S-adic version of Minkowski’s geometry of numbers and Mahler’s compactness criterion✩

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In this note we give a detailed proof of certain results on geometry of numbers in the S-adic case. These results are well-known to experts, so the aim here is to provide a convenient reference for the people who need to use them.

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1. Introduction

The space of unimodular lattices in $\mathbb{R}^n \ (n \geq 2)$ can be identified with the homogeneous space $X = \text{SL}_n(\mathbb{Z}) \backslash \text{SL}_n(\mathbb{R})$ via the correspondence $\mathbb{Z}^n g \leftrightarrow \text{SL}_n(\mathbb{Z})g$ where $g \in \text{SL}_n(\mathbb{R})$. It is proved by Mahler [11] that a subset $R$ of $X$ is relatively compact if and only if nonzero elements of the corresponding unimodular lattices are separated from zero. This phenomenon is called Mahler’s compactness criterion [2, Chapter V]. It has been very useful in dynamical approach to number theory; we refer the readers to survey papers [4, 5] and [7] and references there for details.

Let $S$ be a finite nonempty set of places of a global field $K$. We assume $S$ contains all the archimedean places if $K$ is a number field. For each place $v$ of $K$, let $K_v$ be the completion of $K$ at $v$. Let $K_S = \prod_{v \in S} K_v$ and

$$I_S = \{ a \in K : a \text{ is integral in } K_v \text{ for every place } v \notin S \}. \quad (1.1)$$

We consider $K$ and hence $I_S$ as subrings of $K_S$ via natural embeddings $K \rightarrow K_v$. Then the homogeneous space $\text{SL}_n(I_S) \backslash \text{SL}_n(K_S)$ can be identified with a set of free discrete $I_S$-modules of rank $n$ in $K^n_S$ with fixed covolume. The connection between dynamics and number theory also spreads to the $S$-adic setting, where the corresponding version of Mahler’s criterion plays an important role. The extension of Mahler’s criterion to the $S$-adic case when $K$ is a number field has already been used in several papers, and a proof for $K = \mathbb{Q}$ can be found in [9]. Moreover, a preliminary version [8] of the paper [9], published as a preprint of MPIM (Bonn), contains a proof of the $S$-adic Mahler’s criterion for arbitrary number field $K$. When $K$ is a function field with genus zero and $S$ contains a single place of degree one, Mahler’s criterion is proved in [6]. The general $S$-adic case is known to experts, but it is not easy to find a convenient reference. Here we provide a self-contained proof of an $S$-adic version of Mahler’s criterion.

**Theorem 1.1.** Let $n \geq 2$. A set $R \subset \text{SL}_n(I_S) \backslash \text{SL}_n(K_S)$ is relatively compact if and only if the subset

$$\{ \xi \in I^n_S g : \xi \neq 0, \ g \in \text{SL}_n(K_S) \text{ and } \text{SL}_n(I_S)g \in R \}$$

of $K^n_S$ is separated from zero, i.e. this set has empty intersection with some open neighborhood of zero in $K^n_S$.

Our proof of Theorem 1.1 is based on an $S$-adic version of Minkowski’s lemma of geometry of numbers. Let $\text{vol}$ be the normalized Haar measure on the additive group $K^n_S$ (see §2). For a discrete $I_S$-module $\Gamma$ in $K^n_S$ the covolume of $\Gamma$ (denoted by $\text{cov}(\Gamma)$) is the $\text{vol}$ of a fundamental domain of $\Gamma$ in $K^n_S$. Let $B_v(K^n_S)$ be the closed ball of radius $r$ centered at zero in $K^n_S$ with respect to the normalized norm (see §2). For each integer $1 \leq m \leq n$, the $m$-th minimum of a discrete $I_S$-module $\Gamma$ is defined by
\[ \lambda_m(\Gamma) = \inf \{ r > 0 : \dim_K(\text{span}_K(B_r(K_S^n) \cap \Gamma)) \geq m \}. \tag{1.2} \]

Here \( \text{span}_K \) is the \( K \) linear span of a set and \( \dim_K \) is the dimension of a vector space over \( K \). Similar notations are used when \( K \) is replaced by other rings. We remark here that if \( K = \mathbb{Q} \) and \( S \) contains only the archimedean place, then we get the usual concept of successive minima of lattice points in \( \mathbb{R}^n \).

For two nonnegative real numbers \( s \) and \( t \) the notation \( s \asymp t \) means \( C^{-1}s \leq t \leq Cs \) for some constant \( C \geq 1 \). Let \( \sigma \) and \( \tau \) be the number of real and complex places of \( K \) respectively.\(^1\) Let \( \sharp S = \tau + \text{card}(S) \) where \( \text{card} \) denotes the cardinality of a set. The \( S \)-adic version of Minkowski’s theorem on successive minima (see [12, Chapter IV, §1] for the usual case) is the following theorem.

**Theorem 1.2.** Let \( n \geq 1 \) and let \( \Gamma \subset K_S^n \) be a discrete \( I_S \)-module with finite covolume. Then

\[ (\lambda_1(\Gamma) \ldots \lambda_n(\Gamma))^{\sharp S} \asymp \text{cov}(\Gamma) \]

where the implied constants depend on \( K, S \) and \( n \).

