Simultaneous Diophantine approximation: 
sums of squares and homogeneous polynomials

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Abstract
Let $f$ be a homogeneous polynomial with rational coefficients in $d$ variables. We prove several results concerning uniform simultaneous approximation to points on the graph of $f$, as well as on the hypersurface $\{f(x_1,\ldots,x_d) = 1\}$. The results are first stated for the case $f(x_1,\ldots,x_d) = x_1^2 + \cdots + x_d^2$, which is of particular interest.

1 Diophantine exponents

Let $\Theta = (\theta_1,\ldots,\theta_m)$ be a collection of real numbers. The ordinary Diophantine exponent $\omega = \omega(\Theta)$ for simultaneous rational approximation to $\Theta$ is defined as the supremum over all real $\gamma$ such that the inequality

$$\max_{1 \leq j \leq m} |q\theta_j - a_j| < q^{-\gamma}$$

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has infinitely many solutions in integer points \((q, a_1, \ldots, a_m) \in \mathbb{Z}^{m+1}\) with \(q > 0\).

The uniform Diophantine exponent \(\hat{\omega} = \hat{\omega}(\Theta)\) for simultaneous approximation to \(\Theta\) is defined as the supremum over all real \(\gamma\) such that the system of inequalities
\[
\max_{1 \leq j \leq m} |q \theta_j - a_j| < Q^{-\gamma}, \quad 1 \leq q \leq Q
\]
has a solution \((q, a_1, \ldots, a_m) \in \mathbb{Z}^{m+1}\) for every large enough real \(Q\). It immediately follows from Minkowski’s convex body theorem that \(\hat{\omega}(\Theta) \geq \frac{1}{m}\) for any \(\Theta \in \mathbb{R}^m\). Furthermore, let us say that \(\Theta\) is totally irrational if \(1, \theta_1, \ldots, \theta_m\) are linearly independent over \(\mathbb{Z}\). For such \(\Theta\) it was first observed by Jarník [J38, Satz 9] that
\[
\hat{\omega}(\Theta) \leq 1.
\]
(See also [M10, Theorem 17], as well as [W04, Theorem 5.2] for a proof based on homogeneous dynamics.) In particular for \(m = 1\) one has
\[
(1.1) \quad \hat{\omega}(\theta) = 1 \text{ for all } \theta \in \mathbb{R} \setminus \mathbb{Q}.
\]
On the other hand, for \(m \geq 2\) it is known that for arbitrary \(\lambda\) from the interval \([\frac{1}{m}, 1]\) there exists \(\Theta \in \mathbb{R}^m\) with \(\hat{\omega}(\Theta) = \lambda\).

Moreover it is clear from the definition that
\[
\omega(\Theta) \geq \hat{\omega}(\Theta)
\]
for any \(\Theta \in \mathbb{R}^m\). Here we should mention that in [J54] Jarník gave an improvement of this bound for the collection of \(\Theta\) such that there are at least two numbers \(\theta_i, \theta_j\) linearly independent over \(\mathbb{Z}\) together with 1. In this case he proved the inequality
\[
\frac{\omega}{\hat{\omega}} \geq \frac{\hat{\omega}}{1 - \hat{\omega}}.
\]
This inequality is optimal for \(m = 2\). For arbitrary \(m\) the optimal inequality was obtained recently by Marnat and Moshchevitin [MM18].

**Theorem A.** [MM18, Theorem 1] Let \(\Theta \in \mathbb{R}^m\) be totally irrational, and let \(\omega = \omega(\Theta)\) and \(\hat{\omega} = \hat{\omega}(\Theta)\). Denote by \(G_m\) the unique positive root of the equation
\[
(1.2) \quad x^{m-1} = \frac{\hat{\omega}}{1 - \hat{\omega}}(x^{m-2} + x^{m-3} + \cdots + x + 1).
\]
Then one has
\[
\frac{\omega}{\hat{\omega}} \geq G_m.
\]
In the present paper we study the bounds for the uniform exponent $\hat{\omega}$ for special collections of numbers. Theorem A will be an important ingredient of our proofs.

2 Approximation to several real numbers and sums of their squares

In [DS69] Davenport and Schmidt proved the following theorem:

**Theorem B.** [DS69, Theorem 1a] Suppose that $\xi \in \mathbb{R}$ is neither a rational number nor a quadratic irrationality. Then the uniform Diophantine exponent $\hat{\omega} = \hat{\omega}(\Xi)$ of the vector $\Xi = (\xi, \xi^2) \in \mathbb{R}^2$ satisfies the inequality

$$\hat{\omega} \leq \frac{\sqrt{5} - 1}{2}.$$ 

Here we should note that $\frac{\sqrt{5} - 1}{2}$ is the unique positive root of the equation

$$x^2 + x = 1.$$ 

It is known due to Roy [R03] that the bound of Theorem B is optimal. Davenport and Schmidt proved a more general result [DS69, Theorem 2a] involving successive powers $\xi, \xi^2, \ldots, \xi^m$. However in the present paper we deal with another generalization.

