Tsinghua Mathcamp Summer 2014

A handout about compactness

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1. What we did in class

Compactness is an abstract notion which may have sounded new to most of you. So I decided to write down neatly what we managed to prove in class, and also list some additional facts in the form of problems. Feel free to think about them if you want to familiarize yourself with this notion.

Recall that a metric space is called **compact** if any sequence of its elements has a convergent subsequence. This is not the only way to define compactness. In fact the following fundamental theorem was proved in class:

Theorem 1.1. *The following conditions are equivalent:*

(1) *X is compact;*

(2) *any infinite subset of X has a limit point;*

(3) *if $A_1 \supset A_2 \supset \cdots \supset A_n \supset \dots$ is a nested sequence of nonempty closed sets, then their intersection $\cap_{n=1}^{\infty} A_n$ is not empty;*

(4) *any open covering of X has a finite subcovering.*

We also proved

Proposition 1.2. *Let X be a metric space.*

(1) *If X is compact and $Y \subset X$ is closed, then Y (with the metric induced from X) is compact;*

(2) *if $Y \subset X$ is compact, then it is closed (consequently, a subset of a compact metric space is compact if and only if it is closed).*

What are examples of compact spaces? Compact subsets of $\mathbb{R}^n$ are characterized by the following theorem, due to Bolzano and Weierstrass:

Theorem 1.3. *A subset $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

In particular, $[0,1] \subset \mathbb{R}$ is compact. Another example is given by the space of sequences,

$$\Lambda^{\mathbb{N}} = \{x = (x_n) : x_n \in \Lambda,\ n \in \mathbb{N}\},$$

where $\Lambda = \{0, 1, \dots, k-1\}$ is a finite alphabet, and the distance is defined by

$$d(x,y) = 2^{-m}, \text{ where } m \text{ is the length of the maximal common initial segment of } x \text{ and } y.$$

Proposition 1.4. $\Lambda^{\mathbb{N}}$ *is compact.*

2. Problems

The following is the list of other interesting facts related to compactness, which you can treat as an optional homework. Most of them are proved in Appendices E and F of the lecture notes. However if you are iterated in trying those problems, don't look at the notes right away and try to come up with solutions on your own. You can always check your work later.

Exercise 2.1. Say that a collection $\{A_i\}$ of subsets of X satisfies the **finite intersection property** if for every its finite subcollection $\{A_1, \dots, A_n\}$, the intersection $A_1 \cap \cdots \cap A_n$ is not empty. Prove that a metric space X is compact if and only if for every collection $\{A_i\}$ of closed sets satisfying the finite intersection property, the infinite intersection $\cap_i A_i$ is non-empty.

Exercise 2.2. Let X be a compact metric space, Y an arbitrary metric space, and let $f : X \to Y$ be continuous. Then:

(a) $f(X)$ is compact;

(b) (Heine-Cantor Theorem) f is **uniformly continuous**, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $f\big(B(x,\delta)\big) \subset B\big(f(x),\varepsilon\big)$ for any $x \in X$.

Exercise 2.3. If (X, d_X) and (Y, d_Y) are metric spaces, their direct product

$$X \times Y = \{(x,y) : x \in X,\ y \in Y\}$$

can be viewed as a metric space with the **product metric**

$$d\big((x_1,y_1),(x_2,y_2)\big) \stackrel{\text{def}}{=} \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}\,.$$

Prove that:

(a) a sequence (x_n, y_n) converges to (x,y) in the product metric if and only if $x_n \to x$ in X and $y_n \to y$ in Y;

(b) $X \times Y$ is compact if and only if both X and Y are compact. (This makes it possible to derive the general case of Theorem 1.3 from the case $n = 1$.)

Exercise 2.4. Consider a map f from $\{0,1\}^{\mathbb{N}}$ to $[0,1]$ given by

$$f(x) = 0.x_1x_2\ldots$$

where $x = (x_n)$ and $0.x_1x_2\ldots$ is a binary representation of a real number. Prove that f is continuous. (Thus one can use Proposition 1.4 and Exercise 2.2(a) to give another proof of the compactness of $[0,1]$.)

Exercise 2.5. A metric space is called **separable** if it has a countable sense subset. Prove that compact metric spaces are separable.

Exercise 2.6. Let X be the set of real-valued sequences $x = (x_n)$ such that their squares form a convergent series, that is, $\sum_{n=1}^{\infty} x_n^2 < \infty$. Define a metric on X by

$$d(x,y) = \left(\sum_{n=1}^{\infty}(x_n - y_n)^2\right)^{1/2}$$

(this can be viewed as an infinite-dimensional analogue of the Euclidean metric on $\mathbb{R}^n$). Prove that

(a) the closed unit ball around zero, that is, $B = \{x \in X : d(x,0) \le 1\}$, is not compact;

(b) the so-called **Hilbert cube**

$$H = \{x \in X : |x_n| \le 2^{-n} \text{ for all } n \in \mathbb{N}\}$$

is compact.