A refined version of Theorem 1.2 will be proved in Theorem 4.4 where the implied constants will be explicitly calculated. If \( K \) is a function field of genus zero and \( S \) consists of a single place of degree one, then Theorem 1.2 is established in [10]. The adelic versions of Theorem 1.2 are proved in [1] (resp. [13]) when \( K \) is a number field (resp. function field).

2. Preliminaries: notations

Let \( K \) be a global field and let \( P \) be the set of places of \( K \). Throughout this paper we fix a positive integer \( n \) and a finite nonempty set \( S \subset P \) such that \( S \supset P_0 \) where \( P_0 \) (possibly empty) is the set of archimedean places of \( K \).

For every \( v \in P \) let \( K_v \) be the completion of \( K \) at \( v \). The \( S \)-adic numbers and integers are defined as

\[ K_S \overset{\text{def}}{=} \prod_{v \in S} K_v \quad \text{and} \quad I_S \overset{\text{def}}{=} \{ x \in K : x \text{ is integral for all } v \in P \setminus S \} \]

respectively. We consider \( K \) as a subring of \( K_S \) via the natural inclusions \( K \to K_v \).

For \( v \in P \), let \( | \cdot |_v \) be the normalized absolute value on \( K_v \). If \( v \) is archimedean, we identify \( K_v \) with real or complex numbers where the usual absolute value is \( | \cdot |_v \). If \( v \) is ultrametric then \( |a|_v^{-1} = \text{card}(I_v/aI_v) \) for all \( a \in I_v \) where \( I_v \) is the ring of integers of \( K_v \). For each ultrametric place \( v \in P \) we fix a uniformizer \( \varpi_v \) (a generator of the

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\(^1\) If \( K \) is function field we have \( \sigma = \tau = 0 \).
maximal ideal of $I_v$) and take $q_v = |σ_v|_v^{-1}$. We define the absolute value and content for $x = (x_v)_{v ∈ S} ∈ K_S$ respectively by

$$|x| = \max_{v ∈ S} |x_v|_v \quad \text{and} \quad \text{cont}(x) = \prod_{v ∈ S} |x_v|_v^{ε_v}$$

where $ε_v = 2$ if $K_v = \mathbb{C}$ and $ε_v = 1$ otherwise.

The additive group $K_S^n$ can be naturally identified with $\prod_{v ∈ S} K_v^n$ and we write every $ξ ∈ K_S^n$ as $(ξ_v)_{v ∈ S}$ according to this identification. More precisely, if $ξ = (x_1, \ldots, x_n)$ where $x_i = (x_{i,v})_{v ∈ S}$, then $ξ_v = (x_{1,v}, \ldots, x_{n,v})$. Similarly, the group $GL_n(K_v)$ can be naturally identified with $\prod_{v ∈ S} GL_n(K_v)$ and we write every $g ∈ GL_n(K_v)$ as $(g_v)_{v ∈ S}$ according to this identification. The group $GL_n(K_v)$ (resp. $GL_n(K_S)$) acts on $K_v^n$ (resp. $K_S^n$) by matrix multiplication from the right. Moreover, the action of $g ∈ GL_n(K_v)$ on $ξ ∈ K_S^n$ is consistent with these identifications, that is, $ξg = (ξ_vg_v)_{v ∈ S}$ under previous notations.

For $v ∈ P$ we take vol$_v$ to be the normalized Haar measure on $K_v$. For archimedean $v$, the measure vol$_v$ is the Lebesgue measure. If $v$ is ultrametric, the measure satisfies vol$_v(I_v^n) = 1$. It follows directly from definition that

$$\text{vol}_v(aB) = |a|_v^{nε_v}\text{vol}_v(B)$$

for every $a ∈ K_v$ and any measurable subset $B$ of $K_v^n$. We take the normalized Haar measure vol on $K_S^n$ to be the product measure. In the sequel we will abbreviate dvol(ξ) by dξ for the integration with respect to the volume measure. For a positive integer $m$, we use vol$_m^n$ and vol$_m^n$ to denote the normalized Haar measures on $K_v^n$ and $K_S^n$ respectively.

If $v$ is archimedean we take $|| · ||_v$ to be the Euclidean norm on $K_v^n$. If $v$ is ultrametric we take $|| · ||_v$ to be the sup norm with respect to coordinates, that is

$$||a_1, \ldots, a_n||_v = \max_{1 ≤ i ≤ n} |a_i|_v \quad \text{where} \quad a_i ∈ K_v.$$ 

We define the norm and content for $ξ ∈ K_S^n$ by

$$||ξ|| = \max_{v ∈ S} ||ξ_v||_v \quad \text{and} \quad \text{cont}(ξ) = \prod_{v ∈ S} ||ξ_v||_v^{ε_v}.$$ 

For $a ∈ I_v^n$, where $I_v^n$ is the group of multiplicatively invertible elements of $I_v$, we have that cont$(a) = 1$ and $\text{cont}(aξ) = \text{cont}(ξ)$ where $ξ ∈ K_S^n$. Also for every $g ∈ GL_n(K_v)$ we have

$$d(ξg) = \text{cont}(\text{det}(g)) dξ,$$

(2.1)

where det is the determinant of a matrix.

