Schmidt’s game, fractals, and orbits of toral endomorphisms

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Abstract. Given an integer matrix \( M \in \text{GL}_n(\mathbb{R}) \) and a point \( y \in \mathbb{R}^n / \mathbb{Z}^n \), consider the set
\[
\tilde{E}(M, y) \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : y \not\in \{ M^k x \mod \mathbb{Z}^n : k \in \mathbb{N} \} \}.
\]
S. G. Dani showed in 1988 that whenever \( M \) is semisimple and \( y \in \mathbb{Q}^n / \mathbb{Z}^n \), the set \( \tilde{E}(M, y) \) has full Hausdorff dimension. In this paper we strengthen this result, extending it to arbitrary \( M \in \text{GL}_n(\mathbb{R}) \cap \text{M}_{n \times n}(\mathbb{Z}) \) and \( y \in \mathbb{R}^n / \mathbb{Z}^n \), and in fact replacing the sequence of powers of \( M \) by any lacunary sequence of (not necessarily integer) \( m \times n \) matrices. Furthermore, we show that sets of the form \( \tilde{E}(M, y) \) and their generalizations always intersect with ‘sufficiently regular’ fractal subsets of \( \mathbb{R}^n \). As an application, we give an alternative proof of a recent result [M. Einsiedler and J. Tseng. Badly approximable systems of affine forms, fractals, and Schmidt games. Preprint, arXiv:0912.2445] on badly approximable systems of affine forms.

1. Introduction
Let \( \mathbb{T}^n \overset{\text{def}}{=} \mathbb{R}^n / \mathbb{Z}^n \) be the \( n \)-dimensional torus. Any non-singular \( n \times n \) matrix \( M \) with integer entries defines a continuous surjective endomorphism \( f_M \) of \( \mathbb{T}^n \) given by
\[
f_M(x + \mathbb{Z}^n) \overset{\text{def}}{=} Mx + \mathbb{Z}^n \quad \forall x \in \mathbb{R}^n,
\]
and any continuous surjective endomorphism \( f \) of \( \mathbb{T}^n \) can be obtained in this way. Criteria for ergodicity of \( f \) (with respect to Haar measure on \( \mathbb{T}^n \)) are well known, and ergodicity implies that \( f \)-orbits of almost all points are dense in \( \mathbb{T}^n \). Also, in many cases, it is known that exceptional sets of points with non-dense orbits are rather big. For example, following the notation used in [14], let us define
\[
E(f, y) \overset{\text{def}}{=} \{ x \in \mathbb{T}^n : y \not\in \{ f^k(x) : k \in \mathbb{N} \} \}
\]
for a fixed \( y \in \mathbb{T}^n \) and a self-map \( f \) of \( \mathbb{T}^n \). In 1988, Dani proved the following theorem.

THEOREM 1.1. [5, Theorem 2.1] For any semisimple \( M \in \text{GL}_n(\mathbb{R}) \cap \text{M}_{n \times n}(\mathbb{Z}) \) and any \( y \in \mathbb{Q}^n / \mathbb{Z}^n \), the set \( E(f_M, y) \) is \( \frac{1}{2} \)-winning.
The above winning property is based on a game, introduced by Schmidt in [26], which is usually referred to as Schmidt’s game. This property implies density and full Hausdorff dimension and is stable with respect to countable intersections; see §2 for more detail.

One of the goals of the present paper is to prove a far-reaching generalization of Theorem 1.1. Namely, we remove the assumptions of \( M \) being semisimple and \( y \) being rational. Also, we are able to intersect sets \( E(f, y) \) with many ‘sufficiently regular’ fractal subsets of \( \mathbb{T}^n \). In fact, it will be more convenient to lift the problem to \( \mathbb{R}^n \): denote by \( \pi \) the quotient map \( \mathbb{R}^n \to \mathbb{T}^n \) and, for \( M \in M_{n \times n}(\mathbb{R}) \) and \( y \in \mathbb{T}^n \), consider
\[
\tilde{E}(M, y) \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : y \notin \{ \pi(M^k x) : k \in \mathbb{N} \} \}. \tag{1.2}
\]
Clearly,
\[
\tilde{E}(M, y) = \pi^{-1}(E(f_M, y))
\]
when \( M \in \text{GL}_n(\mathbb{R}) \cap M_{n \times n}(\mathbb{Z}) \); however, the definition (1.2) makes sense even when \( M \) is singular or has non-integer entries.

The ‘sufficient regularity’ of subsets of \( \mathbb{R}^n \) will be characterized by their ability to support so-called absolutely decaying measures; see [16] or §3 for a definition. Examples include \( \mathbb{R}^n \) itself and limit sets of irreducible families of contracting similarities of \( \mathbb{R}^n \) satisfying the open-set condition, such as the Koch snowflake or the Sierpiński carpet. Other interesting examples can be found in [16, 29, 32].

It turns out, as was first observed in [12], that the absolute decay property of a measure can be used for playing Schmidt’s game on its support. Namely, we will say, following [1], that a subset \( S \) of \( \mathbb{R}^n \) is \( \alpha \)-winning on a subset \( K \) of \( \mathbb{R}^n \) if \( S \cap K \) is \( \alpha \)-winning for Schmidt’s game played on the metric space \( K \) with the metric induced from \( \mathbb{R}^n \). From [26], it immediately follows that the intersection of countably many sets \( \alpha \)-winning on \( K \) is also \( \alpha \)-winning on \( K \). We will say that \( S \) is \( \text{winning} \) on \( K \) if it is \( \alpha \)-winning on \( K \) for some \( \alpha > 0 \). Precise definitions are given in §2. As a trivial consequence of Corollary 3.3, if \( S \) is winning on \( K = \text{supp} \mu \), where \( \mu \) is absolutely decaying, then \( S \cap K \) is not contained in a countable union of affine hyperplanes. Furthermore, under some additional assumptions on \( \mu \), for example when \( K = \mathbb{R}^n \) or one of the self-similar sets mentioned above, one can show that the Hausdorff dimension of \( S \cap K \) is equal to \( \text{dim}(K) \) whenever \( S \) is winning on \( K \). See §3 for precise statements.

