications to number theory

Dmitry Kleinbock

Brandeis University

Nachdiplom Lectures, ETH, Zürich

Spring 2022
Esther Serpionova, Odessa
A church is being built, 2019
Transfiguration, 2021
The Creation of Spheres, 2020
Lecture 8, 27/4/22.
Exceptional orbits and number theory.

Recall:

- $X = X_d = \Gamma \backslash G$, where $G = \text{SL}_d(\mathbb{R})$, $\Gamma = \text{SL}_d(\mathbb{Z})$;

- $d = m + n$, $v_Y = \begin{pmatrix} l_m & 0 \\ Y & l_n \end{pmatrix}$, where $Y \in M_{n \times m}(\mathbb{R})$;

- $a_t = \text{diag}(e^{t/m}, \ldots, e^{t/m}, e^{-t/n}, \ldots, e^{-t/n})$
  \[= \begin{pmatrix} e^{t/m}l_m & 0 \\ 0 & e^{-t/n}l_n \end{pmatrix};\]

- $V = \{v_Y : Y \in M_{n \times m}(\mathbb{R})\}$ is the expanding horospherical subgroup with respect to $a_1$;

- $\Lambda_Y = \mathbb{Z}^d v_Y = \{(qY - p, q) : p \in \mathbb{Z}^m, q \in \mathbb{Z}^n\}$. 

\[\text{HomDyn} \quad \text{D.K.}\]

1. Intro
2. HSs, MPTs
3. ETs, ED, M
4. DSbgps
5. Lattices
6. ET on $\Gamma \backslash G$
7. DT
8. EOs and NT
9. STs and NT
Ergodicity and mixing of the $a_t$-action on $X$ imply that:

- $\{\Lambda Y a_t : t \geq 0\}$ is dense in $X$ for Lebesgue-a.e. $Y \in M_{n \times m}(\mathbb{R})$;
- $\Lambda V a_t$ becomes equidistributed in $X$ as $t \to \infty$. 
Also recall (Corollary 7.9, Dani 1985):

\[
\{\Lambda Y a_t : t > 0\} \text{ is bounded in } X
\]

\[\Downarrow \]

\[Y \in BA_{n,m}\]

(badly approximable system of linear forms), where

\[BA_{n,m} = \left\{ Y \in M_{n \times m}(\mathbb{R}) : \exists c > 0 \text{ such that } \|qY - p\| \geq \frac{c}{\|q\|^{n/m}} \quad \forall p \in \mathbb{Z}^m, q \in \mathbb{Z}^n \setminus \{0\} \right\}\]

\[= \left\{ Y \in M_{n \times m}(\mathbb{R}) : \inf_{(p,q) \in \mathbb{Z}^d \setminus \{0\}} \|qY - p\|^m \|q\|^n > 0 \right\}.\]

Here \(\| \cdot \|\) can be the supremum norm or any other norm (no dependence of the choice of norm).

**Conclusion:** this is measure zero condition.
Also recall:

there exists a very special non-empty compact subset of $X$ which we called $Q_1$, the critical locus for the supremum norm $\| \cdot \| = \| \cdot \|_\infty$:

$$Q_1 = \{ \Lambda \in X : \Lambda \cap B_\infty(1) = \{0\} \}.$$
**Proved:** Corollary 7.5, K–Weiss 2006:

the set of forward limit points of \( \{ \Lambda_Y a_t : t \geq 0 \} \) does not intersect \( Q_1 \)
(that is, the trajectory \( \{ \Lambda_Y a_t : t \geq 0 \} \) eventually avoids \( Q_1 \)).

\[ \uparrow \]

\[ Y \in \text{DI}_{n,m} \]

(Dirichlet-improvable system of linear forms), where

\[
\text{DI}_{n,m} = \left\{ Y \in M_{n \times m}(\mathbb{R}) \left| \begin{array}{l}
\exists c < 1 \text{ such that for all large enough } T \\
\text{the system } \|qY - p\| \leq \frac{c}{T^{n/m}}, \|q\| \leq T \\
\text{has a nontrivial integer solution}
\end{array} \right. \right\}
\]

(Here the choice of \( \| \cdot \| = \| \cdot \|_\infty \) is important.)