The set of vectors in $K_S^n$ (resp. $K_v^n$) with norm less than or equal to $r$ is denoted by $B_r(K_S^n)$ (resp. $B_r(K_v^n)$). It can be checked directly that
\[ B_r(K^n_S) = \prod_{v \in S} B_r(K^n_v). \]  

Let \( L \) be a free \( K_S \)-submodule of \( K^n_S \) with rank \( m \leq n \). Then

\[ L = \prod_{v \in S} L_v, \]

where \( L_v \) is an \( m \)-dimensional subspace of \( K^n_v \). There is a unique additive Haar measure \( \text{vol}_L \) on \( L \) (resp. \( \text{vol}_{L_v} \) on \( L_v \)) such that

\[ \text{vol}_L \left( L \cap B_1(K^n_S) \right) = \text{vol}^m \left( B_1(K^n_S) \right) \]

(resp. \( \text{vol}_{L_v} \left( L \cap B_1(K^n_v) \right) = \text{vol}^m \left( B_1(K^n_v) \right) \)).

Moreover, the above definition, (2.2) and (2.3) imply

\[ \text{vol}_L = \prod_{v \in S} \text{vol}_{L_v}. \]  

In the case where \( K = \mathbb{Q} \) and \( \text{card}(S) = 1 \), the measure \( \text{vol}_L \) is the measure given by the inner product on \( L \). Suppose \( \xi = (\xi_v)_{v \in S} \) and \( \xi_v \neq 0 \); then the covolume of \( I_S \xi \) (\( I_S \)-linear span of \( \xi \)) in \( K_S \xi \) (\( K_S \)-linear span of \( \xi \)) with respect to \( \text{vol}_{K_S \xi} \) is equal to \( \text{cont}(\xi) \) multiplied by the covolume of \( I_S \) in \( K_S \). The covolume of a discrete \( I_S \)-module \( \Gamma \) in \( K^n_S \) with respect to the induced measure \( \text{vol}_{K_S \Gamma} \) is called relative covolume of \( \Gamma \) and it is denoted by \( \text{cov}_r(\Gamma) \). The covolume of \( \Gamma \) with respect to \( \text{vol} \) is denoted by \( \text{cov}(\Gamma) \).

3. Discrete \( I_S \)-modules

Let \( \Gamma \subset K^n_S \) be a discrete \( I_S \)-module. In this section, we use ideas of \([9, \S 8]\) to study properties of \( \Gamma \).

Lemma 3.1. Let \( \Gamma \subset K^n_S \) be a discrete \( I_S \)-module and let \( \xi_1, \ldots, \xi_m \in \Gamma \). The following statements are equivalent:

1. \( \xi_1, \ldots, \xi_m \) are linearly independent over \( I_S \);
2. \( \xi_1, \ldots, \xi_m \) are linearly independent over \( K \);
3. \( \xi_1, \ldots, \xi_m \) are linearly independent over \( K_S \).

Proof. It suffices to show that (1) implies (3). We prove it by induction on \( m \). Write \( \xi_i = (\xi_{i,v})_{v \in S} \) as in §2. Suppose that \( \xi_1 \) is linearly dependent over \( K_S \), then there exists \( w \in S \) such that \( \xi_{1,w} = 0 \). According to the strong approximation theorem (see [3, Chapter II, \( \S 15 \)]), there is a sequence \( \{c_i\}_{i \geq 1} \) of \( I_S \) \( \setminus \{0\} \) such that \( |c_i|_v \to 0 \) as \( i \to \infty \) for any \( v \in S \setminus \{w\} \). Therefore \( c_i \xi_1 \to 0 \) which contradicts the assumption that \( \Gamma \) is discrete. This proves (3) in the case where \( m = 1 \).
Now suppose \( m > 1 \) and (1) implies (3) while \( m \) is replaced by \( m - 1 \). By the case for \( m = 1 \), we know \( \xi_{1,v} \neq 0 \) for every \( v \in S \). So there exists \( g \in \text{GL}_n(K_S) \) such that \( \xi_1 g = (1,0,\ldots,0) \). The right multiplication of \( g \) on \( K^n_S \) is a \( K_S \) linear isomorphism, so we can without loss of generality assume that \( \xi_1 = (1,0,\ldots,0) \). Let \( \varphi : K^n_S / K_S \xi_1 \cong K^{n-1}_S \) be the natural quotient map. Since \( I_S \xi_1 \) is a cocompact lattice in \( K_S \xi_1 \) and \( \Gamma \subset K^n_S \) is discrete, the module \( \varphi(\Gamma) \) is discrete and \( \varphi(\xi_2), \ldots, \varphi(\xi_m) \) are linearly independent over \( I_S \). In view of the induction hypothesis, we have \( \varphi(\xi_2), \ldots, \varphi(\xi_m) \) are linear independent over \( K_S \). Therefore \( \xi_1, \ldots, \xi_m \) are linearly independent over \( K_S \). \( \square \)

**Remark 3.2.** The implication (2) \( \Rightarrow \) (3) holds without assuming that \( \xi_1, \ldots, \xi_m \) belong to a discrete \( I_S \)-module, see [9, Lemma 7.1].