In this paper we prove a generalization of Theorem 1.1.

**Theorem 1.2.** For every \( K \subset \mathbb{R}^n \) which supports an absolutely decaying measure there exists \( \alpha = \alpha(K) > 0 \) such that for any \( M \in \text{GL}_n(\mathbb{R}) \cap M_{n \times n}(\mathbb{Z}) \) and any \( y \in \mathbb{T}^n \), the set \( \tilde{E}(M, y) \) is \( \alpha \)-winning on \( K \).

In particular, for any countable subset \( Y \) of \( \mathbb{T}^n \), the set
\[
\bigcap_{y \in Y} \bigcap_{M \in \text{GL}_n(\mathbb{R}) \cap M_{n \times n}(\mathbb{Z})} \tilde{E}(M, y)
\]
is also \( \alpha \)-winning on \( K \). It immediately follows that the sets \( E(f_M, y) \) discussed in Theorem 1.1 and their countable intersections always intersect those subsets of the torus whose pullbacks to \( \mathbb{R}^n \) support absolutely decaying measures. It can also be shown that \( \alpha(\mathbb{R}^n) = 1/2 \), recovering Dani’s result; see §5.1.
The one-dimensional case of Theorem 1.2 appeared recently in [1], and also, independently and for $K = \mathbb{R}$, in [10]; see also [30]. In other words, the sets
\[ \tilde{E}(b, y) \overset{\text{def}}{=} \{ x \in \mathbb{R} : y \notin \{ \pi(b^k x) : k \in \mathbb{N} \} \} \tag{1.3} \]
were shown to be winning on supp $\mu$ for any absolutely decaying measure $\mu$ on $\mathbb{R}$, any integer $b > 1$, and any $y \in \mathbb{T}$. However, the main result of [1] applies to much more general situations, recovering earlier work [6, 23, 24] by Pollington and de Mathan. In particular, $b$ in (1.3) does not have to be an integer, and one can replace the sequence of powers of $b$ by an arbitrary lacunary sequence $t_k$ of real numbers (we recall that $(t_k)$ is called lacunary if $\inf_{k \in \mathbb{N}} t_{k+1}/t_k > 1$).

We now describe an analogous generalization of Theorem 1.2, which is the main result of the present paper. We are going to fix $m, n \in \mathbb{N}$, consider a sequence $\mathcal{M} = (M_k)$ of $m \times n$ matrices and a sequence $\mathcal{Z} = (Z_k)$ of subsets of $\mathbb{R}^m$, and define
\[ \tilde{E}(\mathcal{M}, \mathcal{Z}) \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^n : \inf_{k \in \mathbb{N}} d(M_k x, Z_k) > 0 \right\}. \tag{1.4} \]
(Here $d(\cdot, \cdot)$ stands for the Euclidean distance on $\mathbb{R}^n$.) The sets $\tilde{E}(\mathcal{M}, y)$ defined in (1.2) constitute a special case, with $m = n$, $\mathcal{M} = (M^k)$, and $Z_k = \pi^{-1}(y)$.

Some assumptions on $\mathcal{M}$ and $\mathcal{Z}$ are in order. We will say that a sequence $\mathcal{M}$ of non-zero $m \times n$ matrices is lacunary if so is the sequence $(\|M_k\|_{\text{op}})$ of the values of their operator norms. A subset $Z$ of $\mathbb{R}^n$ will be called $\delta$-uniformly discrete if $\inf_{x, y \in Z, x \neq y} d(x, y) > \delta$. With some abuse of terminology, we say that a sequence $\mathcal{Z} = (Z_k)$ is $\delta$-uniformly discrete if $Z_k$ is $\delta$-uniformly discrete for every $k \in \mathbb{N}$, and that $\mathcal{Z}$ is uniformly discrete if it is $\delta$-uniformly discrete for some $\delta > 0$. For example, for an arbitrary sequence $(y_k)$ of points of $\mathbb{T}^m$, the sequence of sets $Z_k = \pi^{-1}(y_k) \subset \mathbb{R}^m$ is 1-uniformly discrete.

We can now formulate our main result, which is proved in §4.

**Theorem 1.3.** For every $K \subset \mathbb{R}^n$ which supports an absolutely decaying measure there exists a positive $\alpha = \alpha(K)$ such that if $\mathcal{Z}$ is a uniformly discrete sequence of subsets of $\mathbb{R}^m$ and $\mathcal{M}$ is a lacunary sequence of $m \times n$ matrices with real entries, then $\tilde{E}(\mathcal{M}, \mathcal{Z})$ is $\alpha$-winning on $K$.

An important special case is $m = n$ and $\mathcal{M} = (M^k)$, where $M$ is an $n \times n$ matrix with spectral radius strictly greater than 1 (not necessarily invertible and not necessarily with integer entries); this is used to derive Theorem 1.2 from Theorem 1.3: see §4. Our main theorem also generalizes results from [2, 22] dealing with a special case where
\[ \mathcal{M} \text{ is a lacunary sequence of } 1 \times n \text{ integer matrices} \]
and $\mathcal{Z} = (Z_k)$, where $Z_k = \mathbb{Z} = \pi^{-1}(0)$ $\forall k \in \mathbb{N}$. It was observed both in [2] and in [22] that the latter set-up can be used to prove the abundance of badly approximable systems of affine forms. Recall that a pair $(A, \mathbf{x})$, interpreted as a function $\mathbf{q} \mapsto A\mathbf{q} - \mathbf{x}$, $\mathbb{R}^m \to \mathbb{R}^n$ (here $A \in \mathbb{M}_{n \times m}(\mathbb{R})$ and $\mathbf{x} \in \mathbb{R}^n$) is said to be badly approximable if
\[ \inf_{\mathbf{q} \in \mathbb{Z}^m \setminus \{0\}} \|\mathbf{q}\|^{m/n} d(A\mathbf{q} - \mathbf{x}, \mathbb{Z}^n) > 0. \]
This is an inhomogeneous analogue of the notion of badly approximable systems of linear forms; see [27, 28]. It was proved in [15] that the set $\text{Bad}(n, m)$ of badly approximable
pairs \((A, x)\) has full Hausdorff dimension. Then a much easier proof was found in [2], where, for fixed \(A \in M_{n \times m}(\mathbb{R})\), the sets

