Hele–Shaw flows with a free boundary produced by multipoles

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We study Hele–Shaw flows with a moving boundary and multipole singularities. We find that such flows can be defined only on a finite time interval. Using a complex variable approach, we construct a family of explicit solutions for a single multipole. These solutions turn out to have the maximal possible lifetime in a certain class of solutions.

We also discuss the generalized Hele–Shaw model in which surface tension at the moving boundary is considered, and develop a method of finding steady shapes. This method yields new one-parameter families of stationary solutions. In the Appendix we discuss a connection between these solutions and a variational problem of potential theory.

0 Introduction

We deal with a generalization of the classical Hele–Shaw model involving multipole singularities. The integrability of Hele–Shaw flows with time-dependent free boundaries produced by sources and sinks was first observed by Kufarev [1–3] and later independently by S. Richardson [4, 5]. Richardson showed that the moments of the evolving fluid domain depend linearly on time (we review these results in §2). Richardson’s ideas can be expressed using the notions of the Cauchy transform of the fluid domain and the Schwarz function of its boundary [4–12]. The essence of this approach can be briefly formulated as follows: injection or suction of fluid affects the singularities of the Schwarz function. Namely, simple poles are created inside the domain whose residues change linearly in time. In some special cases this information is sufficient to reconstruct \( D(t) \).

This paper has arisen from the question: what happens if we allow poles of higher orders? In this case, the Schwarz function methods are still applicable; however, we prefer to use the language of moments, in which the construction of exact solutions (§4) becomes especially clear.

In the general case of several multipole singularities, a simple Cauchy transform argument (§5) shows that any bounded solution of such a problem breaks down at a finite time. Moreover, this argument gives an estimate for the lifetime of any initial domain of a fixed area.

The very simple form of the solutions constructed in §4 suggests that it should be possible to apply a similar method to Hele–Shaw flows produced by multipoles with a non-zero surface tension. Problems of this sort have been inaccessible to analytic examination for a long time. Nonetheless, in §6 we describe a method of finding steady-state shapes of the gas-
fluid interface in the multipole-driven Hele–Shaw flow with non-zero surface tension on the boundary. This method uses the technique of the Schwarz function.

In the Appendix we describe a connection between the steady shapes for Hele–Shaw flows and a variational problem in potential theory (the maximal conductivity problem).

1 Flows produced by multipoles

Let us recall some basic concepts. A multipole is a certain kind of singular point of an incompressible potential fluid flow. Streamlines near this point are as shown in figure 1 a, b. The number of sections is always even, and it is twice the order of the multipole. If a multipole of order $n$ is located at a point $z_0$ then the complex potential $F(z)$ of the flow has a pole of order $n$ at $z_0$:

$$F(z) = -\frac{M}{2\pi} (z - z_0)^{-n} + \phi(z),$$

(1.1)

$\phi$ being a holomorphic function at $z_0$ and $M$ being the moment of the multipole – a complex number. If a multipole is located at infinity then

$$F(z) = \frac{M}{2\pi} z^n + \phi(z),$$

(1.2)

$\phi$ being a holomorphic function at infinity. A multipole of order 1 is called a dipole, of order 2 a quadrupole. A multipole can be considered as a system of sources and sinks of infinite power lying infinitely close to each other. More precisely, the potential of the system of sources and sinks situated at points $z_0 + \epsilon e^{ik\pi/n}, 0 \leq k < 2n,$ and having powers $\frac{1}{n}(-1)^k M \epsilon^{-n}$, tends to the potential of a multipole of order $n$ and moment $M$ located at $z_0$ as $\epsilon$ tends to 0.

Consider now a Hele–Shaw flow with a moving boundary produced by a multipole. Suppose that the fluid occupies a bounded simply-connected domain $D$ varying in time: $D = D(t)$. The velocity potential $\Phi(z, t) = Re F(z, t)$ is then the solution of the following boundary value problem in $D(t)$:

$$\Delta \Phi = 0 \quad \text{in} \quad D(t), \quad \Phi = 0 \quad \text{on} \quad \partial D(t),$$

(1.3)

$$\Phi = Re \left( -\frac{M}{2\pi} z^{-n} \right) + O(1), z \to 0;$$

(1.4)

here we suppose that the multipole is situated at the origin. The boundary of the fluid domain moves with velocity

$$v_b = \frac{\partial \Phi}{\partial n}.$$  

(1.5)

Equations (1.3)–(1.5) determine the evolution of the domain $D(t)$ in time due to the action of the multipole at the origin of order $n$ and moment $M$ if the initial domain $D(0)$ is given.

The evolution of a bubble produced by a multipole at infinity is described in a similar way. In this case, the fluid occupies the complement $D(t)$ of a bounded simply connected domain $B(t)$ (the bubble), and the velocity potential is determined by (1.3) and

$$\Phi = Re \left( -M z^n / 2\pi \right) + O(1), \quad z \to \infty,$$  

(1.6)
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with (1.5) for the kinematic condition on the boundary. Equations (1.3), (1.5), (1.6) determine the evolution of the bubble if its initial shape \( B(0) \) is given.

**Remark** In a similar fashion one can describe the evolution of the fluid domain under the action of several multipoles, and also sources and sinks (which can be regarded as multipoles of order 0). In particular, one can allow multipoles of different orders to be located at the same point. Then the Laurent expansion of the complex potential at this point will have several non-zero terms of negative degree (and possibly a logarithmic term).

2 Richardson's moments

*Richardson's moments* of a bounded simply connected domain \( D \) are

\[
M_n(D) = \int_D z^n \, dx \, dy, \quad z = x + iy
\]  

(2.1)

for non-negative integers \( n \). They were introduced into the theory of Hele–Shaw flows in 1972 by S. Richardson [4] to study the flows produced by injection of fluid through point
sources: it turned out that $M_n(D(t))$ vary in time linearly, with known coefficients. More precisely, the following statement holds:

**Theorem 2.1** [5] Let $D(T)$ be a solution of the problem of evolution of a bounded simply connected domain produced by $m$ point sources (sinks) of rates $q_j$ located at points $z_j$. Then

$$\frac{dM_n(D(t))}{dt} = \sum_{j=1}^{m} q_j z_j^n.$$  

(2.2)

In particular, if the only source of rate $q$ is located at the origin, then

$$\frac{dM_n(D(t))}{dt} = \begin{cases} q, & n = 0 \\ 0, & n > 0 \end{cases}$$  

(2.3)

i.e. the zero moment (the area of $D(t)$) increases linearly while the higher moments are invariant.