**Conclusion:** also a measure zero condition.
**Also recall:** found a surprising connection between the two sets, namely

\[ \text{BA}_{n,m} \subset \text{DI}_{n,m} \]

(Theorem 7.10, Davenport–Schmidt 1969)

**Proof.** Let us do it again for \( d = 2 \):

- Close to \( \hat{e}_1 \) or \( \hat{e}_2 \)
- Not in \( \text{DI}_{1,1} \)
- Not in \( \text{BA}_{1,1} \)
- Close to \( \hat{0} \) at some (later or earlier) time
- Need:
A curious fact: when $d = 2$, converse is almost true.

**Proposition 8.1** (Also Davenport–Schmidt 1969).

$$\text{DI}_{1,1} = \text{BA}_{1,1} \cup \mathbb{Q}.$$ 

**Proof.** Need: not in $\text{BA}_{1,1} \cup \mathbb{Q} \implies$ not in $\text{DI}_{1,1}$. 

```
\begin{tikzpicture}
  \draw[help lines] (0,0) grid (1,1);
  \draw (0,0) -- (1,1);
  \draw (0,1) -- (1,0);
  \draw (0.5,0.5) -- (0.7,0.3);
  \draw (0.5,0.3) -- (0.7,0.5);
  \draw (0.5,0.7) -- (0.3,0.5);
  \draw (0.3,0.5) -- (0.5,0.3);
  \draw (0,0) node[below left] {a vector close to $\vec{e}_1$} -- (1,1) node[above right] {no small vectors};
\end{tikzpicture}
```
Note. In higher dimensions this argument fails:

(no obstructions to having small vectors)
In fact it is possible to imagine that we could get into $\text{DI}_{n,m}$ in a very drastic way, by keeping a small vector all the time.

**Definition.** Say that $Y \in M_{n \times m}(\mathbb{R})$ is singular ($Y \in \text{Sing}_{n,m}$) if $\forall c > 0$ and large enough $N$ there exists $p \in \mathbb{Z}^m$, $q \in \mathbb{Z}^n \setminus \{0\}$ such that

$$1 \leq \|q\| \leq N \text{ and } \|qY - p\| \leq \frac{c}{N^{n/m}}.$$ 

Equivalently, $\text{Sing}_{n,m} = \bigcap_c c \cdot \text{DI}_{n,m}$

(Dirichlet’s Theorem can be infinitely improved).
Dani’s correspondence (Proposition 7.4) then immediately implies

**Proposition 8.2.** \( Y \in \text{Sing}_{n,m} \)

\[ \Downarrow \]

\[ \forall r > 0, \ \Lambda_{Y a_t} \notin Q_r \text{ for large enough } t \]

(Mahler’s Compactness Criterion) \[ \Downarrow \]

\[ \{\Lambda_{Y a_t} : t \geq 0\} \text{ is divergent} \]

(eventually leaves every compact subset of \( X \))

So we always have \( \text{DI}_{n,m} \supset \text{BA}_{n,m} \cup \text{Sing}_{n,m} \),

with equality if \( m = n = 1 \).
Facts about $\text{Sing}_{n,m}$:

- existence of non-trivial singular matrices: Khintchine '26, '48
- a very nice survey: Moshchevitin '10 (Russian Math. Surveys)
- $\dim \text{Sing}_{n,m} = mn \left(1 - \frac{1}{m+n}\right)$:
  - Cheung '11, Cheung–Chevallier '16,
  - Kadyrov–K–Lindenstrauss–Margulis '17
  - Das–Fishman–Simmons–Urbanski '19
- generalizations to divergent trajectories on other spaces:
  - Dani '85, Weiss '04, Guan–Shi '20,
  - An–Guan–Marnat–Shi '20
So to summarize what we know about

$$DI_{n,m} = \{ Y \in M_{n \times m}(\mathbb{R}) : \Lambda_Y a_t : t \geq 0 \} \text{ eventually avoids } Q_1 \} :$$

- constructed many elements by sheer luck (points with bounded trajectories happen to be there)
- the reason being the special shape of the set $Q_1$, which comes from the fact that we are using the supremum norm

This raises two questions:

- what if we take a subset $Z \subset X$ and study
  $$\{ \Lambda \in X : \Lambda a_t : t \geq 0 \} \text{ eventually avoids } Z \}?$$

- what if we set-up a Diophantine problem with the supremum norm $\| \cdot \|_\infty$ being replaced by another norm – this problem is very sensitive with respect to the choice of the norm!
In fact both questions lead to many interesting developments. Let's start with the second one. Again look at

$$\text{DI}_{n,m} = \left\{ Y \in M_{n \times m}(\mathbb{R}) \mid \exists T > 0 \text{ and } r < 1 \right. \\
\text{such that } t \geq T \implies (qY - p, q)a_t \in B_{\infty}(r) \\
\text{for some } p \in \mathbb{Z}^m, q \in \mathbb{Z}^n \right\}$$

\( r = 1 \) is the critical value given by Minkowski's Theorem)
So what if instead of $\| \cdot \|_\infty$ we take some other norm $\nu$ on $\mathbb{R}^d$?

Then if $r$ is large enough, Minkowski’s Theorem always gives non-trivial vectors in $\Lambda \cap B_\nu(r)$ until $r$ reaches some critical value

$$r_\nu := \sup \{ r : \Lambda \cap B_\nu(r) = \{0\} \text{ for some } \Lambda \in X_d \}$$

$$= \sup \{ r : \Lambda \cap \bar{B}_\nu(r) = \{0\} \text{ for some } \Lambda \in X_d \}$$

(the critical radius of $\nu$).

Here $B_\nu(r) := \{ x \in \mathbb{R}^d : \nu(x) < r \}$; clearly $r_\infty = 1$.

In fact Minkowski gives non-trivial vectors in $\Lambda \cap \bar{B}_\nu(r)$ if

$$c_\nu r^d \geq 2^d \iff r \geq \frac{2}{c_\nu^{1/d}},$$

where $c_\bullet$ is the volume of $B_\nu(1)$. This implies $r_\nu \leq \frac{2}{c_\nu^{1/d}}$.

But is this sharp? (it is for the supremum norm)
**Example.** Take $\nu$ to be the Euclidean norm on $\mathbb{R}^2$; then $c_\nu = \pi$, so $r_\nu \leq \frac{2}{\sqrt{\pi}}$.

On the other hand look at

$$\nu^2 \frac{\sqrt{3}}{2} = 1 \implies r_\nu = \sqrt{\frac{2}{\sqrt{3}}} > \frac{2}{\sqrt{\pi}}.$$
So in general we can define

\[ \mathcal{L}_\nu := \{ \Lambda \in X : \Lambda \cap B_\nu(0, r_\nu) = \{0\} \}, \]

the critical locus for \( \nu \), and

\[ \text{DI}^\nu_{n,m} = \{ Y \in M_{n \times m}(\mathbb{R}) : \{ \Lambda_Y a_t : t \geq 0 \} \text{ eventually avoids } \mathcal{L}_\nu \} \]

\( \text{Dirichlet-improvable w.r.t } \nu \)

\[
= \left\{ Y \in M_{n \times m}(\mathbb{R}) \mid \begin{array}{l}
\exists T > 0 \text{ and } r < r_\nu \\
such \text{ that } t \geq T \implies \left(q Y - p, q\right) a_t \in B_\nu(r) \\
\text{for some } p \in \mathbb{Z}^m, q \in \mathbb{Z}^n
\end{array} \right\}
\]

This looks kind of artificial, but people have actually been studying this recently, starting from [Andersen–Duke, 2020] for some norms on \( \mathbb{R}^2 \).