In the sequel we will consider $m = d$ or $m = d + 1$ numbers. Namely, take

$$\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$$

and introduce the vector

$$\Xi = (\xi_1, \ldots, \xi_d, \xi_1^2 + \cdots + \xi_d^2) \in \mathbb{R}^{d+1}.$$ 

Also let $H_d$ be the unique positive root of the equation

$$x^{d+1} + x^d + \cdots + x = 1.$$ 

Note that $\frac{1}{2} < H_d < 1$, and $H_d \to \frac{1}{2}$ monotonically when $d \to \infty$.

In the present paper we prove the following two theorems dealing with sums of squares.

**Theorem 2.1.** Let $d \geq 1$ be an integer. Suppose that $\Xi$ as in (2.1) is totally irrational. Then the uniform Diophantine exponent of $\Xi$ satisfies the inequality

$$\hat{\omega}(\Xi) \leq H_d.$$
Note that in the case $d = 1$ Theorem 2.1 coincides with Theorem B. The next theorem can be proved by a similar argument.

**Theorem 2.2.** Let $d \geq 2$. Suppose that $\xi = (\xi_1, \ldots, \xi_d)$ is totally irrational and

$$
\xi_1^2 + \cdots + \xi_d^2 = 1.
$$

Then the uniform Diophantine exponent of $\xi$ satisfies the inequality

$$
\hat{\omega}(\xi) \leq H_{d-1}.
$$

Theorems 2.1 and 2.2 are particular cases of more general Theorems 1a and 2a, which we formulate in Section 4.

**Remark 2.3.** It is worth comparing Theorems 2.1 and 2.2, as well as their more general versions, with a lower bound obtained using the methods of [KW05, §5]. It is not hard to derive from [KW05, Corollary 5.2] that for any real analytic submanifold $M$ of $\mathbb{R}^m$ of dimension at least 2 which is not contained in any proper rational affine hyperplane of $\mathbb{R}^m$ there exists totally irrational $\Theta \in M$ with

$$
\hat{\omega}(\Theta) \geq \frac{1}{m} + \frac{2}{m(m^2 - 1)}.
$$

It would be interesting to see if the above estimate could be improved, thus shedding some light on the optimality of our theorems.

## 3 Intrinsic approximation on spheres

Our study of vectors of the form (2.1) was motivated by problems related to intrinsic rational approximation on spheres. In [KM15] Kleinbock and Merrill proved the following result.

**Theorem C.** [KM15, Theorem 4.1] Let $d \geq 2$. There exists a positive constant $C_d$ such that for any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^d$ satisfying (2.3) and for any $T > 1$ there exists a rational vector

$$
\alpha = \left( \frac{a_1}{q}, \ldots, \frac{a_d}{q} \right) \in \mathbb{Q}^d
$$

such that

$$
|\alpha|^2 = \left( \frac{a_1}{q} \right)^2 + \cdots + \left( \frac{a_d}{q} \right)^2 = 1
$$

and

$$
|\xi - \alpha| \leq \frac{C_d}{q^{1/2}T^{1/2}}, \quad 1 \leq q \leq T.
$$
Here and hereafter by $| \cdot |$ we denote the Euclidean norm of a vector. In particular Theorem C implies that in the case $\xi \notin \mathbb{Q}^d$ the inequality

$$|\xi - \alpha| \leq \frac{C_d}{q}$$

has infinitely many solutions in rational vectors (3.1).

See [M16, M17] for effective versions of Theorem C, and [FKMS14, Theorem 5.1] for generalizations. Note that the formulation from [M16] involves sums of squares, while an effective version for an arbitrary positive definite quadratic form with integer coefficients can be found in [M17]. It is also explained in [FKMS14] how the conclusion of Theorem C can be derived from [SV95, Theorem 1] via a correspondence between intrinsic Diophantine approximation on quadric hypersurfaces and approximation of points in the boundary of the hyperbolic space by parabolic fixed points of Kleinian groups; see [FKMS14, Proposition 3.16].