Because a multipole is the limit of a system of sources and sinks, for the multipole-driven evolution one obtains a similar statement.

**Theorem 2.2** Let $D(t)$ be a solution of the problem of evolution of a bounded simply connected domain produced by a multipole of order $n$ and moment $M$ located at the origin. Then

$$\frac{dM_k(D(t))}{dt} = \begin{cases} 0, & k \neq n \\ M, & k = n \end{cases}.$$  

(2.4)

**Proof** For the system of $2n$ sources and sinks described in §1, according to (2.2), one has

$$\frac{dM_m(D(t))}{dt} = \sum_{k=0}^{2n-1} \frac{(-1)^k M}{2e^n} (e^{\pi ik/n})^m = \frac{M}{2e^{n-m}} \sum_{k=0}^{2n-1} (-e^{\pi im/n})^k.$$  

(2.5)

This number is non-zero if and only if $-e^{\pi im/n} = 1$, i.e. if and only if $m = (2l-1)n$, $l$ being a positive integer. If $l > 1$ then $m > n$, and as $\epsilon \to 0$, the value of (2.5) tends to zero. In case $l = 1$ the limit of (2.5) equals $Mn$. So, letting $\epsilon$ tend to zero, one obtains (2.4).

We see that in the problem considered the $n$th moment varies linearly in time while the rest of the moments are invariant

$$M_k(D(t)) = M_k(D(0)), \quad k \neq n; \quad M_n(D(t)) = M_n(D(0)) + Mn.$$  

(2.6)

**Remark** The analogous fact is true for the problem of evolution under action of several sources, sinks, and multipoles: the point of the space of moments corresponding to the domain $D(t)$ moves along a straight line as the domain evolves.

The moments of the bubble are introduced in almost the same way. Let $E(t)$ be the intersection of the fluid domain $D(t)$ with a fixed, sufficiently large disk centred at the origin. We assume that the origin is contained in the air bubble $B$ (figure 2). The moments $M_n(D(t))$ for $n \geq 1$ are defined by the formula

$$M_n(D(t)) = \int_{E(t)} z^n \, dx \, dy.$$  

(2.7)
This integral is independent of the radius of the fixed disk. Define also \( M_0(D) \) as the bubble area with the opposite sign

\[
M_0(D) = -\int_B dxdy. \tag{2.8}
\]

**Theorem 2.3** Let \( D(t) \) be a solution of the problem of bubble evolution under action of a multipole at infinity. Then

\[
\frac{dM_k(D(t))}{dt} = \begin{cases} 
0, & k \neq n \\
M_n, & k = n
\end{cases} \tag{2.9}
\]

This theorem is proved in the same way as Theorem 2.2, using an analogue of the integrability theorem for bubble evolution ([6, 12–14]).

### 3 Moments and conformal mappings

A simply connected bounded domain with piecewise smooth boundary is determined by a conformal mapping of the unit disk onto this domain. Such a mapping is not unique. If the domain contains 0 then there exists the unique mapping \( f \) with properties

\[
f(0) = 0, \quad f'(0) > 0. \tag{3.1}
\]

This mapping will be called canonical.

Richardson found that the canonical mapping of a domain is a polynomial if and only if all its moments, except for a finite number, equal zero.

**Theorem 3.1** [4] Let \( D \) be a simply connected bounded domain, \( f: K \rightarrow D \) be the canonical mapping of the unit disk \( K \) onto \( D \). Then \( f \) is a polynomial of degree \( n \) if and only if \( M_k(D) = 0, k \geq n; M_{n-1}(D) \neq 0 \).

A domain with this property will be called polynomial of degree \( n \).
An analogous statement holds for the bubble case. The canonical mapping then is defined as the one satisfying the conditions

\[ f(0) = \infty, \quad \text{Res}(f(0)) > 0. \]  

**Theorem 3.2** Let \( D \) be an unbounded domain whose complement is bounded and simply connected, and \( f: K \to D \) be the canonical conformal mapping. Then \( zf(z) \) is a polynomial of degree \( n \) if and only if \( M_k(D) = 0, \quad k \geq n + 1; \quad M_n(D) \neq 0. \)

A domain with this property will be called polynomial of degree \( n \).

### 4 Exact solutions

A reason to examine multipole-driven Hele–Shaw flows is that they supply the simplest non-trivial explicit solutions in the Hele–Shaw flow theory. Here we study these solutions and analyze their extremal properties.

Consider the multipole-driven evolution \( D(t) \) of the fluid domain in the case when the initial domain is the disk of radius \( R \) centred at the origin. Since the moments of the disk are

\[ M_0(D(0)) = \pi R^2, \quad M_j(D(0)) = 0, j > 0, \]  

for \( D(t) \), by Theorem 2.2, we have

\[ M_0(D(t)) = \pi R^2, \quad M_n(D(t)) = M_{nt}, \quad M_j(D(t)) = 0, j \neq 0, n, \]  

\( M, n \) being the moment and the order of the multipole. So, by Theorem 3.1, \( D(t) \) is a polynomial domain of degree \( n \) for \( t > 0 \), i.e. the canonical mapping \( f_t: K \to D(t) \) has the following form:

\[ f_t(\xi) = a_1(t) \xi + a_2(t) \xi^2 + \ldots + a_{n+1}(t) \xi^{n+1}. \]  

Moreover, since the flow is obviously invariant with respect to rotation by angle \( 2\pi/n \), the mapping \( f_t \) must have the same property:

\[ f_t(e^{2\pi i/n} \xi) = e^{2\pi i/n} f_t(\xi). \]  

