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Learning, diversification and the nature of risk

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Abstract In many economic settings, faster learning is achievable only through greater exposure to risk. We study this conflict in the context of project choice, where a risk-averse agent must choose whether to invest in two projects of the same type (focus) or of different types (diversification). Focus enables faster learning across periods, but is riskier due to common type-specific shocks. Optimal choice involves balancing these two considerations. We show that focus is preferred for intermediate learning speeds, and that higher prior uncertainty may encourage focus. Thus, what matters for the focus-diversification choice is not only the level of risk, but also whether the risk is permanent or can be "learned away."

Keywords Bayesian learning · Risk · Diversification

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1 Introduction

Models of learning-by-doing in economics typically involve agents learning the optimal action over time by observing the relationship between their actions and

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the resulting output from period to period. In these models, faster learning may be achieved by conducting a greater number of simultaneous trials every period. However, in many situations, simultaneous trials will not be independent because of the presence of common technology shocks in each period. In these cases, conducting simultaneous trials increases the risk, i.e., the variance of aggregate output in each period. A risk-averse agent therefore faces a trade-off between faster learning across periods and greater diversification within each period. As we discuss in the next section, most of the previous literature on Bayesian learning in economics has focused on risk-neutral agents who, by definition, do not face this trade-off. In this paper, we extend the literature by studying the learning-diversification trade-off in the presence of risk-aversion.

We consider the case of a risk-averse agent who must choose two projects for investment. There are two types of projects, and the agent may choose to focus, by investing in two projects of the same type, or to diversify, by investing in projects of two different types. "Type" here may be taken to represent the technology of the project. In the beginning, there is some uncertainty about an underlying technological parameter for each type of project. Over time, through experience, the agent learns this parameter. There are two kinds of risks in the economy—*type-specific risk*, that results from a shock common to all projects of the same type; and *idiosyncratic risk*, that results from a shock which varies from project to project. Focusing on one type of project makes the total output more susceptible to type-specific risk in each period. However, it also enables faster learning-by-doing since each period's outcomes provide two experimental data points rather than one as in the case of diversification. This implies that while focus imposes higher risk in the initial stages of the projects, it may lead to lower risk and higher output in subsequent periods. Thus, for an agent who is concerned with discounted expected utility over the life of projects, the optimal choice of projects depends on the relative importance of risk and learning.

We present two models in this paper. Both models feature similar Bayesian learning processes with normal priors and normal shocks. In the first model, the technology is linear in an unknown parameter which is learnt over time. Learning is passive in that it does not increase productivity, but merely reduces risk. In this case, the expected lifetime income is the same under focus and diversification. But the variance of total income is higher in each period under focus despite the more rapid decline in uncertainty caused by faster learning. Consequently, for any strictly concave utility function, the expected lifetime utility is lower under focus than under diversification. This is true even when consumption is freely transferable across periods and there is no type-specific risk. The model uses a fairly general set-up to highlight the importance of considering the risks involved in specialization.

Next, we consider the target-input model, which features a non-linear technology and has been used to study learning-by-doing by Wilson (1975), Foster and Rosenzweig (1995), and Jovanovich and Nyarko (1995, 1996), among others. In this model, the agent must learn the optimal input level for a particular technology with output decreasing quadratically in the input error. In this case, learning results in greater productivity in subsequent periods. We show that the optimal input choice for a risk-averse agent with any strictly concave utility function is the same as that for a risk-neutral agent. Based on this insight, we derive the expected lifetime utility of an agent with an exponential utility function under focus and diversification, incorporating optimal input choice and savings behavior. We show that it is possible for focus to be superior to diversification when learning occurs neither too fast, nor too slowly.

We then study how the optimal choice of strategy (focus or diversification) depends on the nature of risk, i.e., whether it is ex-ante uncertainty, type-specific risk, or idiosyncratic risk. We show that contrary to intuition, higher levels of prior uncertainty regarding the technology may be associated with greater focus rather than diversification, even with risk-averse agents. When the prior uncertainty is higher, there is more to be learnt about the technology. This increases the relative benefits to focus, though it also increases the costs of focus in the early stages. If the first effect dominates, then we observe the counter-intuitive result. We also show using numerical examples that the effect of type-specific risk is always negative, and that there exists a threshold level of type-specific risk above which diversification is optimal irrespective of the levels of prior variance and idiosyncratic risk.