In this paper we prove a result about uniform intrinsic approximation on the unit sphere. We need some notation. First of all, note that the inequality (3.2) can be rewritten as

$$\frac{|q\xi - a|^2}{q} \leq \frac{C_d^2}{T}, \quad a = (a_1, \ldots, a_d) \in \mathbb{Z}^d, \quad a_1^2 + \cdots + a_d^2 = q^2.$$ 

Now let us define the function

$$\Psi_\xi(T) = \min_{(q, a_1, \ldots, a_d) \in \mathbb{Z}^{d+1}: 1 \leq q \leq T, \quad a_1^2 + \cdots + a_d^2 = q^2} \frac{|q\xi - a|^2}{q}.$$ 

Theorem C states that for any $\xi \in \mathbb{R}^d$ under the condition (2.3) one has

$$T \cdot \Psi_\xi(T) \leq C_d^2 \quad \text{for} \quad T > 1.$$

**Theorem 3.1.** Let $d \geq 2$. Let $\xi \in \mathbb{R}^d \setminus \mathbb{Q}^d$ be such that (2.3) is satisfied. Then for any $\varepsilon > 0$ there exists arbitrary large $T$ such that

$$T \cdot \Psi_\xi(T) \geq \frac{1}{4} - \varepsilon.$$

Theorem 3.1 is an analog of Khintchine’s lemma on rational approximations to one real number (see [K26, Satz 1]). It admits the following corollary. One can try to define the uniform Diophantine exponent of $\xi$ for the intrinsic approximation on the unit sphere as

$$\hat{\omega}_d^1(\xi) = \sup \left\{ \gamma \in \mathbb{R} \left| \begin{array}{l}
\text{the inequalities } |\xi - \alpha| \leq \frac{1}{q^{1/2}T^{\gamma/2}}, \quad 1 \leq q \leq T \\
\text{are solvable in } \alpha \text{ of the form (3.1) for large enough } T
\end{array} \right. \right\}.$$
Then for all vectors $\xi \not\in \mathbb{Q}^d$ satisfying (2.3) we have

$$\hat{\omega}_d^1(\xi) = 1$$

by Theorem 3.1. So here we have an equality similar to (1.1) for the case of approximation to one real number. See also [BGSV16, Theorem 2], where a similar observation was made in the context of Kleinian groups. Theorem 3.1 follows from a more general Theorem 3a which we formulate in Section 4.

## 4 Results on homogeneous polynomials

Given integers $s \geq 2$ and $d \geq 1$, define $H_{d,s}$ to be the unique positive root of the equation

$$(1 - x) = x \cdot \sum_{k=1}^{d} \left( \frac{x}{s - 1} \right)^k.$$

Note that for any $s$ and $d$ one has

$$\frac{s - 1}{s} < H_{d,s} < 1.$$ 

Clearly $H_{d,2} = H_d$, and $H_{d,s}$ monotonically decreases to $\frac{s - 1}{s}$ as $d \to +\infty$.

The results of this section deal with a homogeneous polynomial

$$(4.2) \quad f(x) = \sum_{(s_1, \ldots, s_d) \in \mathbb{Z}_+^d : s_1 + \cdots + s_d = s} f_{s_1, \ldots, s_d} x_1^{s_1} \cdots x_d^{s_d}, \quad \text{where } f_{s_1, \ldots, s_d} \in \mathbb{Q},$$

of degree $s$ in variables $x_1, \ldots, x_d$ (here $\mathbb{Z}_+$ stands for the set of non-negative integers). Theorem 2.1 from the previous section is a corollary of the following general statement.

**Theorem 1a.** Let $s \geq 2$ be an integer, and let $f$ as in (4.2) be such that

$$(4.3) \quad \# \{x \in \mathbb{Q}^d : f(x) = 0 \} < \infty.$$ 

Suppose that

$$(4.4) \quad \Xi_f = (\xi_1, \ldots, \xi_d, f(\xi_1, \ldots, \xi_d))$$

is totally irrational. Then $\hat{\omega}(\Xi_f) \leq H_{d,s}$.

We give a proof of Theorem 1a in Section 8.
Remark 4.1. Let us consider the case $d = 1$. In this case Theorem 1a states that the uniform exponent $\hat{\omega}$ of $(\xi, \xi^s)$ is bounded from above by the positive root of the equation

$$x^2 + (s - 1)x - (s - 1) = 0,$$

that is

$$(4.5) \quad \hat{\omega} \leq \frac{\sqrt{(s - 1)(s + 3)} - (s - 1)}{2}.$$ 

This result was obtained by Batzaya in [B15] for arbitrary vectors of the form $(\xi^l, \xi^s)$ with $1 \leq l < s$. In the case $d = 1, s = 3$ much stronger inequality

$$\hat{\omega} \leq \frac{2(9 + \sqrt{11})}{35}$$

is known due to Lozier and Roy (see [LR12] and the discussion therein). In [B17] Batzaya improved (4.5) and showed that for $(\xi^l, \xi^s)$ with $1 \leq l < s$ one has

$$\hat{\omega} \leq \frac{s^2 - 1}{s^2 - s - 1}$$

for odd $s$. In the case of even $s$ in the paper [B15] he had a better inequality

$$\hat{\omega} \leq \frac{(s - 1)(s + 2)}{s^2 + 2s - 1}.$$ 

Also [B17] contains a better bound for $\hat{\omega}$ when $s = 5, 7, 9$. Thus the inequality of our Theorem 1a is not optimal for $s \geq 3$.