\[
\text{Bad}_A(n, m) = \{x \in \mathbb{R}^n : (A, x) \in \text{Bad}(n, m)\}
\]

were considered, and it was shown that \(\dim(\text{Bad}_A(n, m)) = n\) for any \(A\). The latter result was strengthened by Tseng in the case \(m = n = 1\): he proved [31] that \(\text{Bad}_A(1, 1) \subset \mathbb{R}\) is \(\frac{1}{8}\)-winning for any \(a \in \mathbb{R}\). Shortly thereafter, Moshchevitin concluded [22] that the sets \(\text{Bad}_A(n, m)\) are \(\frac{1}{2}\)-winning for any \(m, n\) and any \(A \in M_{n \times m}(\mathbb{R})\). Our main theorem can be used to deduce the following corollary.

**Corollary 1.4.** Let \(K \subset \mathbb{R}^n\) be the support of an absolutely decaying measure, and let \(\alpha\) be as in Theorem 1.3. Then for any \(A \in M_{n \times m}(\mathbb{R})\), \(\text{Bad}_A(n, m)\) is \(\alpha\)-winning on \(K\).

Independently, in a recent preprint [7], Einsiedler and Tseng provided another proof of this result, with a smaller value of \(\alpha\). We derive Corollary 1.4 in §4. At the end of the paper a remark is made explaining how all our results can be strengthened to replace ‘winning’ with ‘strong winning’, a property introduced recently in [10, 11, 21].

### 2. Schmidt’s game

In this section we describe the game, first introduced by Schmidt in [26]. Let \((X, d)\) be a complete metric space. Consider \(\Omega \overset{\text{def}}{=} X \times \mathbb{R}_+\), and define a partial ordering

\[
(x_2, \rho_2) \leq_s (x_1, \rho_1) \quad \text{if} \quad \rho_2 + d(x_1, x_2) \leq \rho_1.
\]

We associate to each pair \((x, \rho)\) a ball in \((X, d)\) via

\[
B(x, \rho) = \{x' \in X : d(x', x) \leq \rho\}.
\]

Note that \((x_2, \rho_2) \leq_s (x_1, \rho_1)\) implies (but is not necessarily implied by) \(B(x_2, \rho_2) \subset B(x_1, \rho_1)\). However, the two conditions are equivalent when \(X\) is a Euclidean space.

Schmidt’s game is played by two players, whom we will call Alice and Bob, following a convention used previously in [1, 18]. The two players are equipped with parameters \(\alpha\) and \(\beta\), respectively, satisfying \(0 < \alpha, \beta < 1\). Choose a subset \(S\) of \(X\) (a target set). The game starts with Bob picking \(x_1 \in X\) and \(\rho > 0\), hence specifying a pair \((x_1, \rho)\). Alice and Bob then take turns choosing

\[
\omega_k' = (x_k', \rho_k') \leq_s \omega_k \quad \text{and} \quad \omega_{k+1} = (x_{k+1}, \rho_{k+1}) \leq_s \omega_k'
\]

respectively satisfying

\[
\rho_k' = \alpha \rho_k \quad \text{and} \quad \rho_{k+1} = \beta \rho_k'.
\]

As the game is played on a complete metric space and the diameters of the nested balls

\[
B(\omega_1) \supset B(\omega_1') \supset \cdots \supset B(\omega_k) \supset B(\omega_k') \supset \cdots
\]

tend to zero as \(k \to \infty\), the intersection of these balls is a point \(x_\infty \in X\). Call Alice the winner if \(x_\infty \in S\). Otherwise, Bob is declared the winner. A strategy consists of specifications for a player’s choices of centers for his or her balls given the opponent’s previous moves.

If for certain \(\alpha, \beta\), and a target set \(S\) Alice has a winning strategy, i.e. a strategy for winning the game regardless of how well Bob plays, we say that \(S\) is an \((\alpha, \beta)\)-winning set. If \(S\) and \(\alpha\) are such that \(S\) is an \((\alpha, \beta)\)-winning set for all possible \(\beta\), we say that \(S\) is an \(\alpha\)-winning set. Call a set winning if such an \(\alpha\) exists.
Intuitively, one expects winning sets to be large. Indeed, every such set is clearly dense in $X$; moreover, under some additional assumptions on the metric space, winning sets can be proved to have positive, and even full, Hausdorff dimension. For example, the fact that a winning subset of $\mathbb{R}^n$ has Hausdorff dimension $n$ is due to Schmidt [26, Corollary 2]. Another useful result of Schmidt [26, Theorem 2] states that the intersection of countably many $\alpha$-winning sets is $\alpha$-winning.