Both the prior variance and the type-specific risk are aspects of technology-specific uncertainty. But an increase in the first might lead to greater focus while an increase in the second always leads to lesser focus. Thus, our model highlights an important point regarding the role of risk in the focus-diversification choice—it is not only the *level* of risk that matters, but also the *nature* of risk, i.e., whether it is permanent or it can be reduced through learning.

Our theory is applicable in several economic settings. A few examples are given below.

(1) Two members of a poor household, say a father and his son, must decide on which trade to enter in order to maximize household income. If the son enters the same trade as the father, he will be able to pick up the necessary skills faster as he learns from his father's past experience. However, this also makes the income of the household more vulnerable to the shocks affecting that particular trade. If the household lacks access to capital markets and the diversification opportunities provided by them, then the benefits of entering the same trade in terms of faster learning must be weighed against the higher costs in terms of risk.

(2) A pharmaceutical firm deciding on the optimal portfolio of research projects must evaluate the benefits of focused research in one therapeutic area against its costs in terms of lack of diversification. Focusing all research projects in one therapeutic area would enable the firm to exploit learning spill-overs, but expose it to a greater risk of failure.[1]

(3) In a group lending context such as at the Grameen Bank in Bangladesh, all the members of a group share the risks of default, since one agent's non-payment of dues is treated by the lender as default by the whole group. Hence, members much weigh the benefits of learning spillovers within the group against the cost in terms of risk in making their project choice.

(4) A corporate manager who holds a lot of firm-specific human capital and whose financial wealth is also tied to the firm through stock and option holdings (granted for incentive alignment purposes) would likely consider the risk to his wealth as well as the benefits of exploiting learning spillovers in deciding on the areas into which his firm should diversify.

[1] Henderson and Cockburn (1996) provide evidence on the presence of economies of scope in pharmaceutical research.

The paper proceeds as follows: in Sect. 2, we discuss prior research in this area. In Sect. 3, we present our first model of learning-by-doing with risk-averse agents where we show that diversification is always superior to focus when the technology is linear and learning involves normal priors and normal shocks. In Sect. 4, we consider optimal project choice under the target input model and study how the optimal choice of strategy depends on the nature of risk. In Sect. 5, we discuss the relevance of the results and conclude. All proofs are provided in the Appendix.

2 Prior research

Wilson (1975) studies a model of learning under uncertainty in the context of the theory of the firm. Using a target input model (example 1 of the paper), he explores the optimal degree of sampling for information by a firm that follows the mean-variance criterion. In the context of our model, greater sampling implies investing in more projects of the same type. In contrast, we study the issue of whether to invest in projects of the same type or of different types.

To our knowledge, this is the first attempt at studying project choice when Bayesian learning conflicts with risk-aversion. Parente (1994), Jovanovich and Nyarko (1995, 1996) and Karp and Lee (2001) are some of the papers on learning by doing and the choice of technology. In these, the firm is assumed to operate only one technology at any given time, switching to a new technology when the benefits outweigh the costs. Parente (1994) includes risk-averse agents, but the learning process is exogenously specified, non-informational, and deals with only the first moment of productivity. Jovanovich and Nyarko (1995, 1996) and Karp and Lee (2001) include Bayesian learning, but assume that the agents are risk-neutral. Aghion et al. (1993) study learning through experimentation in a dynamic duopoly game. In contrast to these papers, we study the case where a risk-averse agent may focus or diversify across technologies. A separate stream of research including Prescott (1972), Grossman et al. (1977), and Wieland (2000), has focused on the dynamic optimal control problem with learning by doing. While the optimal control problem is not the primary issue in our paper, we do solve this problem for the target input model in order to compare focus and diversification.

Holmstrom (1982, 1999) studies a Bayesian learning problem that is similar to the one in our first model (Sect. 3). Foster and Rosenzweig (1995) use a model of learning-by-doing with village level shocks (analogous to the type-specific risk in our paper) to study the adoption of high-yielding varieties of seeds by farmers in India. Again, these papers deal with risk-neutral agents.