Theorem 2.2 from the previous section is a corollary of the following general statement.

Theorem 2a. Let $s \geq 2$ be an integer, and let $f$ as in (4.2) be such that (4.3) holds. Then

$$(4.6) \quad \xi \in \mathbb{R}^d \text{ is totally irrational and } f(\xi) = 1 \quad \implies \quad \hat{\omega}(\xi) \leq H_{d-1,s}.$$ 

We give a proof of Theorem 2a in Section 7. To get Theorems 2.1, 2.2 from Theorems 1a, 2a one should put $s = 2$ and $f(x) = x_1^2 + \cdots + x_d^2$.

Remark 4.2. The argument used in the proof of Theorem 2a yields (4.6) for any (not necessarily homogeneous) polynomial $f$ with rational coefficients such that the number of rational points on the hypersurface $\{f = 1\}$ is finite. We state it as Theorem 2b in Section 7. For example (cf. (4.5) with $d = 1$ and $s = 6$) it follows that

$$\hat{\omega}(x,y) \leq \frac{\sqrt{45} - 5}{2}$$

for any $(x, y) \in \mathbb{R}^2$ such that $y^2 - x^2 - x^6 = 1$. 
Now for $\xi \in \mathbb{R}^d$ under the condition $f(\xi) = 1$ consider the function

$$
\Psi_{f,\xi}(T) = \min_{(q,a)=(q,a_1,\ldots,a_d) \in \mathbb{Z}^{d+1}: 1 \leq q \leq T, f\left(\frac{a_1}{q},\ldots,\frac{a_d}{q}\right) = 1} \frac{|q\xi - a|^s}{q}.
$$

It is clear that $\Psi_{f,\xi}(T)$ is a non-increasing piecewise constant function. Here we do not suppose that it tends to zero as $T \to +\infty$.

**Theorem 3a.** Let $s \geq 2$ be an integer, and let $f$ as in (4.2) be such that (4.3) holds. Take $\xi \not\in \mathbb{Q}^d$ with

$$
f(\xi) = 1,
$$

and let $D = D(f) \in \mathbb{Z}_+$ be the common denominator of all rational numbers $f_{s_1,\ldots,s_d}$. Also define

$$
K = K(f) = \sup_{x \in \mathbb{R}^d: |x|=1} |f(x)|.
$$

Then for any positive $\varepsilon$ there exists arbitrary large $T$ such that

$$
T^{s-1} \cdot \Psi_{f,\xi}(T) \geq \frac{1}{2^s DK} - \varepsilon.
$$

For $f(x) = x_1^2 + \cdots + x_d^2$ we have $s = 2$ and $D(f) = K(f) = 1$. Thus Theorem 3.1 is a direct corollary of Theorem 3a. We give a proof of Theorem 3a in Section 9.

## 5 The main lemma

The next lemma is a polynomial analogue of the classical simplex lemma in simultaneous Diophantine approximation going back to Davenport [D64]. See also [KS18] for a version for arbitrary quadratic forms, [BGSV16, Lemma 1] for a similar statement in the context of Kleinian groups, and [FKMS18, Lemma 4.1] for a general simplex lemma for intrinsic Diophantine approximation on manifolds.

**Lemma 5.1.** Let $s \geq 2$ be an integer, and let $f$ be as in (4.2). Let $D = D(f)$ and $K = K(f)$ be defined as in Theorem 3a, and take two rational vectors

$$
\alpha = \left(\frac{a_1}{q}, \ldots, \frac{a_d}{q}\right) \quad \text{and} \quad \beta = \left(\frac{b_1}{r}, \ldots, \frac{b_d}{r}\right)
$$

such that

$$
f(\alpha - \beta) \neq 0.
$$
(i) Suppose that
\[ f(\alpha) = \frac{A}{q} \]
with an integer \( A \). Then
\[ |\alpha - \beta|^s \geq \frac{1}{DKq^{s-1}r^s}. \]

(ii) Suppose
\[ f(\alpha) = \frac{A}{q}, \quad f(\beta) = \frac{B}{r} \]
with integers \( A, B \). Then
\[ |\alpha - \beta|^s \geq \frac{1}{DKq^{s-1}r^s-1}. \]

