Exercise 1. HW6 This exercise fills in the details we sketched in class. Show that any section \( \sigma \) of a rank \( r \) vector bundle \( p : E \to M \) embeds a copy of \( M \) into \( E \) as a submanifold. (Hint: Realize \( \sigma(M) \) locally as a level set.)

Exercise 2. HW6 Let \( F : M \to M \) be a diffeomorphism of \( M \). Show that the fiberwise induced linear isomorphisms on tangent vectors 
\[ DF_a : T_a M \to T_{F(a)} M \]
defines diffeomorphism \( F_* \) of \( TM \) (not a vector bundle homomorphism!) Prove that this further induces a linear isomorphism
\[ \Gamma F_* : \Gamma(M, TM) \to \Gamma(M, TM), \quad X \mapsto \tilde{X} \]
where \( \tilde{X}_a f = DF_a(X_a)(f \circ F^{-1}) \). Moreover, if \( F, G \) are diffeomorphisms of \( M \) then under composition
\[ (FG)_* = F_* G_*, \quad \Gamma(FG)_* = \Gamma F_* \Gamma G_* \]
Therefore, conclude that there is group homomorphism
\[ \text{Diff}(M) \to \text{Aut}(\Gamma(M, TM)), \quad F \mapsto \Gamma F_* \]
where the domain is the group of diffeomorphisms of \( M \).

Exercise 3. HW6 Let \( S(V) \) be the ideal of \( \otimes V \) generated by all quadratic tensors of the form \( u \otimes v - v \otimes u \in \otimes^2 V \) and let \( S^* V = \otimes V / S(V) \)
be the quotient algebra.
(a) Show that the \( \mathbb{Z} \)-grading on the tensor algebra induced one on this quotient algebra.
(b) Show that it is isomorphic to the algebra \( \mathbb{R}[V^*] \) of polynomial functions on \( V^* \), i.e. the algebra of smooth functions generated by the linear functions.
(c) Prove the formal power series identity
\[ \sum_{p \geq 0} (\dim S^p V) z^p = (1 - z)^{-\dim V} \]
where the right side is treated the formal binomial series. Thus for \( n = \dim V \)
\[ (1 - z)^{-n} = 1 + nz + \frac{n(n+1)}{2} z^2 + \cdots + \binom{n+p-1}{p} z^p + \cdots \]
The formula is sometimes called a bosonic partition function. Contrast this with the fermionic partition function
\[ \sum_{p \geq 0} (\dim \wedge^p V) z^p = (1 + z)^{\dim V} \]

Exercise 4. HW6 In class we show that \( H^\bullet(\mathbb{R}^n) = 0 \) except in degree 0. Adapt the argument to show that for any manifold \( M \)
\[ H^\bullet(M \times \mathbb{R}^n) \simeq H^\bullet(M). \]

Exercise 5. HW6 Construct a volume form on \( \mathbb{R}P^3 \).