Substituting (4.3) into (4.4), one obtains

\[ a_2(t) = a_3(t) = \ldots = a_n(t) = 0, \]  

hence

\[ f_t(\xi) = A(t) \xi + B(t) \xi^{n+1}. \]  

To find \( A(t) \) and \( B(t) \), consider the representation of the moments in terms of the canonical mapping

\[ M_s(D) = \frac{i}{2} \int_K (f(\xi))^s \frac{df(\xi)}{d\bar{\xi}}. \]  

This formula is obtained by the substitution \( z = f(\xi) \) in (2.1). For the case (4.6) formula (4.7) reduces to

\[ |A|^2 + (n+1)|B|^2 = R^2, \quad A^{n+1} B = M_{nt}/\pi. \]  

The first equation corresponds to \( k = 0 \) in (4.7), the second one to \( k = n \).
Suppose that $M > 0$. Then $A = f'(0) > 0$ and $B > 0$ (from (4.8)), hence (4.8) can be written in the form

$$A^2 + (n+1)B^2 = R^2,$$  \hfill (4.9)

$$A^{n+1}B = Mnt/\pi.$$  \hfill (4.10)

Equation (4.9) describes an ellipse $\Gamma_1$, and equation (4.10) a hyperbola-like curve $\Gamma_2(t)$, in the $(A, B)$ plane (figure 3). Let $t^*$ be the time at which $\Gamma_2(t)$ meets $\Gamma_1$. It is easy to find $t^*$:

$$t^* = \frac{\pi R^{n+2}(n+1)^{(n+1)/2}}{Mn(n+2)^{(n+2)/2}}.$$ \hfill (4.11)

It is seen from figure 3 that system (4.9), (4.10) has two positive solutions for $t < t^*$, one for $t = t^*$, and none for $t > t^*$. For $t \leq t^*$, only one of the solutions corresponds to a real domain, namely, the one with smaller $B$. (For the other solution the mapping (4.6) is not one-to-one on $K$). So, the evolution of the fluid domain $D(t)$ on $[0, t^*]$ is uniquely determined by (4.9), (4.10).

At $t = t^*$ one obtains readily $A = (n+1)B$. This implies that $\partial D(t)$ is an epicycloid formed by rolling a disk around another of $n$ times larger radius (figure 4). This curve has $n$ cusps at points $(A-B)e^{(2k+1)(n/\pi)}$, $k = 0, 1, \ldots, n-1$. These cusps indicate a breakdown of
the solution of the moving boundary problem (cf. [15, 16]). The evolution, hence, cannot be extended to $t > t^\ast$.

Now consider the same evolution problem when the initial domain is polynomial of degree $n$. By theorems 2.2 and 3.1, then $D(t)$ is a polynomial domain of degree $n + 1$, i.e. the canonical mapping $f_\xi: K \to D(t)$ has the form

$$f_\xi(\xi) = a_1(\xi) \xi + a_2(\xi) \xi^2 + \ldots + a_{n+1}(\xi) \xi^{n+1}. \quad (4.12)$$

Since $M_0(D(t)) = S$ is the area of $D(0)$, $M_n(D(t)) = Mnt$, using (4.7) for $k = 0$ and $k = n$, one obtains

$$\sum_{j=1}^{n+1} j |a_j|^2 = S/\pi, \quad \frac{a_{n+1}^n}{a_1^n} a_1^n = \frac{Mnt}{\pi}. \quad (4.13)$$

Suppose that $M > 0$. Then $a_1 = f'(0) > 0$, hence $a_{n+1} > 0$, and (4.13) implies

$$a_1^2 + (n + 1) a_{n+1}^2 \leq S/\pi; \quad a_1^n a_{n+1} = \frac{Mnt}{\pi}. \quad (4.14)$$

It has been shown that these relations are compatible if and only if $t \leq t^\ast$ where $t^\ast$ is given by (4.11) for $R^2 = S/\pi$. Thus (4.12) breaks down not later than $t = t^\ast$. In other words, the following extremal property of solution (4.6) holds.

**Theorem 4.1** Let $t^\ast(D)$ be the time of breakdown of the evolution of the initial domain $D$ under action of a multipole of order $n$ and moment $M$ at $0$. Then among all polynomial domains $D$ of degree $\leq n$ and area $S$ the disk centred at $0$ has the maximal $t^\ast(D)$.

In other words, among all polynomial solutions of degree $\leq n$ and area $S$, the solution starting from the disk has the maximal lifetime. Thus we have an estimate

$$t^\ast(D) \leq t^\ast = \frac{\pi^{-n/2} S^{(n+2)/2} (n + 1)^{(n+1)/2}}{Mn(n+2)^{(n+2)/2}}. \quad (4.15)$$

For polynomial solutions of degree $n + 1$ and area $S$, the maximal possible lifetime is $2t^\ast$. Namely, let $G$ be the domain of area $S$ with cusps developing from the disk under action of a multipole of order $n$ and moment $-M$ for the time $t^\ast$. We regard $G$ as an initial domain for the action of a multipole of order $n$ situated at the same point but now having the moment $M$ instead of $-M$.

**Theorem 4.2** Among all polynomial domains of degree $\leq n + 1$ and area $S$, the domain $G$ has the maximal possible time $t^\ast(G)$ of regular evolution under the action of a multipole of order $n$ and moment $M$. Moreover, $t^\ast(G) = 2t^\ast$.