Proof. (i) First of all we observe that
\[ |f(\alpha - \beta)| \geq \frac{1}{Dq^{s-1}r^s}. \]
Indeed, for any \( s_1, \ldots, s_d \in \mathbb{Z}_+ \) under the condition \( s_1 + \cdots + s_d = s \) consider the product
\[ \Pi_{s_1, \ldots, s_d} = \prod_{k=1}^{d} \left( \frac{a_k}{q} - \frac{b_k}{r} \right)^{s_k}. \]
It is clear that
\[ \Pi_{s_1, \ldots, s_d} = \prod_{k=1}^{d} \frac{a_k^{s_k}}{q^s} + \frac{W_{s_1, \ldots, s_d}}{q^{s-1}r^s} \]
with an integer \( W_{s_1, \ldots, s_d} \). Now from (5.1) we see that
\[ 0 \neq f(\alpha - \beta) = \sum_{(s_1, \ldots, s_d) \in \mathbb{Z}_+^d, \ s_1 + \cdots + s_d = s} f_{s_1, \ldots, s_d} \Pi_{s_1, \ldots, s_d} \]
\[ = f(\alpha) + \frac{W}{Dq^{s-1}r^s} = \frac{A}{q} + \frac{W}{Dq^{s-1}r^s} = \frac{W_1}{Dq^{s-1}r^s}, \]
with \( W, W_1 \in \mathbb{Z} \), and (5.4) is proved. Then from the definition (4.7) we see that
\[ |f(\alpha - \beta)| \leq K|\alpha - \beta|^s. \]

Now (5.4) and (5.5) give (5.2).

(ii) The proof here is quite similar. From the conditions on \( f(\alpha) \) and \( f(\beta) \) we see that
\[ 0 \neq f(\alpha - \beta) = f(\alpha) \pm f(\beta) + \frac{W'}{Dq^{s-1}r^s-1} = \frac{A}{q} \pm \frac{B}{r} + \frac{W'}{Dq^{s-1}r^s-1} = \frac{W'_1}{Dq^{s-1}r^s-1}, \]
with $W', W'_1 \in \mathbb{Z}$. So we get

$$\tag{5.6} |f(\alpha - \beta)| \geq \frac{1}{Dq^{r-1}r_s-1}.$$  

Now (5.6) together with (5.5) give (5.3).  

\hspace{1cm} \Box

6 Best approximation vectors

If $\Theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m \setminus \mathbb{Q}^m$, let us say that a vector $(q, a_1, \ldots, a_m) \in \mathbb{Z}^{m+1}$ is a best simultaneous approximation vector of $\Theta$ if

$$\text{dist}(q\Theta, \mathbb{Z}^m) < \text{dist}(k\Theta, \mathbb{Z}^m) \quad \forall k = 1, \ldots, q - 1,$$

and

$$\text{dist}(q\Theta, \mathbb{Z}^m) = \max_{i=1, \ldots, m} |q\theta_i - a_i|.$$  

Here 'dist' stands for the distance induced by the supremum norm on $\mathbb{R}^m$. Best approximation vectors of $\Theta$ form an infinite sequence $(q_\nu, a_{1, \nu}, \ldots, a_{m, \nu})$, $\nu \in \mathbb{N}$, and satisfy the inequalities

$$q_{\nu-1} < q_\nu, \quad \zeta_{\nu-1} > \zeta_\nu, \quad \nu \in \mathbb{N},$$

where one defines

$$\zeta_\nu = \max_{i=1, \ldots, m} |q_\nu \theta_i - a_{i, \nu}|.$$  

It is important that

$$\text{g.c.d.}(q_\nu, a_{1, \nu}, \ldots, a_{m, \nu}) = 1, \quad \nu \in \mathbb{N}.$$  

So for any two successive rational approximation vectors

$$\alpha_j = \left(\frac{a_{1, j}}{q_j}, \ldots, \frac{a_{m, j}}{q_j}\right) \in \mathbb{Q}^m, \quad j = \nu - 1, \nu$$

we have

$$\alpha_{\nu-1} \neq \alpha_\nu.$$  

Some detailed information about best approximation vectors may be found for example in papers [C13] and [M10]. In particular, the following property of the uniform exponent $\hat{\omega} = \hat{\omega}(\Theta)$ is well known (see e.g. [M10, Proposition 1]). Suppose that $\gamma < \hat{\omega}$. Then for all $\nu$ large enough one has

$$\tag{6.1} \zeta_{\nu-1} \leq q_\nu^{-\gamma}.$$
7 Proof of Theorem 2a

We take \( m = d \) and consider best approximation vectors 
\[
z_{\nu} = (q_{\nu}, a_{1,\nu}, \ldots, a_{d,\nu}) \in \mathbb{Z}^{d+1}
\]
of \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \), together with distances 
\[
\zeta_{\nu} = \max_{1 \leq j \leq d} |q_{\nu}\xi_j - a_{j,\nu}|,
\]
and the corresponding rational approximants 
\[
\alpha_{\nu} = \left( \frac{a_{1,\nu}}{q_{\nu}}, \ldots, \frac{a_{d,\nu}}{q_{\nu}} \right) \in \mathbb{Q}^d.
\]

Under the condition \( \gamma < \hat{\omega}(\xi) \) we have (6.1) for all large \( \nu \).