For a discrete \( I_S \)-module \( \Gamma \subset K^n_S \) let \( K \Gamma \) (resp. \( K_S \Gamma \)) be the \( K \)-linear (resp. \( K_S \)-linear) span of \( \Gamma \) in \( K^n_S \). We call the dimension over \( K \) of \( K \Gamma \) the rank of \( \Gamma \). It follows from Lemma 3.1 that the rank of \( \Gamma \) is less than or equal to \( n \) and the equality holds if and only if \( \Gamma \) has finite covolume.

In the following lemma we prove a Gram–Schmidt orthogonalization process for ultrametric local fields.

**Lemma 3.3.** Let \( K_v \) be a ultrametric local field. For any \( K_v \)-linearly independent vectors \( \xi_1, \ldots, \xi_m \in K^n_v \) there exist linearly independent vectors \( \eta_1, \ldots, \eta_m \in K^n_v \) such that \( \eta_1, \ldots, \eta_r \) are in the \( K_v \)-linear span of \( \xi_1, \ldots, \xi_r \) for all \( r \leq m \), and

\[
\|a_1 \eta_1 + \cdots + a_m \eta_m\|_v = \max_{1 \leq i \leq m} |a_i|_v \text{ for all } a_i \in K_v.
\]  

(3.1)

**Remark 3.4.** In the sequel we call a basis of \( L_v \) \( \text{def} \) \( \text{span}_{K_v} \{ \xi_1, \ldots, \xi_m \} \) which satisfies (3.1) an *orthonormal basis* of \( L_v \). The map

\[
\varphi : K_v^m \to L_v \text{ where } \varphi_v(a_1, \ldots, a_m) = a_1 \eta_1 + \cdots + a_m \eta_m
\]

is an isometric embedding sending \( \text{vol}_v^m \) to \( \text{vol}_{L_v}^m \).

**Proof.** Contrary to the archimedean case, here we choose an entry with maximal absolute value for the corresponding vector. Write

\[
\xi_i = (x_{i1}, \ldots, x_{in}), \text{ where } x_{ij} \in K_v \text{ and } 1 \leq i \leq m.
\]

First we choose \( j_1 \leq n \) such that \( \|\xi_1\|_v = |x_{1j_1}|_v \) and set \( \eta_1 = x^{-1}_{1j_1} \xi_1 \). Next we take \( \eta_2 = \xi_2 - x_{2j_1} \eta_1 = (y_{21}, \ldots, y_{2n}) \). We choose \( j_2 \leq n \) such that \( \|\eta_2\|_v = |y_{2j_2}|_v \) and set \( \eta_2 = y_{2j_2} \eta_2 \). In general after \( r \) steps we have \( r \) different integers \( j_1, \ldots, j_r \) and unit norm vectors \( \eta_1, \ldots, \eta_r \) such that \( \eta_i \) has \( j_i \)-th entry 1 and \( j_s \)-th entry zero for \( s < i \). We take

\[
\eta_{r+1} = \xi_{r+1} - x_{r+1,j_i} \eta_1 - \cdots - x_{r+1,j_r} \eta_r = (y_1, \ldots, y_n)
\]
and choose \( j_{r+1} \) such that \( \|\eta'_{r+1}\|_v = |y_{jr+1}|_v \). We set \( \eta_{r+1} = y_{jr+1}^{-1}\eta'_{r+1} \). Then it has \( j_{r+1} \)-th entry 1 and \( j_s \)-th entry 0 for \( s < r+1 \). This induction process gives \( m \) unit norm vectors \( \eta_1, \ldots, \eta_m \).

For \( (a_1, \ldots, a_m) \in K_v^m \) let
\[
k = \min\{1 \leq r \leq m : |a_r|_v = \max_{1 \leq i \leq m} |a_i|_v \}.
\]

It is clear from the construction that
\[
\|a_1\eta_1 + \cdots + a_m\eta_m\|_v = |a_k|_v,
\]
which proves (3.1). \( \Box \)

The next lemma appeared as [9, Corollary 8.4] for \( K = \mathbb{Q} \) and as [8, Corollary 5.8] for \( K \) a number field.

**Lemma 3.5.** Suppose \( \Gamma \) and \( \Gamma' \) are discrete \( I_S \)-modules in \( K_S^m \) with
\[
K\Gamma \cap K\Gamma' = \{0\}. \tag{3.2}
\]

Then
\[
\text{cov}_r(\Gamma + \Gamma') \leq \text{cov}_r(\Gamma)\text{cov}_r(\Gamma'). \tag{3.3}
\]

**Proof.** Let \( L = K_S\Gamma, L' = K_S\Gamma' \) and \( L'' = L + L' \). In view of (3.2), Lemma 3.3 implies that \( L'' \) is a direct sum of \( L \) and \( L' \). The right (resp. left) hand side of (3.3) is the covolume of \( \Gamma + \Gamma' \) with relative to \( \text{vol}_L \times \text{vol}_{L'} \) (resp. \( \text{vol}_{L''} \)). Let \( L = \prod_{v \in S} L_v \) and \( L' = \prod_{v \in S} L'_v \) according to (2.3). In view of (2.4), it suffices to prove that for each \( v \in S \) there is a positive Haar measure set \( R_v \) of \( L_v + L'_v \) such that
\[
\text{vol}_{L_v + L'_v}(R_v) \leq (\text{vol}_{L_v} \times \text{vol}_{L'_v})(R_v). \tag{3.4}
\]