Thus, for each polynomial domain $D$ of degree $\leq n + 1$,

$$t^\ast(D) \leq \frac{2\pi^{-n/2} S^{(n+2)/2} (n + 1)^{(n+1)/2}}{Mn(n+2)^{(n+2)/2}}. \quad (4.16)$$

**Proof** Let $D(t)$ be a polynomial solution of degree $n + 1$. Then the canonical mapping has the form (4.12), and instead of (4.13) one has

$$\sum_{j=1}^{n+1} j |a_j|^2 = S/\pi, \quad \frac{a_{n+1}^n}{a_1^n} a_1^n = (M_n(D(0)) + Mnt)/\pi, \quad (4.17)$$

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which implies

\[ |a_i|^2 + (n+1)|a_{n+i}|^2 \leq S/\pi, \quad |a_{n+i}| = |(M_n(D(0)) + Mnt)/\pi|, \]  

\[(4.18)\]

We know that breakdown occurs when

\[ |(M_n(D(0)) + Mnt)/\pi| = Mnt^*/\pi, \]  

\[(4.19)\]

and, so as long as the solution exists, \(|M_n(D(0))/Mn + t| \leq t^*\). This inequality holds for the longest time when \(M_n(D(0))/Mn = -t^*\), and the time of its breakdown is then \(2t^*\), which proves estimate (4.16). It is easy to show that inequality (4.16) turns to an equality if the initial domain is \(G\).

In other words, the extremal flow can be obtained from that of Theorem 4.1 by a kind of reflection: it starts at \(t = 0\) with the domain \(G\), passes through the circular disk shape at \(t = t^*\), and finally ceases to exist at \(t = 2t^*\) when the fluid domain has evolved to \(G\) rotated through the angle \(\pi/n\) about the origin. This reflection is possible due to the time-reversal properties of such flows: changing the sign of \(M\) is equivalent to changing the direction of \(t\)-axis.\(^1\)

**Remark** It is obvious that the constructed extremal solutions are unique.

The analogous exact solutions exist for the problems of evolution of an unbounded domain and have similar properties. Let us summarize the main results.

Consider the flow produced by a multipole at infinity of order \(n\) and moment \(M\). Let the initial domain be the exterior of the circle of radius \(R\) centred at the origin. The moments of the fluid domain \(D(t)\) are

\[ M_0(D(t)) = -nR^2; \quad M_n(D(t)) = Mnt; \quad M_k(D(t)) = 0, k \neq 0, n, \]  

\[(4.20)\]

hence the canonical mapping \(f: K \rightarrow D\) has the form

\[ f_\xi(\zeta) = A(t)/\xi + B(t)\xi^{-1/n}, \quad A(t) \in \mathbb{R}^+. \]

\[(4.21)\]

Suppose that \(M > 0\). Then \(B(t) < 0\), and \(A(t), B(t)\) can be derived from equations similar to (4.9), (4.10):

\[ A^2 - (n-1)B^2 = R^2, \]  

\[(4.22)\]

\[ B = -MntA^{n-1}/\pi. \]  

\[(4.23)\]

Since the case \(n = 1\) is trivial (a circular bubble moves without changing its shape), consider \(n \geq 2\). There are two essentially different cases.

**Case 1:** \(n = 2\) Equation (4.22) describes the hyperbola \(\Gamma_1\), and equation (4.23) the straight line \(\Gamma_n(t)\) whose slope depends linearly on time (figure 5). \(t^* = \pi/2M\) is the time at which \(\Gamma_n(t)\) is an asymptote of the hyperbola \(\Gamma_1\). It is clear that system (4.22), (4.23) has one solution in the region \(A > 0\) for \(t < t^*\) and none for \(t \geq t^*\). For \(t < t^*\), the solution is

\[ A = R(1-(2Mt/\pi)^2)^{-\frac{1}{2}}, \quad B = -2MRt(1-(2Mt/\pi)^2)^{-\frac{1}{2}}/\pi. \]  

\[(4.24)\]

\(^1\) We are indebted to the referee for this remark.
and the bubble domain $\overline{D(t)}$ is the interior of the ellipse

$$\frac{x^2}{(A+B)^2} + \frac{y^2}{(A-B)^2} = 1. \quad (4.25)$$

As $t \to t^*$, this ellipse tends to the imaginary axis (figure 6).

**Case 2: $n > 2$** This case is analogous to the case of a bounded fluid domain. Equation
(4.22) describes a hyperbola $\Gamma_1$, and equation (4.23) a hyperbola-like curve $\Gamma_2(t)$ (figure 7). The time $t^*$ at which $\Gamma_2(t)$ touches $\Gamma_1$ is given by the formula

$$t^* = \frac{\pi R^{n-2}(n-2)^{(n-3)/2}}{M n(n-1)^{(n-1)/2}}.$$  

The system (4.22), (4.23) has two solutions in region $A > 0$ if $t < t^*$, one for $t = t^*$, and none for $t > t^*$. For $t \leq t^*$, only one of the solutions corresponds to a real domain, namely the one with smaller $A$. (For the other solution the mapping (4.21) is not one-to-one on $K$).

Thus, the system (4.22), (4.23) uniquely determines the evolution on the interval $[0, t^*].$

At $t = t^*$ one has $A = -(n-1)B$, which implies that $\partial D(t^*)$ is a hypocycloid with $n$ cusps (figure 8). The evolution cannot be extended for times $t > t^*$.

Solution (4.21) has extremal properties analogous to Theorems 4.1, 4.2.

**Theorem 4.3** Let $t^*(D)$ be the time of breakdown of the solution with initial domain $D = \bar{B}$ ($B$ is bounded and simply connected).
1. Among all polynomial domains $D$ of a degree $n - 1$, $n > 2$, with $B$ having area $S$, the exterior of the circle has the maximal $t^*(D)$. In other words, for any $D$

$$t^*(D) \leq t^* = \frac{\pi^{2-n/2} S^{(n-2)/2}(n-2)^{(n-2)/2}}{M n(n-1)^{(n-1)/2}}. \quad (4.27)$$

2. Among all polynomial domains $D$ of a degree $< n$ with area $S$ of $B = D$, the domain with cusps that develops from the exterior of the circle under action of a multipole of order $n$ and moment $-M$ for the time $t^*$ has the maximal $t^*(D)$, which equals $2t^*$. In other words, for each $D$ of degree $n$

$$t^*(D) \leq \frac{2n^{2-n/2} S^{(n-2)/2}(n-2)^{(n-2)/2}}{M n(n-1)^{(n-1)/2}}. \quad (4.28)$$

These formulas also hold for $n = 2$ if we use the convention $0^0 = 1$.