Schmidt himself used the machinery of the game he invented to prove that certain subsets of $\mathbb{R}$ or $\mathbb{R}^n$ are winning, and hence have full Hausdorff dimension. Now let $K$ be a closed subset of $X$. Following an approach initially introduced in [12], we will say that a subset $S$ of $X$ is $(\alpha, \beta)$-winning on $K$ (respectively, $\alpha$-winning on $K$, winning on $K$) if $S \cap K$ is $(\alpha, \beta)$-winning (respectively, $\alpha$-winning, winning) for Schmidt’s game played on the metric space $K$ with the metric induced from $(X, d)$. In the present paper we let $X = \mathbb{R}^n$ and take $K$ to be the support of an absolutely decaying measure. In other words, since the metric is induced, playing the game on $K$ amounts to choosing balls in $\mathbb{R}^n$ according to the rules of a game played on $\mathbb{R}^n$, but with an additional constraint that the centers of all the balls lie in $K$. Since the first appearance of this approach in [12], where it was used to show that sufficiently regular fractals meet with a countable intersection of non-singular affine images of the set of badly approximable vectors in $\mathbb{R}^n$, it has been utilized in [8, 13], and most recently in [1], of which the present paper is a following and a generalization.

3. Absolutely decaying measures

In this section we describe in detail the class of absolutely decaying measures and discuss other related properties and their applications. Following a terminology introduced in [16, 25], we say that a locally finite Borel measure $\mu$ on $\mathbb{R}^n$ is $(C, \gamma)$-absolutely decaying if there exists $\rho_0 > 0$ such that

$$\mu(B(x, \rho) \cap L^{(\varepsilon)}) < C(\varepsilon/\rho)^\gamma \mu(B(x, \rho))$$

for any affine hyperplane $L \subset \mathbb{R}^n$ and any $x \in \text{supp } \mu$, $0 < \rho < \rho_0$, $\varepsilon > 0$. (3.1)

Here $B(x, \rho)$ stands for the closed Euclidean ball in $\mathbb{R}^n$ of radius $\rho$ centered at $x$, and

$L^{(\varepsilon)} \overset{\text{def}}{=} \{x \in \mathbb{R}^n : d(x, L) \leq \varepsilon\}$

is the closed $\varepsilon$-neighborhood of $L$. We say that $\mu$ is absolutely decaying if it is $(C, \gamma)$-absolutely decaying for some $C, \gamma > 0$. (This terminology differs slightly from the one introduced in [16], where a less uniform version was considered.) If $\mu$ is $(C, \gamma)$-absolutely decaying, we will denote by $\rho_{C, \gamma}(\mu)$ the supremum of $\rho_0$ for which (3.1) holds.

Another property, which often comes in a package with absolute decay, is the so-called doubling, or Federer, condition. One says that $\mu$ is $D$-Federer if there exists $\rho_0 > 0$ such that

$$\mu(B(x, 2\rho)) < D \mu(B(x, \rho)) \quad \forall x \in \text{supp } \mu, \forall 0 < \rho < \rho_0,$$

and $\text{Federer}$ if it is $D$-Federer for some $D > 0$. Measures which are both absolutely decaying and Federer are called absolutely friendly, a term coined in [25].
Many examples of absolutely friendly measures can be found in [16, 17, 29, 32]. The Federer condition is very well studied; it obviously holds when \( \mu \) satisfies a power law, i.e. there exist positive \( \delta, c_1, c_2, \rho_0 \) such that
\[
c_1 \rho^\delta \leq \mu(B(x, \rho)) \leq c_2 \rho^\delta \quad \forall x \in \text{supp } \mu, \forall 0 < \rho < \rho_0. \tag{3.3}\]
Such measures are often referred to as \( \delta \)-Ahlfors regular. However, it is not hard to construct absolutely friendly measures not satisfying a power law; see [17] for an example. Also, when \( n = 1 \) the Federer property is implied by the absolute decay, which in its turn is implied by a power law (see [1] for a thorough discussion of equivalent definitions of absolute friendliness in the one-dimensional case). However, these implications fail to hold in higher dimensions. In particular, the volume measures on smooth \( k \)-dimensional submanifolds of \( \mathbb{R}^n \) obviously are \( k \)-Ahlfors regular but not absolutely decaying unless \( k = n \).

The goal of the current work, as well as in several earlier papers [8, 12, 13, 17, 19], is to use measures in order to construct points in their supports with prescribed (dynamical or Diophantine) properties. Our attention will therefore be focused on closed subsets \( K \) of \( \mathbb{R}^n \) which support absolutely decaying and absolutely friendly measures. For example, this is the case when \( K = \mathbb{R}^n \), or when \( K \) is the limit set of an irreducible family of contracting self-similar [16] or self-conformal [32] transformations of \( \mathbb{R}^n \) satisfying the open-set condition. More examples can be found in [17, 29]. Note that the paper [2] established full Hausdorff dimension of \( \tilde{E}(\mathcal{M}, \mathcal{Z}) \cap K \) for \( \mathcal{M}, \mathcal{Z} \) as in (1.5) and under an assumption that \( K \subset \mathbb{R}^n \) supports an absolutely decaying, \( \delta \)-Ahlfors regular measure with \( \delta > n - 1 \). It is not hard to show, using an elementary covering argument, that (3.3) with \( \delta > n - 1 \) implies (3.1) with \( \gamma = \delta - n + 1 \). Hence, the sets considered in [2] support absolutely decaying measures.