Then in several papers by myself and Anurag Rao.
Example. Take $\nu$ to be the Euclidean norm on $\mathbb{R}^2$; then

$$D\Gamma_{1,1}^\nu = \left\{ \alpha \in \mathbb{R} \mid \exists \gamma < \frac{2}{\sqrt{3}} \text{ such that} \right\}
\begin{align*}
(e^t|\alpha q - p|)^2 + (e^{-t}q)^2 &< \gamma \\
\text{is solvable in } p &\in \mathbb{Z}, \ q \in \mathbb{N} \\
\text{for all large enough } t &> 0.
\end{align*}$$

$\gamma < \frac{2}{\sqrt{3}}$

$\rightsquigarrow \mathcal{L}_\nu = \times_2$
**Remark.** Critical lattices are in one-to-one correspondence with the densest lattice packings of $\mathbb{R}^d$ by translates of $B_\nu(1)$. Indeed, suppose that

$$\Lambda \cap B_\nu(r) = \{0\} \quad (\ast)$$

Then it is easy to see that the collection of sets

$$\{2\nu + B_\nu(r) : \nu \in \Lambda\}$$

is pairwise disjoint.

Conversely, if such a collection of sets is pairwise disjoint for some $\Lambda$, then (\ast) holds. Maximizing the value of $r$ while keeping (\ast) thus corresponds to maximizing the relative area covered by the collection of the above sets.
Andersen and Duke considered strongly symmetric norms $\nu$, that is satisfying

$$\nu((x_1, x_2)) = \nu((|x_1|, |x_2|)) \text{ for all } (x_1, x_2) \in \mathbb{R}^2,$$

and proved

**Theorem 8.3.** For any strongly symmetric norm $\nu$ on $\mathbb{R}^2$, the set $D\text{I}_{1,1}'$

- has Lebesgue measure zero, and
- is uncountable.
Examples of strongly symmetric norms:

![Graphs](image)

**Figure 1.** $B^p$ for $p = 1, 2, 4, \infty$ and $B^{oct_1}$ and $B^{oct_2}$.

approximation theorem can be improved, while uncountable, is small in the sense of measure theory.

**Theorem 1.** Fix a strongly symmetric norm $F$. Then the set of all real irrationals for which Minkowski’s approximation theorem can be improved is uncountable and has Lebesgue measure zero.
Now we can clearly see that the Lebesgue measure part trivially follows from ergodicity; and more generally we have

**Theorem 8.4.** For any norm $\nu$ on $\mathbb{R}^d$, $\text{Leb}(\text{DI}_{n,m}^\nu) = 0$.

And as far as the second part of Theorem 8.3 goes, we have

**Conjecture 8.5.** For any norm $\nu$ on $\mathbb{R}^d$, the set $\text{DI}_{n,m}^\nu$ has full Hausdorff dimension.

\[
\text{(still open)} \quad \text{(ok if } m=n=1) \quad [K-Rao '20]
\]
The difficult feature of the above conjecture is that in general very little is known about critical loci $\mathcal{L}_\nu$. Moreover, even critical radii $r_\nu$ can be tricky to compute in higher dimensions.

Still, Conjecture 8.5 is proved in some special cases. I will prove

**Theorem 8.6.** For the Euclidean norm $\| \cdot \|_2$ on $\mathbb{R}^d$, the set

$$\mathbf{D}^2_{n,m} = \left\{ \alpha \in \mathbb{R} \left| \exists \gamma < \gamma_d \text{ such that } \right. \begin{array}{l}
(e^{t/m}\|Yq - p\|_2)^2 + (e^{-t/n}\|q\|_2)^2 < \gamma \\
\text{is solvable in } p \in \mathbb{Z}^m, \ q \in \mathbb{Z}^n \setminus \{0\}
\end{array} \right\},$$

has full Hausdorff dimension.

Here $\gamma_d = r_2^2$ is the Hermite constant.
Examples of known values of Hermite constants:

Hermite constant

From Wikipedia, the free encyclopedia

In mathematics, the Hermite constant, named after Charles Hermite, determines how short an element of a lattice in Euclidean space can be.

The constant \( \gamma_n \) for integers \( n > 0 \) is defined as follows. For a lattice \( L \) in Euclidean space \( \mathbb{R}^n \) unit covolume, i.e. \( \text{vol}(\mathbb{R}^n/L) = 1 \), let \( \lambda_1(L) \) denote the least length of a nonzero element of \( L \). Then \( \sqrt{\gamma_n} \) is the maximum of \( \lambda_1(L) \) over all such lattices \( L \).

The square root in the definition of the Hermite constant is a matter of historical convention.