5 An estimate for the breakdown time

We have seen in the previous section that each of the solutions we constructed has evolved to a blow-up. This leads to a natural question whether this behaviour is typical for any initial domain. Since the area of the fluid domain (bubble) remains unchanged in the course of evolution, one might expect that some solutions would exist for all $t > 0$. To show that this is impossible for bounded flows, we need to consider the Cauchy transform of the domain $D(t)$ [4–6, 11, 12]:

$$h_D(w) = \frac{1}{\pi} \int_{\partial D(t)} \frac{dx \, dy}{w - z}. \quad (5.1)$$

Using the expansion

$$h_{D(0)}(w) = \frac{1}{\pi} \sum_{j=0}^{\infty} M_j(D(t)) \frac{1}{w^{j+1}}, \quad (5.2)$$

valid for sufficiently large $|w|$, one can derive from Theorem 2.2 the following statement:

**Theorem 5.1** If $D(t)$ is a solution of the problem (1.3)–(1.5) then for large $|w|$

$$\frac{\partial}{\partial t} h_{D(0)}(w) = \frac{M n}{\pi w^{n+1}}. \quad (5.3)$$

Note that this also follows directly from (1.4), and the equation

$$\frac{\partial g(z, t)}{\partial t} = 2 \frac{\partial F(z, t)}{\partial z} \quad (5.4)$$

([7, 8, 10], where $g(z, t)$ denotes the Schwarz function of $\partial D(t)$, to be defined in the next section), and the connection between $g(z, t)$ and the Cauchy transform of $D(t)$ [4, 5, 11].

In the case of several multipoles one has

$$\frac{\partial}{\partial t} h_{D(0)}(w) = R(w), \quad (5.5)$$

where $R(w)$ is a rational function determined by the moments and orders of the multipoles.

Now we can prove the following:
Theorem 5.2 The breakdown time $t^*(D)$ of evolution of any bounded simply-connected domain under the action of any number of multipoles is finite.

Proof. We use the following estimate [11, 17] of the Cauchy transform of a domain $D$ of area $\pi r^2$:

$$|h_D(w)| \leq r$$

(5.6)

([11] also contains an easy proof of a weaker estimate $|h_D| \leq 2r$). Let $D(t)$ be the evolution of $D$ during $[0, t]$. Then, integrating (5.5), one has, for large $|w|$,

$$h_{D(t)}(w) - h_{D(0)}(w) = R(w) \tau.$$  

(5.7)

Choose a point $w$ far enough from the origin so that $R(w) = 0$. From (5.6) it follows that $\tau \leq 2r/(|R(w)|)$ which proves the theorem.

To obtain an explicit estimate for the problem (1.3)–(1.5), observe that by analytic continuation (5.7) holds for all $w \not\in D(0) \cup D(t)$; since the domains $D(0)$ and $D(t)$ have area $\pi r^2$, the point $w$ can be chosen so that $|w| \leq \sqrt{2}r$. This gives the estimate

$$t^*(D) \leq \frac{2^{s+3/2}\pi r^{n+2}}{Mn}.$$  

(5.8)

This estimate is obviously very rough. The problem of finding sharper estimates or the exact one (with the corresponding extremal domain) remains open.

Remark 1 One can see that Theorem 5.2 holds if the action of any number of sources and sinks (and even a continuous distribution of sources and sinks, as in [11]) is allowed, with the only condition that their total rate be equal to zero.

Remark 2 The similar statement about bubbles is not true: an elliptic bubble in a steady flow (dipole at infinity) gives an example of a solution which does not blow up.

Remark 3 Theorem 5.2 does not hold if we allow the singularities to move. It is easy to show that the evolution of the unit disk under the action of a travelling dipole of moment $M$ located at the point $Mt/\pi$ at a time $t$ is just the uniform motion of this disk at the speed of $M/\pi$ without changing its shape. Obviously, this solution never blows up.

6 Steady boundary shapes for flows with non-zero surface tension

The problem of §1 and similar problems with several sources and multipoles are limiting cases of the more general and much more important problem of boundary dynamics with surface tension. Some versions of this problem have been studied for more than 30 years by many authors [7, 16, 18–20], but the first nontrivial exact solutions were those found quite recently by Kadanoff [21, 22]. The existence of the simple explicit solutions of §4 encourages one to examine analytically the boundary behaviour in the presence of surface tension under the action of a multipole at $0$ or $\infty$. It can be conjectured that the presence of surface tension may prevent the process of cusp formation or infinite lengthening of an elliptic bubble, thus making the boundary approach a stationary shape. So the problem
arises to find all equilibrium shapes corresponding to a given area of the fluids domain or air bubble. This problem happens to be unexpectedly easy, so that all the shapes are found readily in elementary functions, for multipoles of order \( n > 1 \).

The statement of the problem with nonzero surface tension has one difference from that in §1: instead of the zero boundary condition for the velocity potential one has

\[
\Phi = -\kappa \beta \quad \text{on} \quad \partial D
\]

(6.1)

for evolution of a bounded fluid domain, and

\[
\Phi = \kappa \beta \quad \text{on} \quad \partial D
\]

(6.2)

for evolution of a bubble, \( \beta \) being a positive constant proportional to the surface tension coefficient, and \( \kappa \) the boundary curvature.

If \( D \) is an equilibrium shape then the stream function is constant on \( \partial D \), because \( \partial D \) is evidently a streamline (figure 9a, b). So, the complex potential \( F(z) \) in \( D \) has the boundary values

\[
F(z) = \pm \kappa \beta \quad \text{on} \quad \partial D.
\]

(6.3)

Thus, the analytic continuation of the function \( \pm \kappa(z) \beta \) from \( \partial D \) to \( D \) has inside \( D \) just one singularity – a pole of order \( n \) at \( \infty \) or 0. This information turns out to be sufficient to reconstruct \( D \).
Let us now introduce the important notion of the Schwarz function of an analytic curve $\Gamma \subset \mathbb{C}$. It is a holomorphic function $g(z)$ in a neighbourhood of $\Gamma$ such that

$$g(z) = \bar{z}, \quad z \in \Gamma. \quad (6.4)$$

This function always exists and is unique. It has the following properties:

1. The tangent vector to $\Gamma$ at a point $z$ is $(g'(z))^{-\frac{1}{2}}$.
2. The curvature of $\Gamma$ at a point $z$ is

$$\kappa(z) = -i \frac{d}{dz} (g'(z))^{-\frac{1}{2}} = \frac{ig''(z)}{2(g'(z))^{\frac{3}{2}}}. \quad (6.5)$$

3. If $f$ is the canonical mapping of a domain $D$ and $\Gamma = \partial D$ then

$$g(f(\zeta)) = f(\bar{\zeta}) \quad (6.6)$$

for any $\zeta$ at which both sides are well defined.