Let \( r \) and \( m \) be the rank of \( L \) and \( L'' \) respectively. For each \( v \in S \) we choose an orthonormal basis \( \xi_{1,v}, \ldots, \xi_{r,v} \) of \( L_v \) and an orthonormal basis \( \xi_{r+1,v}, \ldots, \xi_{m,v} \) of \( L'_v \). We will show that (3.4) holds for
\[
R_v \overset{\text{def}}{=} \{a_1\xi_{1,v} + \cdots + a_m\xi_{m,v} : a_i \in B_1(K_v)\}.
\]

For all \( v \in S \)
\[
(\text{vol}_{L_v} \times \text{vol}_{L'_v})(R_v) = \text{vol}_v^1(B_1(K_v))^m. \tag{3.5}
\]
If \( v \) is archimedean, then it is clear from Euclidean geometry (i.e. volume of parallelootope) that
Moreover, vectors see (1.2).

Proof. Let \( K \) be a discrete \( I_S \)-module with finite covolume and let \( R \subset K^n_S \) be a measurable subset. Then there exists \( \xi \in K^n_S \) such that

\[
\text{card}(\{\xi + R \cap \Gamma\}) \geq \frac{\text{vol}(R)}{\text{cov}(\Gamma)}.
\]

(4.1)

Proof. Let \( \chi_R \) be the characteristic function of \( R \), and let \( F \subset K^n_S \) be a fundamental domain for \( \Gamma \). Then

\[
\int_F \text{card}(\{\xi + R \cap \Gamma\}) \, d\xi = \int \sum_{\gamma \in \Gamma} \chi_R(\gamma - \xi) \, d\xi = \text{vol}(R).
\]

Therefore there exists \( \xi \in F \) such that (4.1) holds.

4. Successive minima

The aim of this section is to prove Theorem 1.2.

Lemma 4.1. Let \( \Gamma \subset K^n_S \) be a discrete \( I_S \)-module with finite covolume and let \( R \subset K^n_S \) be a measurable subset. Then there exists \( \xi \in K^n_S \) such that

\[
\text{card}(\{\xi + R \cap \Gamma\}) \geq \frac{\text{vol}(R)}{\text{cov}(\Gamma)}.
\]

(4.1)

Proof. Let \( \chi_R \) be the characteristic function of \( R \), and let \( F \subset K^n_S \) be a fundamental domain for \( \Gamma \). Then

\[
\int_F \text{card}(\{\xi + R \cap \Gamma\}) \, d\xi = \int \sum_{\gamma \in \Gamma} \chi_R(\gamma - \xi) \, d\xi = \text{vol}(R).
\]

Therefore there exists \( \xi \in F \) such that (4.1) holds.

Lemma 4.2. Let \( \Gamma \subset K^n_S \) be a discrete \( I_S \)-module with finite covolume. Let \( R_1 \) be a centrally symmetric convex subset of \( K^n_{P_0} \) and let \( R_2 \) be a closed additive subgroup of \( K^n_{S \times P_0} \). Suppose \( R \subset K^n_S \) is equal to \( R_1 \times R_2 \) with the natural identification of \( K^n_S \) with \( \text{vol}(R) > 2^{n(\sigma + 2r)} \text{cov}(\Gamma) \), then \( R \) contains a nonzero element of \( \Gamma \).

Proof. Let \( R' = (\frac{1}{2}R_1) \times R_2 \). It follows from the assumption on \( \text{vol}(R) \) that \( \text{vol}(R') > \text{cov}(\Gamma) \). According to Lemma 4.1, we can find two distinct elements \( \gamma_1, \gamma_2 \in \Gamma \) and \( \xi \in K^n_S \) such that \( \gamma_i - \xi \in R' \) for \( i = 1, 2 \). Therefore the nonzero element \( \gamma_1 - \gamma_2 \) belongs to \( R \).

Recall that \( \lambda_m(\Gamma) \) (\( 1 \leq m \leq n \)) is the \( m \)-th minimum of a discrete \( I_S \)-module \( \Gamma \), see (1.2). It follows directly from the definition that there exist \( K \)-linearly independent vectors \( \xi_1, \ldots, \xi_n \in \Gamma \) with

\[
\|\xi_m\| = \lambda_m \quad \text{for all} \ 1 \leq m \leq n.
\]

Moreover, by Lemma 3.1 these vectors are also linearly independent over \( K_S \).
According to Lemma 4.2 for any $0 < t < \lambda_1(\Gamma)$ we have
\[
\left( \prod_{v \in S \setminus P_0} q_v^{-1} \right)^t \text{vol}(B_1(K^n_S)) \leq \text{vol}(B_t(K^n_S)) \leq 2^{n(\sigma + 2\tau)} \text{cov}(\Gamma).
\] (4.2)
Since $B_1(\mathbb{R}^m)$ contains $\{(x_1, \ldots, x_m) \in \mathbb{R}^m : -m^{-1/2} \leq x_i \leq m^{-1/2}\}$, for archimedean $v \in P$ we have
\[
\text{vol}_v(B_1(K^n_S)) \geq 2^{n_\e n - n_\e / 2},
\] (4.3)
where for complex place we use $B_1(\mathbb{C}^n) = B_1(\mathbb{R}^{2n})$. By (4.2), (4.3) and (2.2) we have
\[
\lambda_1(\Gamma)^n \leq n^{n(\sigma + 2\tau) / 2} \left( \prod_{v \in S \setminus P_0} q_v^{2} \right) \text{cov}(\Gamma).
\] (4.4)