Here we should note that 
\[
\max_{1 \leq j \leq d} \left| \frac{\xi_j - a_{j,\nu} q_{\nu}}{q_{\nu}} \right| = \zeta_{\nu}.
\]

So for large \( \nu \) we see that 
\[
\left( \frac{a_{1,\nu}}{q_{\nu}} \right)^{s_1} \cdots \left( \frac{a_{d,\nu-1}}{q_{\nu}} \right)^{s_d} = \xi_1^{s_1} \cdots \xi_d^{s_d} + O \left( \frac{\zeta_{\nu}}{q_{\nu}} \right).
\]

We consider two cases.

Case 1. \( f(\alpha_{\nu}) = 1 \) for infinitely many \( \nu \).

Here, since \( \alpha_{\nu} \neq \alpha_{\nu-1} \), we may apply Lemma 2.1(i) with \( A = q_{\nu} \). Take \( \alpha = \alpha_{\nu}, \beta = \alpha_{\nu-1} \); then (5.1) follows from (4.3) when \( \nu \) is large enough, and from (5.2) we deduce that
\[
\frac{1}{(DK)^{\frac{1}{2}} q_{\nu}^{\frac{s-1}{2}} q_{\nu-1}} \leq \sqrt{d} \max_{1 \leq k \leq d} \left| \frac{a_{k,\nu}}{q_{\nu}} - \frac{a_{k,\nu-1}}{q_{\nu-1}} \right|
\]
\[
\leq \sqrt{d} \left( \max_{1 \leq k \leq d} \left| \frac{a_{k,\nu}}{q_{\nu}} - \xi_k \right| + \max_{1 \leq k \leq d} \left| \frac{a_{k,\nu-1}}{q_{\nu-1}} - \xi_k \right| \right)
\]
\[
\leq \frac{2\sqrt{d} \zeta_{\nu-1}}{q_{\nu-1}} \leq \frac{2\sqrt{d}}{q_{\nu-1} q_{\nu}}.
\]

So with some positive \( c_1 \) we have \( q_{\nu}^{s} \leq c_1 q_{\nu}^{\frac{s-1}{s}} \) for infinitely many \( \nu \) and \( \hat{\omega}(\xi) \leq \frac{s-1}{s} < H_{d-1,s} \).
**Case 2.** $f(\alpha_\nu) \neq 1$ for all $\nu$ large enough.

For such $\nu$ the difference $f(\alpha_\nu) - 1$ is a nonzero rational number with denominator $Dq_\nu^s$. Therefore we have

$$|f(\alpha_\nu) - 1| \geq \frac{1}{Dq_\nu^s}.$$  

Now from (7.1) we see that

$$(7.3) \quad \frac{1}{Dq_\nu^s} \leq |f(\alpha_\nu) - 1| = |f(\alpha_\nu) - f(\xi)| = O \left( \frac{\zeta_\nu}{q_\nu} \right)$$

and

$$(7.4) \quad \zeta_\nu \geq \frac{c_2}{q_\nu^{s-1}}$$

with some positive constant $c_2$ depending on $f$ and $\xi$. As this inequality holds for all large $\nu$, we conclude that $\omega(\xi) \leq s - 1$ and

$$\frac{s - 1}{\hat{\omega}(\xi)} \geq \frac{\omega(\xi)}{\hat{\omega}(\xi)} \geq G_d.$$  

Recall that $G_d$ is a root of equation (1.2). This means that the upper bound for $\hat{\omega}(\xi)$ is given by the unique positive root of the equation

$$\left( \frac{s - 1}{x} \right)^{d-1} = \frac{x}{1 - x} \left( \left( \frac{s - 1}{x} \right)^{d-2} + \left( \frac{s - 1}{x} \right)^{d-3} + \cdots + \frac{s - 1}{x} + 1 \right)$$

which coincides with (4.1) if $d$ is replaced by $d - 1$.

We close the section by observing that the argument used in the proof of Case 2 above does not rely on the homogeneity of $f$. Thus the following result can be established.

**Theorem 2b.** Suppose that $f$ is an arbitrary polynomial of degree $s$ in $d$ variables with rational coefficients such that

$$\# \{ x \in \mathbb{Q}^d : f(x) = 1 \} < \infty,$$

and let $\xi = (\xi_1, ..., \xi_d) \in \mathbb{Q}^d$ be such that $f(\xi) = 1$. Then:

(i) $\omega(\xi) \leq s - 1$;

(ii) if $\xi$ is totally irrational, then $\hat{\omega}(\xi) \leq H_{d-1,s}$.