Recall that the lower point-wise dimension of a measure \( \mu \) at \( x \in \text{supp } \mu \) is defined as
\[
d_{\mu}(x) \overset{\text{def}}{=} \liminf_{\rho \to 0} \frac{\log \mu(B(x, \rho))}{\log \rho}.
\]
For an open \( U \) with \( \mu(U) > 0 \), let
\[
d_{\mu}(U) \overset{\text{def}}{=} \inf_{x \in \text{supp } \mu \cap U} d_{\mu}(x). \tag{3.4}
\]
It is well known, see e.g. [9, Proposition 4.9], that (3.4) constitutes a lower bound for the Hausdorff dimension of \( \text{supp } \mu \cap U \) (where this bound is sharp when \( \mu \) satisfies a power law). It is also easy to see that \( d_{\mu}(x) \geq \gamma \) for every \( x \in \text{supp } \mu \) whenever \( \mu \) is \( (C, \gamma) \)-absolutely decaying: indeed, take \( \rho < \rho_0 < \rho_{C, \gamma}(\mu) \) and \( x \in \text{supp } \mu \); then, using (3.1) and noting that \( B(x, \rho) \subset \mathcal{L}(\rho) \) for some hyperplane \( \mathcal{L} \), one has
\[
\mu(B(x, \rho)) < C \left( \frac{\rho}{\rho_0} \right)^\gamma \mu(B(x, \rho_0)).
\]
Thus, for \( \rho < 1 \),
\[
\frac{\log \mu(B(x, \rho))}{\log \rho} \geq \gamma + \frac{\log C - \gamma \log \rho_0 + \log \mu(B(x, \rho_0))}{\log \rho},
\]
and the claim follows.
The following proposition [18, Proposition 5.1] makes it possible to estimate the Hausdorff dimension of sets winning on supports of Federer measures.

**Proposition 3.1.** Let $K$ be the support of a Federer measure $\mu$ on $\mathbb{R}^n$, and let $S$ be winning on $K$. Then for any open $U \subset \mathbb{R}^n$ with $\mu(U) > 0$, one has

$$\dim(S \cap K \cap U) \geq d_\mu(U).$$

In particular, if in addition $\mu$ is $(C, \gamma)$-absolutely decaying, in the above proposition one can replace $d_\mu(U)$ with $\gamma$, and with $\dim(K)$ if $\mu$ satisfies a power law. Note that this generalizes estimates for the Hausdorff dimension of winning sets due to Schmidt [26] for $\mu$ being Lebesgue on $\mathbb{R}^n$, and to Fishman [12, §5] for measures satisfying a power law.

The next lemma exhibits a crucial feature of sets supporting absolutely decaying measures, namely the fact that while playing Schmidt’s game on such a set, Alice can distance herself from hyperplanes ‘efficiently’. This observation is the cornerstone of the proof of our main theorem. The argument has been adapted from the one in [22], where the case $K = \mathbb{R}^n$ was proved with $\alpha = \frac{1}{2}$ (see §5.1 for more detail), and then refined using an observation from [7].

**Lemma 3.2.** For every $C, \gamma > 0$, and

$$\alpha < \frac{1}{2C^{1/\gamma} + 1},$$

there exists $\varepsilon = \varepsilon(C, \gamma, \alpha) \in (0, 1)$ such that if $K$ is the support of a $(C, \gamma)$-absolutely decaying measure $\mu$ on $\mathbb{R}^n$, $0 < \rho < \rho_{C, \gamma}(\mu)$, $x_1 \in K$, $N \in \mathbb{N}$, and $\mathcal{L}_1, \ldots, \mathcal{L}_N$ are hyperplanes in $\mathbb{R}^n$, there exists $x_2 \in K$ with

$$B(x_2, \alpha \rho) \subset B(x_1, \rho)$$

and

$$d(B(x_2, \alpha \rho), \mathcal{L}_i) > \alpha \rho \quad \text{for at least } \lceil \varepsilon N \rceil \text{ of the hyperplanes } \mathcal{L}_i.$$  \hspace{1cm} (3.7)

**Proof.** Let

$$A_i = B(x_1, (1 - \alpha)\rho) \setminus \mathcal{L}_i^{(2\alpha \rho)}.$$  \hspace{1cm} (3.6)

By (3.1) and (3.5), for each $1 \leq i \leq N$, 

$$\frac{\mu(A_i)}{\mu(B(x_1, (1 - \alpha)\rho))} > 1 - C \left( \frac{2\alpha}{1 - \alpha} \right)^\gamma \equiv \varepsilon > 0.$$  \hspace{1cm} (3.6)

We claim that there exist $j_1, \ldots, j_k$, where $k \geq \lceil \varepsilon N \rceil$, such that

$$K \cap \bigcap_{i=1}^k A_{j_i} \neq \emptyset.$$  \hspace{1cm} (3.7)

To see this, let

$$f(x) = \sum_{i=1}^N \chi_{A_i}(x).$$

Then

$$\int_{B(x_1, (1 - \alpha)\rho)} f(x) \, d\mu(x) \geq N\varepsilon \mu(B(x_1, (1 - \alpha)\rho)),$$

so clearly there exists some $x_2 \in K$ with $f(x_2) \geq N\varepsilon$. Since $f(x_2) \in \mathbb{Z}$, there must exist $j_1, \ldots, j_k$ as above. Hence, $x_2$ satisfies (3.6) and (3.7). \hfill \Box
We will also need the following corollary of the above lemma.

**Corollary 3.3.** Let $K$ be the support of a $(C, \gamma)$-absolutely decaying measure on $\mathbb{R}^n$, let $\alpha$ be as in (3.5), let $S \subset \mathbb{R}^n$ be $\alpha$-winning on $K$, and let $S' \subset S$ be a countable union of hyperplanes. Then $S \setminus S'$ is also $\alpha$-winning on $K$.

**Proof.** In view of the countable intersection property, it suffices to show that for any hyperplane $L \subset \mathbb{R}^n$, the set $\mathbb{R}^n \setminus L$ is $(\alpha, \beta)$-winning on $K$ for any $\beta$. Let $\mu$ be a $(C, \gamma)$-absolutely decaying measure with $K = \text{supp} \mu$. We let Alice play arbitrarily until the radius of a ball chosen by Bob is less than $\rho(C, \gamma)(\mu)$. Then apply Lemma 3.2 with $N = 1$ and $L_1 = L$, which yields a ball disjoint from $L$. Afterwards, she can keep playing arbitrarily, winning the game.