**Remark** If $\Gamma = \partial D$ is a closed simple curve then $g'(z) = d\bar{z}/dz$ turns around twice as $z$ goes around $\partial D$, so $(g'(z))^{-\frac{1}{2}}$ is single-valued along $\Gamma$; the sign of this square root is chosen according to the orientation of $\Gamma$, which will be taken to be anti-clockwise.

All these and many other properties of the Schwarz function are described in detail by Davis [23].

Equalities (6.3) and (6.5) imply that if $D$ is an equilibrium shape then

$$F(z) = \pm \frac{i\beta g''(z)}{2(g'(z))^{\frac{3}{2}}}. \quad (6.7)$$

So, the function $(i\beta g''(z))/(2(g'(z))^{\frac{3}{2}})$ has just one singularity inside $D$ – a pole of order $n$ at zero or infinity. Thus any stationary shape is shown to be a solution of an inverse problem for a special class of Schwarz functions, similar to that of Millar [24].

Now consider the cases of bounded and unbounded $D$ separately.

**Case 1: Equilibrium shapes of a bounded fluid domain**

According to the above statement, the function $-i(g'(z))^{-\frac{1}{2}}$ has the only pole of order $n - 1$ at zero; it is holomorphic in $D \setminus \{0\}$ and has asymptotics

$$-i(g'(z))^{-\frac{1}{2}} \sim Mz^{1-n}/2\pi\beta(n-1), \quad z \to 0. \quad (6.8)$$

By the argument principle, this implies that $(g'(z))^{-\frac{1}{2}}$ has $n$ zeros inside $D$, because it turns around 0 once as $z$ goes along $\partial D$. These zeros are situated symmetrically at vertices of a regular polygon: $Z_k = Z \exp(2\pi ik/n)$. Therefore, the function $(g'(z))^{\frac{1}{2}}$ is meromorphic inside $D$; it has simple poles at $Z_k$ and no other singularities. On the other hand, differentiating (6.6) by $\zeta$ and taking square root, one gets

$$(g'(f(\zeta)))^{\frac{1}{2}} = \frac{-i}{\zeta} \left(\frac{f''(1/\zeta)}{f'(\zeta)}\right)^{\frac{1}{2}}. \quad (6.9)$$
So, the function
\[
\frac{-i \left( f'(1/\zeta) \right)^{1/2}}{\zeta}
\]
is meromorphic inside the unit circle, and has there \( n \) simple poles at vertices of a regular polygon and no other singularities. Since \( f'(\zeta) \) is nonzero and holomorphic in the unit disk, the same statement about singularities is true for the function \( f'(1/\zeta) \); moreover, this function must vanish at \( \zeta = 0 \). Hence, the function \( f'(\zeta) \) is meromorphic outside the unit circle: it has \( n \) simple poles at points \( a \exp(2\pi ik/n) \), \( a > 0 \), \( 0 \leq k \leq n - 1 \), and vanishes at infinity. Since this function is regular inside the disk, we conclude that it is a rational function of the form
\[
f'(\zeta) = \frac{A}{a^n - \xi^n}, \quad A/a^{2n} > 0, |a| > 1.
\]
(6.10)
Hence
\[
f(\zeta) = A \left( \frac{\zeta}{a^n - \xi^n} - \frac{n-1}{na^n} \phi_n(a^n, \zeta) \right),
\]
(6.11)
where
\[
\phi_n(a^n, \zeta) = -na^n \int_0^\zeta \frac{dt}{a^n - t^n} = \sum_{k=1}^n a e^{2\pi ik/n} \ln(1 - e^{-2\pi ik/n} \zeta/a).
\]
(6.12)

From equation (6.9), by comparison of the principal parts of both sides at \( \zeta = 0 \), one derives the following relationship between parameters \( x = a^n \) and \( A \) (here we assume \( M > 0 \), which yields \( A > 0 \) and \( x > 1 \):
\[
\frac{M}{2\pi(n-1)\beta} \left( \frac{x^2}{\beta} \right)^{n-1} = \frac{1}{x}.
\]
(6.13)
Thus one obtains a smooth one-parameter family of steady-state solutions
\[
f(\zeta) = x^{n((n-1)}/n \left( \frac{M}{2\pi(n-1)\beta} \right)^{1/(n-1)} \left( \frac{\zeta}{x^n - \xi^n} - \frac{n-1}{nx} \phi_n(x, \zeta) \right).
\]
(6.14)

Direct computer evaluations show that this function is univalent in the unit disk for all \( x > 1 \) if \( n = 2 \), and for \( x \geq x_*(n) \) if \( n > 2 \) (in the latter case it ceases to be univalent when \( x < x_*(n) \)); the values of \( x_*(n) \) for \( n \leq 7 \) are presented in Table 1. So, a one-parameter family of exact steady-state solutions is constructed for any order of the multipole and any values of \( M \) and \( \beta \).