The proof is left to the reader. In particular, the conclusion of Theorem 2b holds when $\{ f = 1 \}$ is an algebraic curve over $\mathbb{Q}$ of genus at least 2, such as the one mentioned in Remark 3.1.
8 Proof of Theorem 1a

The proof of Theorem 1a is similar to the proof of Theorem 2a. We take $m = d + 1$ and consider a sequence of best simultaneous approximation vectors

$$z_\nu = (q_\nu, a_{1,\nu}, \ldots, a_{d,\nu}, A_\nu) \in \mathbb{Z}^{d+2}, \quad \nu \in \mathbb{N},$$

of $\Theta = \Xi_f$ as in (4.4), and the corresponding distances from $q_\nu \Xi_f$ to $\mathbb{Z}^{d+1}$:

$$\zeta_\nu = \max \left( |q_\nu \xi_1 - a_{1,\nu}|, \ldots, |q_\nu \xi_d - a_{d,\nu}|, |q_\nu f(\xi) - A_\nu| \right).$$

We also need “shortened” rational approximation vectors

$$\alpha_\nu = \left( \frac{a_{1,\nu}}{q_\nu}, \ldots, \frac{a_{d,\nu}}{q_\nu} \right) \in \mathbb{Q}^d.$$ 

Note that now it may happen that

$$\alpha_{\nu-1} = \alpha_\nu \tag{8.1}$$

for some $\nu$.

**Lemma 8.1.** Suppose that (8.1) holds and

$$f(\alpha_\nu) = \frac{A_\nu}{q_\nu} \tag{8.2}$$

Then

$$\Delta = \gcd(q_\nu, a_{1,\nu}, \ldots, a_{d,\nu}) = O \left( \frac{s^{\frac{s+1}{2}}}{q_\nu^s} \right). \tag{8.3}$$

**Proof.** We know that

$$\gcd(q_\nu, a_{1,\nu}, \ldots, a_{d,\nu}, A_\nu) = 1$$

and thus

$$\gcd(\Delta, A_\nu) = 1.$$

From (8.2) we see that

$$DA_\nu q_\nu^{s-1} = Dq_\nu^s f(\alpha_\nu) \in \mathbb{Z}.$$ 

But $\Delta \mid a_{j,\nu}$ for any $j$. As $Dq_\nu^s f \left( \frac{\cdot}{q_\nu} \right)$ is a homogeneous polynomial of degree $s$ with integer coefficients, we deduce that

$$\Delta^s \mid Dq_\nu^{s-1}.$$

This gives (8.3).
To prove Theorem 1a we consider three cases.

**Case 1.1.** For infinitely many $\nu$ (8.1) and (8.2) hold.

In this case the vectors 

$$(q_{\nu-1}, a_{1,\nu-1}, \ldots, a_{d,\nu-1}), \quad (q_{\nu}, a_{1,\nu}, \ldots, a_{d,\nu})$$

are proportional, but the vectors

$$(q_{\nu-1}, a_{1,\nu-1}, \ldots, a_{d,\nu-1}, A_{\nu-1}), \quad (q_{\nu}, a_{1,\nu}, \ldots, a_{d,\nu}, A_{\nu})$$

are not proportional. This means that (8.4)

$$\left| \begin{array}{cc} q_{\nu-1} & A_{\nu-1} \\ q_{\nu} & A_{\nu} \end{array} \right| \neq 0.$$ 

There exists a primitive vector

$$(q_*, a_{1,*}, \ldots, a_{d,*}) \in \mathbb{Z}^{d+1}, \quad \text{g.c.d.}(q_*, a_{1,*}, \ldots, a_{d,*}) = 1, \quad q_* \geq 1,$$

such that $(q_{\nu}, a_{1,\nu}, \ldots, a_{d,\nu}) = \Delta \cdot (q_*, a_{1,*}, \ldots, a_{d,*})$ and $(q_{\nu-1}, a_{1,\nu-1}, \ldots, a_{d,\nu-1}) = \Delta' \cdot (q_*, a_{1,*}, \ldots, a_{d,*})$, where

$$\Delta = \text{g.c.d.}(q_{\nu}, a_{1,\nu}, \ldots, a_{d,\nu}), \quad \Delta' = \text{g.c.d.}(q_{\nu-1}, a_{1,\nu-1}, \ldots, a_{d,\nu-1}).$$

In particular

$q_{\nu} = \Delta q_*, \quad q_{\nu-1} = \Delta' q_*$

and

$$\left| \begin{array}{cc} q_{\nu-1} & A_{\nu-1} \\ q_{\nu} & A_{\nu} \end{array} \right| \equiv 0 \pmod{q_*}.$$ 

Now from (8.4) we deduce

$$\frac{q_{\nu}}{\Delta} = q_* \leq \left| \frac{q_{\nu-1}}{q_{\nu}} \frac{A_{\nu-1}}{A_{\nu}} \right| \leq 2q_{\nu}|q_{\nu-1} f(\xi) - A_{\nu-1}| \leq 2q_{\nu} \zeta_{\nu-1} \leq 2q_{\nu}^{1-\gamma}$$

by (6.1). Thus we get

$q_{\nu}^{\gamma} \leq 2\Delta.$

We apply Lemma 2.2 to see that $\gamma \leq \frac{s-1}{s}$, and hence $\hat{\omega}(\Xi_f) \leq \frac{s-1}{s} < H_{d,s}$.