**Lemma 4.3.** Let $\Gamma$ be a discrete $I_S$-module with finite covolume. Suppose that $\xi_1, \ldots, \xi_n \in \Gamma$ are $K$-linearly independent vectors and $\|\xi_m\| = \lambda_m(\Gamma)$ for all $1 \leq m \leq n$. Then there exists $g \in \text{GL}_n(K_S)$ such that
\[
\text{cont}(\text{det}(g)) = \prod_{i=1}^{n} \text{cont}(\xi_i)^{-1},
\] (4.5)
and any nonzero vector of $\Gamma' \overset{\text{def}}{=} \Gamma g$ has norm greater than or equal to one.

**Proof.** Suppose that $\xi_i = (\xi_{i,v})_{v \in S}$ where $\xi_{i,v} \in K^n_v$ (the notation here is the same as §2). By Lemma 3.1, for every $v \in S$ the vectors $\xi_{1,v}, \ldots, \xi_{n,v}$ are $K_v$-linearly independent in $K^n_v$. Using Gram–Schmidt orthogonalization process (see Lemma 3.3 for the ultrametric case), for each $v \in S$ we can find an orthonormal basis $\eta_{1,v}, \ldots, \eta_{n,v}$ such that for every $1 \leq m \leq n$ the $K_v$-linear span of $\eta_{1,v}, \ldots, \eta_{m,v}$ is the same as that of $\xi_{1,v}, \ldots, \xi_{m,v}$. Let $b_i = (b_{i,v})_{v \in S} \in K_S$ ($1 \leq i \leq n$) such that $|b_{i,v}|_v = \|\xi_{i,v}\|_v$. It follows from the definition of content that
\[
\text{cont}(b_i) = \text{cont}(\xi_i).
\] (4.6)
Since $\eta_i \overset{\text{def}}{=} (\eta_{i,v})_{v \in S}$ ($1 \leq i \leq n$) is a $K_S$-basis of $K^n_S$, there is a unique $g \in \text{GL}_n(K_S)$ such that $\eta_i g = b_i^{-1} \eta_i$. We claim that this $g$ satisfies the requirement of the lemma.

The equation (4.5) follows from
\[
\text{cont}(\text{det}(g)) = \text{cont}(b_1^{-1} \cdots b_n^{-1}) = \prod_{i=1}^{n} \text{cont}(b_i)^{-1} = \prod_{i=1}^{n} \text{cont}(\xi_i)^{-1},
\]
where in the last equality we use (4.6). For the other conclusion suppose that $\zeta = c_1 \eta_1 + \cdots + c_m \eta_m \in \Gamma' = \Gamma g$ where $c_i \in K_S$ and $c_m \neq 0$. We have
\[
\zeta g^{-1} = c_1 b_1 \eta_1 + \cdots + c_m b_m \eta_m \in \Gamma.
\]
Since for every $v \in S$ the basis $\eta_{1,v}, \ldots, \eta_{n,v}$ is orthonormal, we have
\[
\|\zeta g^{-1}\| \leq \|\zeta\| \max_{1 \leq i \leq m} |b_i| = \|\zeta\| \cdot \lambda_m(\Gamma).
\] (4.7)

On the other hand for any $1 \leq j \leq m$, the $K_S$-linear span of $\eta_1, \ldots, \eta_j$ is the same as that of $\xi_1, \ldots, \xi_j$. Since $c_m b_m \neq 0$, Lemma 3.1 implies that $\xi_1, \ldots, \xi_{m-1}, \zeta g^{-1}$ are $K$-linearly independent. Thus it follows from the definition of the $m$-th minimum of $\Gamma = \Gamma' g^{-1}$ that $\|\zeta g^{-1}\| \geq \lambda_m(\Gamma)$. This estimate together with (4.7) imply $\|\zeta\| \geq 1$, which completes the proof.  \[\square\]

**Theorem 4.4.** Let $\Gamma$ be a discrete $I_S$-module with finite covolume. Let $\xi_1, \ldots, \xi_n \in \Gamma$ be $K$ linearly independent vectors with $\|\xi_m\| = \lambda_m(\Gamma)$ for all $1 \leq m \leq n$. Then we have
\[
\text{cov}(I_S^n)^{-1} \text{cov}(\Gamma) \leq \prod_{i=1}^n \text{cont}(\xi_i) \leq n^{n(\sigma+2\tau)/2} \left( \prod_{v \in S \setminus P_0} q_v^n \right) \text{cov}(\Gamma).
\] (4.8)

**Proof.** We first prove the upper bound of (4.8). Suppose that $\Gamma' = \Gamma g$, where $g \in \text{GL}_n(K_S)$, satisfies the conclusion of Lemma 4.3. Then $\lambda_1(\Gamma') \geq 1$. Applying (4.4) for $\Gamma'$ we have
\[
1 \leq \lambda(\Gamma') n^d S \leq n^{n(\sigma+2\tau)/2} \left( \prod_{v \in S \setminus P_0} q_v^n \right) \text{cov}(\Gamma').
\] (4.9)

On the other hand by (2.1) and (4.5)
\[
\text{cov}(\Gamma') = \text{cov}(\Gamma) \cdot \text{cont}(\det(g)) = \text{cov}(\Gamma) \cdot \prod_{i=1}^n \text{cont}(\xi_i)^{-1}.
\] (4.10)

The upper bound of (4.8) follows from (4.9) and (4.10).