4. Proofs

Let us now state a more precise version of Theorem 1.3.

**Theorem 4.1.** Let $K$ be the support of a $(C, \gamma)$-absolutely decaying measure on $\mathbb{R}^n$, and let $\alpha$ be as in (3.5). Then for any uniformly discrete sequence $Z$ of subsets of $\mathbb{R}^m$ and any lacunary sequence $M$ of $m \times n$ real matrices, the set $\tilde{E}(M, Z)$ is $\alpha$-winning on $K$.

**Proof.** Write $M = (M_k)$, let $t_k \overset{\text{def}}{=} \|M_k\|_{\text{op}}$, and let $v_k$ be a unit vector satisfying $\|M_k v_k\| = t_k$. Take $\delta > 0$ such that $Z$ is $\delta$-uniformly discrete, and let $\inf_k \frac{t_{k+1}}{t_k} = Q > 1$. (4.1)

Now pick an arbitrary $0 < \beta < 1$, take $\varepsilon$ as in Lemma 3.2, and choose $N$ large enough that

$$(\alpha \beta)^{-r} \leq Q^N, \quad \text{where } r = \lfloor \log_{1/(1-\varepsilon)} N \rfloor + 1. \quad (4.2)$$

We will denote by $M_k^{-1}(Z)$ the preimage of a set $Z \subset \mathbb{R}^n$ under $M_k$. Notice that for each $k \in \mathbb{N}$, $M_k^{-1}(Z_k)$ is contained in a countable union of hyperplanes, so, applying Corollary 3.3 a finite number of times, we may assume that $t_1 \geq 1$.

By playing arbitrary moves if needed, we may assume without loss of generality that $B(\omega_1)$ has radius

$$\rho < \min\left(\frac{\alpha \beta \delta}{4}, \rho_{C, \gamma}\right). \quad (4.3)$$

Now let

$$c = \min\left(\rho(\alpha \beta)^{2r-1}, \frac{\delta}{4}\right). \quad (4.4)$$

We will describe a strategy for Alice to play the $(\alpha, \beta)$-game on $K$ and to ensure that for all $j \in \mathbb{N}$, for all $x \in B(\omega_{r(j+1)})$, and for all $k$ with $1 \leq t_k < (\alpha \beta)^{-rj}$, one has $d(M_k x, Z_k) > c$. This will imply that

$$\bigcap_k B(\omega'_k) \in \tilde{E}(M, Z) \cap K,$$

finishing the proof.
To satisfy the above goal, Alice can choose \( \omega'_i \) arbitrarily for \( i < r \). Now fix \( j \in \mathbb{N} \). By (4.1) and (4.2), there are at most \( N \) indices \( k \in \mathbb{N} \) for which
\[
(\alpha \beta)^{-r(j-1)} \leq t_k < (\alpha \beta)^{-rj}.
\] Let \( k \) be one of these indices. For any \( x \in \mathbb{R}^n \),
\[
\|x\| \geq \frac{1}{t_k} \|M_k(x)\|.
\]
Thus, if \( y_1, y_2 \) are two different points in \( Z_k \), then by (4.3) and (4.5)
\[
d(M_k^{-1}(B(y_1, c)), M_k^{-1}(B(y_2, c))) \geq \frac{\delta - 2c}{t_k} \geq \frac{\delta}{2t_k} > \frac{\delta}{2(\alpha \beta)^{rj}} \geq 2(\alpha \beta)^{rj-1};
\]
therefore, \( B(\omega_{rj}) \) intersects with at most one set of the form \( M_k^{-1}(B(y, c)) \), where \( y \in Z_k \). Hence, for each \( k \) satisfying (4.5),
\[
B(\omega_{rj}) \cap M_k^{-1}(Z_k(c)) \subset M_k^{-1}(B(y, c)) \quad \text{for some } y \in Z_k.
\]
We will now show that the preimage of such a ball is contained in a ‘small enough’ neighborhood of some hyperplane, so that we can apply the decay condition. Toward this end, let \( V \subset \mathbb{R}^m \) the hyperplane perpendicular to \( M_kv_k \) and passing through \( 0 \). Then
\[
W \overset{\text{def}}{=} M_k^{-1}(V)
\]
is a hyperplane in \( \mathbb{R}^n \) passing through \( 0 \).

If \( x \notin W^{(c/t_k)} \), then \( x = w + \eta v_k \) for some \( \eta > c/t_k \) and \( w \in W \); thus,
\[
\|M_kx\| = \|M_kw + M_k\eta v_k\| \geq \eta \|M_kv_k\| = t_k \eta > c.
\]
Hence,
\[
M_k^{-1}(B(0, c)) \subset W^{(c/t_k)};
\]
which clearly implies that for each \( y \in Z_k \),
\[
M_k^{-1}(B(y, c)) \subset \mathcal{L}^{(c/t_k)}
\]
for some hyperplane \( \mathcal{L} \subset \mathbb{R}^n \). By (4.4) and (4.5),
\[
\frac{c}{t_k} \leq (\alpha \beta)^{r(j+1)-1} \rho \overset{\text{def}}{=} \xi.
\]
Therefore, by (4.7),
\[
\bigcup_{t_k \text{ satisfies (4.5)}} B(\omega_{rj}) \cap M_k^{-1}(Z_k(c)) \subset \bigcup_{i=1}^N \mathcal{L}_i^{(\xi)};
\]
where \( \mathcal{L}_i \) are hyperplanes. Noticing that by (4.2), \( (1-\varepsilon)^r N < 1 \), Alice can utilize Lemma 3.2 \( r \) times to distance herself by \( \xi \) from each of the hyperplanes \( \mathcal{L}_i \) after \( r \) turns. Thus, for \( k \) satisfying (4.5), we have
\[
B(\omega'_{r(j+1)}) \cap M_k^{-1}(Z_k(c)) = \varnothing.
\]
We conclude that \( d(M_kx, Z_k) \geq c \) for any \( x \in B(\omega'_{r(j+1)}) \), which implies the desired statement. \( \Box \)