Alternatively, the Hermite constant \( \gamma_n \) can be defined as the square of the maximal systole of a flat \( n \)-dimensional torus of unit volume.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_n )</td>
<td>1</td>
<td>( \frac{4}{3} )</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>( \frac{64}{3} )</td>
<td>64</td>
<td>( 2^8 )</td>
<td>( 4^{24} )</td>
</tr>
</tbody>
</table>

For \( n = 2 \), one has \( \gamma_2 = \frac{2}{\sqrt{3}} \). This value is attained by the hexagonal lattice of the Eisenstein integers.\(^1\)

Estimates

It is known that\(^2\)

\[
\gamma_n \leq \left( \frac{4}{3} \right)^{\frac{n-1}{2}}.
\]

A stronger estimate due to Hans Frederick Blichfeldt\(^3\) is\(^4\)

\[
\gamma_n \leq \left( \frac{2}{\pi} \right) \Gamma\left( 2 + \frac{n}{2} \right)^{\frac{1}{2}},
\]

where \( \Gamma(x) \) is the gamma function.
Question: now that we have the general conjecture, could it be the case that it can be proved similarly to the supremum norm case, that is, by establishing the inclusion

$$BA_{n,m} \subset DI_{n,m}^\nu?$$

Answer: not really! In fact we have the following

**Theorem 8.7.** [K–Rao ’22] For any $n, m \in \mathbb{N}$,

$$\bigcap_{\nu \text{ a norm on } \mathbb{R}^d} DI_{n,m}^\nu = \text{Sing}_{n,m}.$$ 

Recall: $DI^{\infty} \supset BA \cup \text{Sing}$

In fact, for any fixed norm $\nu$ on $\mathbb{R}^d$, we have

$$\text{Sing}_{n,m} = \bigcap_{g \in SL_d(\mathbb{R})} DI_{n,m}^{\nu \circ g}.$$
Proof. Recall: $Y \in \text{Sing}_{n,m}$

$\uparrow$

$\Lambda_Y$ is divergent under $\{a_t : t \geq 0\}$.

By divergence, the trajectory must avoid any given critical locus after a certain time. Thus $\text{Sing}_{n,m}$ is contained in each of the intersections in the theorem.

To complete the proof, it now suffices to show that, for a fixed $\nu$,

$$\bigcap_{g \in \text{SL}_d(\mathbb{R})} \text{DI}^\nu_{n,m} \subset \text{Sing}_{n,m}$$

Key observation: for $g \in \text{SL}_d(\mathbb{R})$, we have $\mathcal{L}^\nu_{o\circ g} = g^{-1} \mathcal{L}_\nu$. 
Thus, given any $\Lambda \in X$, there is some $g \in \text{SL}_d(\mathbb{R})$ such that $\Lambda \in \mathcal{L}_{\nu^\circ g}$.

Now say we have $Y \in \bigcap_{g \in \text{SL}_d(\mathbb{R})} D\mathcal{I}_{n,m}^{\nu^\circ g}$.

If $Q \subset X$ is compact, then, given any $\Lambda \in Q$, there is some

1. $g \in \text{SL}_d(\mathbb{R})$
2. $t_0 > 0$
3. open $U \ni \Lambda$

such that $\Lambda_Y a_t$ is not in $U$ for all $t > t_0$.

By the compactness of $Q$, there is a time after which the $a_t$-orbit of $\Lambda_Y$ leaves $Q$.

Thus $\Lambda_A$ is forward divergent, and we are done.
Conclusion: for a general norm \( \nu \) there is no reason for \( \text{DI}_{n,m}^\nu \) to contain \( \text{BA}_{n,m} \).

Thus if someone wants to prove Conjecture 8.5, one needs to

(i) get some understanding of \( \mathcal{L}_\nu \);

(ii) do something else to construct trajectories eventually avoiding it.

(i) is actually very difficult! there are may results for \( d = 2 \), mostly due to Mahler and his followers, and very few in higher dimensions. But at least for the Euclidean norm some sort of description of the critical locus is well known.
Theorem 8.8. [Korkine–Zolotareff 1877, see Theorem 3.4.5 in HADAMARD, Volume 327, p. 1-100]  

Let $\nu$ be the Euclidean norm on $\mathbb{R}^d$, and let $\Lambda \in \mathcal{L}_\nu$. Then there exists a neighborhood of $\Lambda$ in $X_d$ on which the length of the smallest nonzero vector of any lattice not isometric to $\Lambda$ is strictly smaller than that of $\Lambda$. 