**Case 1.2.** For infinitely many $\nu$ (8.2) holds with $a_{\nu-1} \neq a_{\nu}$. 

We proceed similarly to Case 1 from the proof of Theorem 2a by applying Lemma 5.1(i) with $A = q_{\nu}$ for $a = a_{\nu} \neq a = a_{\nu-1}$. From (5.2), similarly to (7.2), for large enough $\nu$ we get

$$\frac{1}{(DK)^{1/2} q_{\nu}^{-\epsilon_1}} \leq 2 \sqrt{d} \max_{1 \leq k \leq d} \left| \frac{a_{k,\nu}}{q_{\nu}} - \frac{a_{k,\nu-1}}{q_{\nu-1}} \right| \leq \frac{2\sqrt{d}}{q_{\nu-1} q_{\nu}}.$$
Again $q_\nu^s = O \left( \frac{s-1}{q_\nu^s} \right)$ for infinitely many $\nu$, and $\hat{\omega}(\Xi_f) \leq \frac{s-1}{s} < H_{d,s}$.

**Case 2.** $f(\alpha_\nu) \neq \frac{A_\nu}{q_\nu}$ for all $\nu$ large enough.

This case is similar to Case 2 from the proof of Theorem 2a. Now the difference $f(\alpha_\nu) - \frac{A_\nu}{q_\nu}$ is a nonzero rational number with denominator $Dq_\nu^s$. Therefore we have

$$|f(\alpha_\nu) - \frac{A_\nu}{q_\nu}| \geq \frac{1}{Dq_\nu^s}.$$

Analogously to (7.3) we now get (7.4), which leads to $\omega(\Xi_f) \leq s - 1$. Then, applying Theorem A to the $(d+1)$-dimensional vector $\Xi_f$, we obtain

$$\frac{s-1}{\hat{\omega}(\Xi_f)} \geq \frac{\omega(\Xi_f)}{\hat{\omega}(\Xi_f)} \geq G_{d+1}.$$

This shows that the positive root of equation

$$\left( \frac{s-1}{x} \right)^d = \frac{x}{1-x} \left( \left( \frac{s-1}{x} \right)^{d-1} + \left( \frac{s-1}{x} \right)^{d-2} + \cdots + \frac{s-1}{x} + 1 \right)$$

gives an upper bound for $\hat{\omega}(\Xi_f)$. Theorem 1a is proved.

**9 Proof of Theorem 3a**

Let

$$q_1 < q_2 < \ldots < q_\nu < q_{\nu+1} < \ldots$$

be the sequence of points where the function $\Psi_{f,\xi}(T)$ is not continuous. Without loss of generality we may suppose that this sequence is infinite. We consider the corresponding best approximation vectors

$$\alpha_\nu = \left( \frac{a_{1,\nu}}{q_\nu}, \ldots, \frac{a_{d,\nu}}{q_\nu} \right) \in \mathbb{Q}^d,$$

where $a_{j,\nu}$ realize the minima in the definition of the function $\Psi_{f,\xi}(T)$. They satisfy $f(\alpha_\nu) = 1$. By definition of the function $\Psi_{\xi}(T)$ and numbers $q_\nu$ we see that

$$\Psi_{\xi}(T) = \Psi_{\xi}(q_{\nu-1}) \quad \text{for} \quad q_{\nu-1} \leq T < q_\nu.$$

Now, since $\alpha_\nu \neq \alpha_{\nu-1}$, we may apply Lemma 5.1(ii) with $A = q_\nu$ and $B = q_{\nu-1}$. Indeed, $\alpha = \alpha_\nu$ and $\beta = \alpha_{\nu-1}$ satisfy (5.1) for large enough $\nu$, and from (5.3) we have

$$\frac{1}{DKq_{\nu-1}^s q_\nu^s} \leq |\alpha_{\nu-1} - \alpha_\nu|^s \leq (|\alpha_{\nu-1} - \xi| + |\alpha_\nu - \xi|)^s \leq 2^s|\alpha_{\nu-1} - \xi|^s,$$
since $|\alpha_\nu - \xi| \leq |\alpha_{\nu-1} - \xi|$. Thus
\[
\frac{1}{2^s DK} \leq q_{\nu}^{s-1} q_{\nu-1}^{s-1} |\alpha_{\nu-1} - \xi|^s \leq q_{\nu}^{s-1} \Psi_{f,\xi}(q_{\nu-1}).
\]
Thus this means that
\[
\lim_{T \to q_{\nu-1}^{-1}} T^{s-1} \cdot \Psi_{f,\xi}(T) = q_{\nu}^{s-1} \Psi_{f,\xi}(q_{\nu-1}) \geq \frac{1}{2^s DK}.
\]

Theorem 3a is proved.

\[\square\]

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**References**


Simultaneous approximation and homogeneous polynomials