**Proof of Theorem 1.2.** Recall that we are given \( M \in \text{GL}_m(\mathbb{R}) \cap \text{M}_n(\mathbb{Z}) \). If all the eigenvalues of \( M \) have modulus less than or equal to 1, then obviously every eigenvalue of \( M \) must have modulus 1. By a theorem of Kronecker [20], they must be roots of unity,
so there exists an $N \in \mathbb{N}$ such that the only eigenvalue of $M^N$ is 1. Let $J = L^{-1}M^NL$ be the Jordan normal form of $M^N$, and let $v_i = Le_i$, $i = 1, \ldots, n$, be the Jordan basis for $M^N$. Then, since $M^N$ is an integer matrix, we have $v_i \in \mathbb{Q}^n$ for each $1 \leq i \leq n$. Hence, letting $V = \text{span}(v_1, \ldots, v_{n-1})$, $V + \mathbb{Z}^n$ is a union of positively separated parallel hyperplanes. Since $J$ fixes the last coordinate of any vector, if $a_1, \ldots, a_n \in \mathbb{R}$, then
\[ M^N \left( \sum_{i=1}^{n} a_i v_i \right) \in a_n v_n + V. \]

Therefore, for $x, y \in \mathbb{R}^n$ with $x - y \notin V + \mathbb{Z}^n$ and any $k \in \mathbb{N}$, one has
\[ d(M^N x, y + \mathbb{Z}^n) \geq c_0 d(x - y, V + \mathbb{Z}^n) > 0, \]
where $c_0$ is a positive constant depending only on $v_1, \ldots, v_n$. Hence, for any $y \in \mathbb{T}^n$,
\[ \tilde{E}(M^N, y) \supset \mathbb{R}^n \setminus (\pi^{-1}(y) + V) = \mathbb{R}^n \setminus (y + V + \mathbb{Z}^n), \]
where $y$ is an arbitrary vector in $\pi^{-1}(y)$. Thus, $\tilde{E}(M^N, y)$ is $\alpha$-winning on $K$ by Corollary 3.3. Hence, $\tilde{E}(M^N, z)$ is $\alpha$-winning on $K$ whenever $z \in f_M^{-i}(y)$, where $0 \leq i < N$. Thus, the intersection
\[ \tilde{E}(M, y) = \bigcap_{i=0}^{N-1} \bigcap_{z \in f_M^{-i}(y)} \tilde{E}(M^N, f_M^{-i}(y)) \]
is also $\alpha$-winning on $K$.

In the case where at least one of the eigenvalues is of absolute value strictly greater than 1, we will show that the sequence $\|M^k\|_{\text{op}}$ is a finite union of lacunary sequences, which will clearly imply that $\tilde{E}((M^k), Z)$ is $\alpha$-winning on $K$. Let $J = L^{-1}ML$ be the Jordan normal form of $M$. Since the operator norm of $M$ as a real transformation is equal to its operator norm as a complex transformation and
\[ \|J^k\|_{\text{op}} \leq \|L\|_{\text{op}} \|L^{-1}\|_{\text{op}} \|M^k\|_{\text{op}} \quad \text{and} \quad \|M^k\|_{\text{op}} \leq \|L\|_{\text{op}} \|L^{-1}\|_{\text{op}} \|J^k\|_{\text{op}}, \]
letting $c = \|L\|_{\text{op}} \|L^{-1}\|_{\text{op}}$, we have
\[ \frac{1}{c} \|M^k\|_{\text{op}} \leq \|J^k\|_{\text{op}} \leq c \|M^k\|_{\text{op}} \quad \text{for all} \ k \in \mathbb{N}. \]
Hence, if $\|J^k\|_{\text{op}}$ is eventually lacunary, then there exist $\ell, N \in \mathbb{N}$, and $Q > 1$ such that, for all $k \geq N$,
\[ \frac{\|M^{k+\ell}\|_{\text{op}}}{\|M^k\|_{\text{op}}} \geq \frac{1}{c^2} \frac{\|J^{k+\ell}\|_{\text{op}}}{\|J^k\|_{\text{op}}} \geq Q. \]
Thus, it will suffice to show that $\|J^k\|_{\text{op}}$ is eventually lacunary.

Let $B$ be an $m \times m$ block of $J$ associated to an eigenvalue $\lambda$ and write $B^k = (b_{ij}(k))$. Direct computation shows that, for $0 \leq j - i \leq k$,
\[ b_{ij}(k) = \binom{k}{j - i} \lambda^{k-(j-i)}, \quad (4.9) \]
and $b_{ij}(k) = 0$ otherwise. Since $|b_{ij}(k)| = o(|b_{1m}(k)|)$ as functions of $k$ for all $(i, j) \neq (1, m)$,
\[ \lim_{k \to \infty} \frac{\|B^k\|_{\text{op}}}{|b_{1m}(k)|} = 1. \quad (4.10) \]
Hence,
\[
\lim_{k \to \infty} \frac{\|B^{k+1}\|_{\text{op}}}{\|B^k\|_{\text{op}}} = |\lambda|,
\]
(4.11)
so clearly if $|\lambda| > 1$ then $(\|B^k\|_{\text{op}})$ is eventually lacunary. Write $J = B_1 \oplus \cdots \oplus B_s$, where $s \in \mathbb{N}$ and $B_i$ are the Jordan blocks, with associated eigenvalues $\lambda_i$. Let $\lambda_{\text{max}} = \max |\lambda_i|$, and let $B_{\text{max}}$ be a block with associated eigenvalue having absolute value $\lambda_{\text{max}}$ and of maximal dimension among such blocks. By (4.9) and (4.10), for any $i$,
\[
\lim_{k \to \infty} \frac{\|B_{\text{max}}^{k+1}\|_{\text{op}}}{\|B_i^k\|_{\text{op}}} \geq 1.
\]
Hence, by (4.11),
\[
\lim_{k \to \infty} \frac{\|J^{k+1}\|_{\text{op}}}{\|J^k\|_{\text{op}}} = \lim_{k \to \infty} \frac{\|B_{\text{max}}^{k+1}\|_{\text{op}}}{\|B_i^k\|_{\text{op}}} = \lambda_{\text{max}}.
\]
Since by assumption $M$ (and therefore $J$) has an eigenvalue with absolute value greater than 1, $(\|J^k\|_{\text{op}})$ is eventually lacunary. \(\square\)