Table 1

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_{\text{min}}(n) )</th>
<th>( \lambda_{\text{min}}(n) )</th>
<th>( x_*(n) )</th>
<th>( \lambda_*(n) )</th>
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<td>4.608279</td>
</tr>
</tbody>
</table>
It is easy to obtain the relationship between the parameter $x$ and the area of the fluid domain $S$:

$$
\frac{S}{n} \left( \frac{2\pi \beta (n-1)}{M} \right)^{2/(n-1)} = \frac{1}{n} \left( x^2 \right)^{(n-1)/n} \left( \frac{2n-1}{n} \frac{1}{x^2 - 1} + \frac{1}{(x^2 - 1)^2} + \frac{(n-1)^2}{n} \int_0^1 \frac{dt}{x^2 - t^n} \right) \quad (6.15)
$$

The graphs of the right hand side functions of (6.15) for $n = 2, 3, 4, 5, 6, 7$ are presented in figure 10. One can see that all these functions tend to infinity when $x$ goes to infinity, and attain their minimal positive values $\lambda_{\min}(n)$ at some finite points $x = x_{\min}(n)$. Therefore,
there are no solutions when the dimensionless parameter \( \lambda = S/(2\pi\beta(n-1))/M^{2(n-1)} \) is small enough: \( \lambda < \lambda_{\text{min}}(n) \).

Furthermore, one can prove that for all \( n > 2 \), \( x_{\text{min}}(n) > x_*(n) \) (see Table 1). If we denote by \( \lambda_*(n) \) the value of the right hand side of (6.15) at \( x_*(n) \) (the values of \( \lambda_*(n) \) and \( \lambda_{\text{min}}(n) \) are also given in Table 1; it is clear that \( \lambda_{\text{min}}(n) < \lambda_*(n) \) for all \( n \geq 2 \), so we see that the problem has two solutions when \( \lambda_{\text{min}}(n) < \lambda < \lambda_*(n) \), and that it has one solution when \( \lambda = \lambda_{\text{min}}(n) \) or \( \lambda > \lambda_*(n) \).

Two stationary fluid domains with equal areas are shown in figure 11 a, b (\( n = 2 \) and 4). Figure 12 a, b (\( n = 4,5 \)) illustrate a situation when one of the solutions of (6.15) corresponds to a multivalued conformal mapping.

**Case 2: Equilibrium shapes of a bubble**

If \( f(\zeta) \) is the canonical mapping of the unit disk \( K \) onto the unbounded fluid domain \( D \), equation (6.9) also holds. Reasoning analogously to case 1, one can easily obtain that the function \( f'(\zeta)^2 \) has \( n \) simple poles outside \( K \) and vanishes at infinity. So,

\[
f'(\zeta)^2 = \frac{(\lambda_0)^2}{(\zeta - a^n)^2}, \quad |a| > 1, \lambda/a^n > 0,
\]

\( (6.16) \)

and

\[
f(\zeta) = \frac{A}{na^2n} \left( -\frac{\zeta^{n-1}}{a^n - \zeta^n} + \frac{n}{\lambda_0} + (n+1) \psi_n (a^n, \zeta) \right),
\]

\( (6.17) \)

where

\[
\psi_n (a^n, \zeta) = -\int_0^\infty \left( \frac{t^n - \zeta^n}{a^n - t^n} \right)^n \sum_{k=1}^n a^{-1} e^{-2\pi ik/n} \ln(1 - e^{-2\pi ik/n} \zeta/a).
\]

\( (6.18) \)

Thus, in both cases the steady shapes, if they exist, can be expressed in elementary functions.

From (6.9), after the substitution \( a^n = x \) one has, assuming \( M > 0 \),

\[
\frac{M}{2\pi(n+1)\beta} \left( \frac{x^2}{A} \right)^{n-1} = \frac{1}{x}, \quad A > 0, x > 1,
\]

\( (6.19) \)
Hele–Shaw flows with a free boundary produced by multipoles

Table 2

<table>
<thead>
<tr>
<th>n</th>
<th>$x_{\text{max}}(n)$</th>
<th>$\mu_{\text{max}}(n)$</th>
<th>$x_{*}(n)$</th>
<th>$\mu_{*}(n)$</th>
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<td>0.186627</td>
</tr>
</tbody>
</table>

so again it is possible to obtain a smooth one-parameter family of steady-state solutions

$$f(\zeta) = \frac{x^{-1/(n+1)}}{n} \left( \frac{2\pi(n+1)}{M} \right)^{1/(n+1)} \left( \frac{n}{\zeta} - \frac{x}{x-\zeta} + (n+1) \psi_n(x, \zeta) \right). \quad (6.20)$$

In this case, the critical values $x_{*}(n)$ can also be calculated (Table 2), so that the functions (6.20) are univalent in the unit disk when $x \geq x_{*}(n)$. The bubble area can be expressed in terms of $x$ as follows:

$$S = \frac{2\pi\beta(n+1)}{n^2} \left( n^2 - \frac{n+1}{x^2-1} - \frac{n}{(x^2-1)^2} - (n+1)^2 \int_0^1 \frac{t^{n-2}}{x^2-t^2} dt \right). \quad (6.21)$$

The graphs on the right hand side functions of (6.21) are shown in figure 13. These functions tend to zero as $x$ goes to infinity, remaining positive, and attain their maximal values $\mu_{\text{max}}(n)$ at some finite $x = x_{\text{max}}(n)$. So, there are no solutions when the dimensionless parameter $\mu = S/\pi((2\pi\beta(n+1))/M)^{2/(n+1)}$ is sufficiently large: $\mu > \mu_{\text{max}}(n)$. 

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As in case 1, one can prove that $x_*(n) < x_{\text{max}}(n)$ (see Table 2), and if one denotes by $\mu_*(n)$ the value of the right hand side of (6.21) at $x_*(n)$, one can see that there are two equilibrium shapes when $\mu_*(n) < \mu < \mu_{\text{max}}(n)$, and only one stationary shape when $\mu = \mu_{\text{max}}(n)$ or $\mu \leq \mu_*(n)$ (the values of $\mu_*(n)$ and $\mu_{\text{max}}(n)$ for $n \leq 7$ are also in Table 2).