Let $\Gamma''$ be the $I_S$-linear span of $\xi_1, \ldots, \xi_n$. Since $\Gamma''$ is a submodule of $\Gamma$, by Lemma 3.5 we get
\[
\text{cov}(\Gamma) \leq \text{cov}(\Gamma'') \leq \prod_{i=1}^n \text{cov}_r(I_S \xi_i) = \text{cov}(I_S^n) \cdot \prod_{i=1}^n \text{cont}(\xi_i),
\]
which implies lower bound of (4.8). \[\square\]

To prove Theorem 1.2 we need a balance between contents and norms of vectors in $K^n_S$. The following lemma is a generalization of [9, Lemma 8.6] and [8, Lemma 5.9], and the proof is the same.

**Lemma 4.5.** For any $\xi \in K^n_S$ with $\text{cont}(\xi) \neq 0$, there exists $a \in I_S^n$ such that $\|a\xi\|^{\sharp S} \asymp \text{cont}(\xi)$ where the implied constants depend on $K$ and $S$. 
**Proof.** Suppose that $S = \{v_1, \ldots, v_m\}$ where $m = \text{card}(S)$. Let $\mathbb{R}_+$ be the multiplicative group of positive real numbers. We define a map

$$\varphi : K \to \mathbb{R}^m_+ \text{ by } \varphi(a) = (|a|_{v_1}, \ldots, |a|_{v_m}).$$

Let

$$H = \{(r_1, \ldots, r_m) \in \mathbb{R}^m_+ : \prod_{i=1}^m r_i^{\xi_{v_i}} = 1\}.$$

It follows from Dirichlet’s unit theorem (see [3, Chapter II, §18]) that the group $\varphi(I^*_S) \subset H$ is a cocompact lattice in $H$. So there exists $A \geq 1$ which depends on $K$ and $S$ such that for any $(r_1, \ldots, r_m) \in H$ we can find $a \in I^*_S$ with

$$A^{-1} \leq r_i|a|_{v_i} \leq A. \quad (4.11)$$

Suppose $\xi = (\xi_v)_{v \in S}$. It follows from the definition that

$$(\|\xi_{v_1}\|_{v_1} \cdot \text{cont}(\xi)^{-1/(m+\tau)}, \ldots, \|\xi_{v_m}\|_{v_m} \cdot \text{cont}(\xi)^{-1/(m+\tau)}) \in H.$$

By (4.11) one can find $a \in I^*_S$ such that for all $1 \leq i \leq m$

$$A^{-1} \leq \|\xi_{v_i}\|_{v_i} \text{cont}(\xi)^{-1/(m+\tau)}|a|_{v_i} \leq A.$$

Therefore

$$A^{-m-\tau} \text{cont}(\xi) \leq \|a\xi\|^{m+\tau} \leq A^{m+\tau} \text{cont}(\xi). \quad \square$$

**Proof of Theorem 1.2.** Let $\xi_1, \ldots, \xi_n \in \Gamma$ be $K$-linearly independent vectors with $\|\xi_i\| = \lambda_i(\Gamma)$. By Theorem 4.4

$$\prod_{i=1}^n \text{cont}(\xi_i) \asymp \text{cov}(\Gamma), \quad (4.12)$$

where the implied constants depend on $K, S$ and $n$. The definitions of content and norm imply

$$\text{cont}(\xi_i) \leq \lambda_i(\Gamma)^{2S}. \quad (4.13)$$

According to Lemma 4.5 there exists $a_1, \ldots, a_n \in I^*_S$ such that

$$\text{cont}(\xi_i) \gg \|a_i\xi_i\|^{2S}, \quad (4.14)$$

where the implied constant depends on $K$ and $S$. Note that elements $a_1\xi_1, \ldots, a_n\xi_n \in \Gamma$ are linear independent over $K$. So the definition of successive minima implies
Therefore the conclusion of Theorem 1.2 follows from (4.12), (4.13), (4.14) and (4.15).

5. Mahler’s compactness criterion

Let \( X = \text{SL}_n(I_S) \backslash \text{SL}_n(K_S) \). There is a one-to-one correspondence between \( X \) and

\[
\{ I_S^n g : g \in \text{SL}_n(K_S) \}
\]

via the map \( \text{SL}_n(I_S)g \to I_S^n g \). In this section \( e_1, \ldots, e_n \) denotes the standard basis of \( K^n_S \), i.e. \( e_i \) has \( i \)-th entry 1 and other entries 0. Before proving Theorem 1.1 we need the following lemma. See [9, Corollary 8.6] for \( K = \mathbb{Q} \) and [8, Corollary 5.1] for \( K \) a number field.

**Lemma 5.1.** Let \( M > 0 \). Then there are only finitely many \( I_S^n \)-submodules \( \Gamma \) of \( I_S^n \) such that \( \text{card}(I_S^n / \Gamma) \leq M \).