Consequently, $\mathcal{L}_\nu$ is a finite union of $\text{SO}(d)$-orbits. 

(Note: no information on the # of orbits or the value of the critical radius.)
Let me now try to construct trajectories avoiding $\mathcal{Z}$, such as $\mathcal{L}_\nu$.

For that it will be useful to recall what we did last time to construct bounded orbits using mixing.

relative measure of $(- - - - -) \approx 1$
\{ \forall t \, a_{kt} \colon k \in \mathbb{N} \} \text{ is bounded}
A difference between the previous case and the new set-up:

\[ Z \text{ is not fixed by } a_t. \]

A solution: consider

\[ Z_t := Za_{[0,t]}; \]

if we succeed in constructing many \( \Lambda \in X \) with \( \Lambda a_t \) being a certain distance away from \( Z_t \), we can iterate the procedure and construct trajectories staying away from \( Z \).
An important feature of the set $Z = \mathcal{L}_\nu$: its **transversality** both to the flow direction $F = \{a_t\}$, and to the orbits of the expanding horospherical subgroup $V = \left\{ \begin{pmatrix} I_m & 0 \\ Y & I_n \end{pmatrix} \right\}$.

It turns out that the transversality considerations are enough to construct exceptional trajectories.
Say that $Z$ is $(F, V)$-transversal at $x \in Z$ if the following holds:

1. $T_x(F) \not\subset T_x Z$
2. $T_x(V) \not\subset T_x Z \oplus T_x(F)$

We will say that $Z$ is $(F, V)$-transversal if it is $(F, V)$-transversal at its every point.

Note that in a special case when $Z$ is an orbit of a Lie subgroup $L$ of $G$, the above conditions can be restated as

$$\text{Lie}(F) \not\subset \text{Lie}(L)$$

and

$$\text{Lie}(V) \not\subset \text{Lie}(L) \oplus \text{Lie}(F)$$

respectively.
Below is a special case of a general theorem [K,'99]:

**Theorem 8.9.** Let $Z$ be a $C^1$ compact $(F, V)$-transversal submanifold of $X$; then for any $\Lambda \in X$, the set

$$\left\{ Y : \{ a_Y a_t : t \geq 0 \} \cap Z = \emptyset \right\}$$

has full Hausdorff dimension.

This clearly applies to the Euclidean critical locus, as well as to many other known special cases.

(for example, to norms whose critical locus is finite).

$$L_s = \exp(\text{skew-symm. matrices})$$

$$\mathcal{V} = \exp \left( \left\{ \left( \begin{array}{cc} 0 & \theta \\ -\theta & 0 \end{array} \right) \right\} \right)$$

$$F = \exp \left( \left\{ \left( \begin{array}{c} t \\ \frac{1}{t} \end{array} \right) \right\} \right)$$
Sketch of proof:

- Pick a large $t > 0$ and let $Z_t = Z_{a[0,t]}$
- Using compactness and smoothness of $Z$, choose a small $\sigma$ such that the intersection of any $\sigma$-ball with $Z_t$ lies in a very thin neighborhood of its tangent hyperplane.
- Then pick a cube $I_r$ of radius $r$ in $V$ such that its expansion by $a_t$ has diameter less than $\sigma$.
- Using transversality, choose $\eta$ such that for all $z \in Z_t$, the intersection of this expanded cube with an $\eta$-neighborhood of $Z_t$ has very small relative measure, hence can be covered by a very small number of translates of $I_r$.

\[\Rightarrow \text{iterate}\]
\[\Rightarrow \text{get a Cantor-like subset of large Hausdorff dimension}\]
**Remark.** Is it possible to find $\nu$ such that $\mathcal{L}_\nu$ does not satisfy the transversality assumptions? Yes – look at the supremum norm.

![Diagram](image)

But there we had a different reason for being able to avoid it. This justifies the conjecture.

Yet, what is lacking is an understanding of the structure of critical loci in higher dimensions (no classification theorem).