In the remaining part of this section we apply Theorem 1.3 to badly approximable systems of affine forms.

**Proof of Corollary 1.4.** Recall that we need to fix $A \in M_{n \times m}(\mathbb{R})$ and study the set
\[
\text{Bad}_A(n, m) = \left\{ x \in \mathbb{R}^n : \inf_{q \in \mathbb{Z}^m \setminus \{0\}} \|q\|^{m/n} d(Aq - x, \mathbb{Z}^n) > 0 \right\}.
\]
First observe that the above set is easy to understand in the ‘rational’ case when there exists a non-zero $u \in \mathbb{Z}^n$ such that $A^T u \in \mathbb{Z}^m$ (or, equivalently, when the rank of the group $A^T \mathbb{Z}^n + \mathbb{Z}^m$ is strictly smaller than $m + n$). In this case, by a theorem of Kronecker, see \([4, \text{Ch. III, Theorem IV}], \inf_{q \in \mathbb{Z}^m} d(Aq - x, \mathbb{Z}^n)\) is positive if and only if the value of $u \cdot x$ is not an integer. Therefore,
\[
\text{Bad}_A(n, m) \supset \{ x \in \mathbb{R}^n : u \cdot x \notin \mathbb{Z} \}.
\]
Since the right-hand side is the complement of a countable union of hyperplanes, in view of Corollary 3.3 $\text{Bad}_A(n, m)$ is $\alpha$-winning on $K$ whenever $K$ is absolutely decaying and $\alpha$ is as in Theorem 1.3.

In the more interesting ‘irrational’ case when rank($A^T \mathbb{Z}^n + \mathbb{Z}^m$) = $m + n$, one can utilize the theory of best approximations to $A$ as developed by Cassels \([4, \text{Ch. III}]\) and recently made more precise by Bugeaud and Laurent \([3]\). In \([2, \S\S 5–6]\), it is shown that if rank($A^T \mathbb{Z}^n + \mathbb{Z}^m$) = $m + n$, then there exists a lacunary sequence of vectors $y_k \in \mathbb{Z}^n$ (a subsequence of the sequence of best approximations to $A$) such that whenever $x \in \mathbb{R}^n$ satisfies
\[
\inf_{k \in \mathbb{N}} d(y_k \cdot x, \mathbb{Z}) > 0,
\]
it follows that $x \in \text{Bad}_A(n, m)$. In other words,
\[
\bar{E}(\mathcal{Y}', \mathcal{Z}) \subset \text{Bad}_A(n, m),
\]
where $\mathcal{Y}' \overset{\text{def}}{=} (y_k)$ and $\mathcal{Z} = (Z_k)$ with $Z_k = \mathbb{Z}$ for each $k$. (See also \([22, \S 2]\) for an alternative exposition.) Therefore, in this case $\text{Bad}_A(n, m)$ is $\alpha$-winning on $K$ by Theorem 1.3. \(\square\)
5. Concluding remarks

5.1. Playing on $\mathbb{R}^n$ with $\alpha = 1/2$. As was mentioned before, the special case $K = \mathbb{R}^n$ of our main theorem is essentially contained in [22]. In fact, arguing as in §4 and using [22, Lemma 2] (the analogue of our Lemma 3.2) and [22, Lemma 3] (Schmidt’s escaping lemma, cf. [28, Ch. 3, Lemma 1B]), one can show that for $\mathcal{Z}$ and $\mathcal{M}$ as in Theorem 1.3 and any $\alpha, \beta > 0$ with $1 + \alpha \beta - 2\alpha > 0$, the sets $\tilde{E}(\mathcal{M}, \mathcal{Z})$ are $(\alpha, \beta)$-winning. In particular, this shows that one can take $\alpha(\mathbb{R}^n) = 1/2$ in Theorems 1.2 and 1.3.

5.2. Strong winning. Recently in [10, 11] and independently in [21] a modification of Schmidt’s game has been introduced, where condition (2.1) is replaced by

$$\rho'_k \geq \alpha \rho_k \quad \text{and} \quad \rho_{k+1} \geq \beta \rho'_k. \quad (5.1)$$

Following [21], a subset $S$ of a metric space $X$ is said to be $(\alpha, \beta)$-strong winning if Alice has a winning strategy in the game defined by (5.1). Analogously, one defines $\alpha$-strong winning and strong winning sets. It is not hard to verify that strong winning implies winning (see [11] for a proof), and that a countable intersection of $\alpha$-strong winning sets is $\alpha$-strong winning. Furthermore, this class has stronger invariance properties, e.g. it is proved in [21] that strong winning subsets of $\mathbb{R}^n$ are preserved by quasisymmetric homeomorphisms.

It is not hard to modify the proofs given above to show that in Theorem 1.3 (and therefore in all of its corollaries), $\alpha$-winning may be replaced by $\alpha$-strong winning. This is done by adding ‘dummy moves’ in order to accommodate the possibly slower decrease in radii of the chosen balls. Details will appear elsewhere.

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References