**Proof.** Let \( \Gamma \subset I_S^n \) be an \( I_S^n \)-submodule with \( \text{card}(I_S^n / \Gamma) \leq M \). For every \( 1 \leq i \leq n \) there is an ideal \( J_i \) of \( I_S \) such that

\[
I_S e_i \cap \Gamma = J_i e_i \quad \text{and} \quad \text{card}(I_S / J_i) \leq M.
\]

Therefore

\[
J_1 \times \cdots \times J_n \subset \Gamma \subset I_S^n.
\]

Note that \( I_S \) is a Dedekind domain. It follows from the structure theory of ideals in \( I_S \) that there are only finitely many ideals \( J \) in \( I_S \) such that \( \text{card}(I_S / J) \leq M \). So the conclusion of the lemma holds.

**Proof of Theorem 1.1.** Let \( \pi : \text{SL}_n(K_S) \to X \) be the natural quotient map and let

\[
r = \inf\{\|\xi g\| : \xi \in I_S^n, \xi \neq 0, g \in \text{SL}_n(K_S), \pi(g) \in R\}.
\]

(5.1)

Suppose \( R \) is relatively compact. There exists a relatively compact subset \( F \subset \text{SL}_n(K_S) \) with \( \pi(F) = R \). Therefore there exists \( C > 0 \) such that

\[
\|\xi g\| \leq C\|\xi\| \quad \text{for every } \xi \in K^n_S \text{ and } g \in F.
\]

(5.2)

The discreteness of \( \Gamma \) and (5.2) imply \( r > 0 \).

Now we assume \( r > 0 \) and prove that \( R \) is relatively compact. Let \( \{g_i\}_{i \geq 1} \) be a sequence in \( \pi^{-1}(R) \). It suffices to show that there exists \( g \in \text{SL}_n(K_S) \) such that \( \pi(g) \) is
a limit point of a subsequence of \( \{\pi(g_i)\}_{i \geq 1} \). By Theorem 1.2 there exists \( C \geq 1 \) such that for any free \( I_S \)-module \( \Gamma \in R \) one has

\[
    r \leq \lambda_1(\Gamma) \leq \lambda_n(\Gamma) \leq C. \quad (5.3)
\]

For every \( i \geq 1 \) let \( \xi_1^{(i)}, \ldots, \xi_n^{(i)} \in I_S^n \) be \( K \)-linearly independent vectors such that \( \|\xi_j^{(i)} g_i\| \) equals to the \( j \)-th minimum of \( I_S^n g_i \). By (5.3) we have

\[
    \|\xi_j^{(i)} g_i\| \leq C \quad \forall \ i \geq 1 \text{ and } 1 \leq j \leq n. \quad (5.4)
\]

Let

\[
    \Gamma_i = \text{span}_{I_S} \{\xi_j^{(i)} : 1 \leq j \leq n\}.
\]

According to (5.3) and Theorem 1.2 there exists \( M > 0 \) such that

\[
    \text{cov}(I_S^n) \leq \text{cov}(\Gamma_i) = \text{cov}(\Gamma_i g_i) \leq M \quad \forall \ i \geq 1.
\]

(5.5)

By Lemma 5.1 and (5.5), the set \( \{\Gamma_i : i \geq 1\} \) is finite. Therefore by possibly passing to a subsequence we may assume that there exists \( h \in \text{GL}_n(K) \) such that \( \Gamma_i = I_S^n h \) for all \( i \geq 1 \). It follows that there is a sequence \( \{f_i\}_{i \geq 1} \) in \( \text{GL}_n(I_S) \) such that \( e_j f_i h = \xi_j^{(i)} \) for all \( i \geq 1 \) and \( 1 \leq j \leq n \). By (5.4) there is a subsequence \( \{g_{ik}\}_{k \geq 1} \) of \( \{g_i\}_{i \geq 1} \) and \( g \in \text{GL}_n(K_S) \) such that

\[
    f_{ik} h g_{ik} \to g \quad \text{as} \quad k \to \infty. \quad (5.6)
\]

Since \( \det \) is continuous and \( I_S \) is discrete in \( K_S \), for \( k \) sufficiently large we have \( \det(f_{ik}) = \det(f_{ik+1}) \in I_S^c \). Therefore by possibly passing to a subsequence and multiplying the first row of \( h \) by some element of \( I_S^c \), we assume without loss of generality that \( f_{ik} \in \text{SL}_n(I_S) \) for all \( k \).

The group \( h^{-1} \text{SL}_n(I_S)h \cap \text{SL}_n(I_S) \) has finite index in \( h^{-1} \text{SL}_n(I_S)h \). So by possibly passing to a subsequence we can find \( f \in h^{-1} \text{SL}_n(I_S)h \) and a sequence \( \{h_k\}_{k \geq 1} \) of \( \text{SL}_n(I_S) \) such that

\[
    h^{-1} f_{ik} h = f h_k \quad \forall \ k \geq 1.
\]

(5.7)

By (5.6) and (5.7) we have \( hf_{ik} g_{ik} \to g \) as \( k \to \infty \). Therefore

\[
    h_{ik} g_{ik} \to f^{-1} h^{-1} g \quad \text{as} \quad k \to \infty.
\]

Since \( h_k \in \text{SL}_d(I_S) \) we have \( \pi(g_{ik}) \to \pi(f^{-1} h^{-1} g) \) as \( k \to \infty \). \( \square \)
References