Two stationary solutions with equal areas of the bubble are shown in figure 14a, b ($n = 2$ and 3). Figure 15a, b ($n = 2$ and 6) illustrate a situation when one of the solutions of (6.21) gives a multivalued function (6.20).
In both cases the main question is whether the evolution of a given domain in a multipole-driven flow with surface tension can lead to a stationary shape. We have proved that it is impossible when the area of the blob (bubble) is small (large) enough. However, there is a range of areas where formulas (6.14), (6.20) provide one or two stationary shapes of the boundary for a given value of the area. In this case it is not clear how to tell if a given initial domain will tend to a stationary shape. It is strongly suggested by intuition that if a stationary shape is unique, it is stable, and if there are two stationary shapes, one of them is stable and the other is not, so that the linearization of the flow in its neighbourhood has a single eigenvalue with positive imaginary part. These two equilibrium points collide and annihilate as the area attains its critical value. However, the details of this stability analysis are yet to be worked out, so the above statements should be considered as conjectures.

Remark 1 In the formulas of this section, we could take n to be any real number greater than 1. In this case \( f(\zeta) \) should be interpreted as the conformal mapping of the sector \( \{ re^{i\theta} \mid r \leq 1, 0 \leq \theta \leq 2\pi/n \} \) onto a domain \( D \) bounded by straight lines \( \theta = 0 \) and \( \theta = 2\pi/n \) and a curve (figure 16a, b). This domain turns out to be a stationary shape of the fluid domain in the problem of dipole-driven evolution inside the angle \( 2\pi/n \) with impervious sides, the dipole being situated at 0 or \( \infty \). The approach obtained above remains valid for this problem. The function \( f(\zeta) \) is a hypergeometric series with respect to \( \zeta \).

Remark 2 The analysis of this section can be generalized to problems with an arbitrary number of multipoles of order \( n \geq 2 \). Then, if \( D \) is an equilibrium shape of the fluid domain, the canonical mapping \( K \to D \) is given by a function \( f(\zeta) \) whose derivative \( f'(\zeta) \) is square of a rational function. For many multipoles, the number of steady shapes may also be large. The parameters of \( f(\zeta) \) can be found from a system of transcendental equations.
Conclusion

We have shown that complex variable methods developed for Hele–Shaw flows produced by sources and sinks can be applied to flows with multipole singularities, and the simplest exact solutions can be derived analytically. We have proved that, in the case of bounded flows, solutions always blow up at a finite time. On the other hand, the presence of surface tension changes the behaviour of the boundary radically, creating the possibility of stationary flows. We have constructed the corresponding steady shapes by solving a certain problem for Schwarz functions. These shapes seem to be the first examples of analytically obtained nontrivial solutions of a nonzero surface tension Hele–Shaw problem with a bounded gas-fluid interface.

The authors hope that all this has convinced the reader that, although this generalization of the standard Hele–Shaw model does not seem to have much practical relevance, it is worthy of being studied.

Acknowledgement

The authors are very grateful to Stella Kleinbock who performed computations and drew pictures for this paper on a personal computer.

Appendix: Steady shapes and calculus of variations

The steady shapes of §6 turn out to be solutions of an important variational problem of hydrodynamics. The meaning of this coincidence is not quite clear to us, and this section should be considered as an observation rather than a part of the mathematical theory.

Consider a fluid domain $D$ in Hele–Shaw cell or porous medium surrounded by an inviscid fluid. This domain can be either bounded or unbounded with bounded complement. Let the fluid be pumped out through $N$ small circular openings centred at points $z_1, \ldots, z_n$ under pressures $p_1, \ldots, p_n$, the pressure on $\partial D$ is constant and equals $p_0$. Denote by $Q(D)$ the total rate of injection.

**Variational problem** Among all domains $D$ of area $S$ (area $S$ of the complement) find the domain with the minimal value of $Q(D)$, the injection points $z_j$ and pressures $p_j$ being fixed.

The solution $D$ of this problem is called the extremal domain for the given system of sinks and pressures. It has been used to estimate the rate of an oil-producing well [25]. Critical domains (i.e. providing a critical point of $Q(D)$), are also of interest. The extremal domain is obviously a critical domain. For a given system of sinks and pressures, there may exist a few critical domains.

It can be shown that a critical domain is characterized by the property: the flow velocity along $\partial D$ has constant absolute value. This property allows us to find critical domains analytically.

**Theorem A1** A stationary shape $D$ (defined in §6) is a critical domain for the system of $n$ sinks $z_k = Ze^{2\pi k/n}$, $Z = f(1/a)$, $f$ being the canonical mapping of $D$ and $a > 0$ being the singular point of $f(\zeta)$.
Remark This theorem can be generalized to the case of several multipoles.

Sketch of proof Let $D$ be a stationary shape, and let $g(z)$ be the Schwarz function of its boundary. Then $g'(z)$ is a meromorphic function inside $D$. It has $n$ simple poles at $z_k$ but no other singularities. So,

$$\int_{\partial D} \frac{g'(z)dz}{w-z} = \frac{n\lambda w^{n-1}}{w^n - Z^n}, \quad \lambda > 0, w \notin D.$$

It is obvious that, on $\partial D g'(z)dz = dl$, the element of length. Hence,

$$\int_{\partial D} \frac{dl}{w-z} = \frac{n\lambda w^{n-1}}{w^n - Z^n} = \sum_{k=0}^{n-1} \frac{\lambda}{w-z_k}, \quad w \notin D.$$

Let $V(\sigma), \sigma \in \partial D$, be the absolute value of the flow velocity at $\sigma$ for the flow produced by suction from $D$ through the sinks $z_k$ with equal pressure. Then (see §2)

$$\int_{\partial D} \frac{V(\sigma)dl}{w-z} = \sum_{k=0}^{n-1} \frac{q}{w-z_k}, \quad w \notin D,$$

$q$ being the rate of each sink. Thus,

$$\int_{\partial D} \frac{(V(\sigma)/q)dl}{w-z} = \sum_{k=0}^{n-1} \frac{\lambda}{w-z_k}, \quad w \notin D.$$

This implies $V(\sigma)/q = 1$, i.e. $V(\sigma) = q/\lambda = \text{const.}$ So, the absolute value of the flow velocity is constant along the boundary, hence $D$ is a critical domain.

References