Jovanovich (1993) develops a model in which diversification is driven by the desire to exploit R&D spillovers, rather than to reduce risk. He uses the model to explain the empirical regularity that more R&D intensive firms are also more diversified. In our model, spillovers are present only when the agent focuses. Hence, spillover benefits can be captured only at the expense of greater risk.

3 Learning-by-doing with risk-averse agents

We begin with a model that incorporates risk in a standard model of learning-by-doing. There are two types of projects, say poultry farming and rice cultivation, denoted by P and R respectively. There is one agent, the proprietor-entrepreneur, who plans to invest in exactly two projects and must choose the type(s) of projects to invest in. Investment in the projects must be made at the beginning (at time $t = 0$). The agent learns the technology of a particular type only by investing in that type of project.

Each project lasts for two periods. The cash flows $y_{ij,t}$ from project $j \in \{1, 2\}$ of type $i \in \{P, R\}$ in period t are given by:

$$y_{ij,t} = a_i + u_{i,t} + e_{ij,t} \tag{1}$$

where a_i is a measure of the underlying project quality, which does not change over time. $u_{i,t}$ is a type-specific shock which affects all projects of type i in period t, and $e_{ij,t}$ is an idiosyncratic shock that varies from project to project. The agent knows that the two shocks are distributed as follows:[2]

$$u_{i,t} \sim \mathrm{N}\left(0, \sigma_{\mathrm{u}}^2\right); \quad e_{ij,t} \sim \mathrm{N}\left(0, \sigma_{\mathrm{e}}^2\right) \tag{2}$$

An alternative but equivalent formulation that combines both types of risk into a single shock is:

$$\begin{aligned} y_{ij,t} &= a_i + \nu_{ij,t} \\ (\nu_{i1,t}, \nu_{i2,t}) &\sim \mathrm{BVN}(0, 0, \sigma_\nu^2, \sigma_\nu^2, \rho) \\ \rho &= \frac{\mathrm{Cov}(u_{i,1} + e_{ij,1}, u_{i,2} + e_{ij,2})}{\sqrt{\mathrm{Var}(u_{i,1} + e_{ij,1})\mathrm{Var}(u_{i,2} + e_{ij,2})}} = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_e^2} \end{aligned} \tag{3}$$

We will mostly use the formulation in (1)–(2) as it is more intuitive, except in the proof of Proposition 2 where (3) is mathematically more convenient.

The underlying project quality is initially unknown (but is learnt over time). The agent begins with a prior belief that

$$a_i \sim \mathrm{N}(\mathrm{a}_0, \sigma_0^2) \quad \forall\, i \tag{4}$$

Over time, the agent learns the project quality from experience. Since the priors and shocks are distributed normally, the Bayesian process for updating beliefs is well known (DeGroot 1989). Let τ denote the precision of a random variable. Thus,

$$\tau_0 = \frac{1}{\sigma_0^2}; \quad \tau_u = \frac{1}{\sigma_u^2}; \quad \text{and} \quad \tau_e = \frac{1}{\sigma_e^2}$$

Learning involves an increase in the precision of beliefs. This increase is faster under focus than under diversification, since the agent is effectively conducting two trials in each period under focus as against one (per project type) under diversification. However, the two trials under focus are not independent due to the presence of type-specific risk. This implies that while learning is faster under focus, it is not as fast as would have been the case with two independent trials. Specifically, the

[2] We are assuming here that the distribution of the type-specific shock is identical across types. This is to emphasize that the difference in risk between focus and diversification is due to the role of correlation in shocks across projects of the same type, and not due to differences in the profitability of the types themselves. The extension to the case where $u_{it} \sim N(\mu_i, \sigma_u^2)$ is straightforward.

precision of beliefs increases in each period by $\tau_{u+e} = (\sigma_u^2 + \sigma_e^2)^{-1}$ under diversification and by $2\tau_{2u+e} = 2(2\sigma_u^2 + \sigma_e^2)^{-1}$ under focus, with $\tau_{u+e} < 2\tau_{2u+e} < 2\tau_{u+e}$. The following lemma characterizes the distribution of the posterior beliefs of the agent under focus and under diversification.

Lemma 1.1 *After the first period, the posterior beliefs of the agent in the case of focus are given by*:

$$a_i \sim \mathrm{N}\,(a_{i,1F}, \tau_{1F}^{-1}) \tag{5}$$

$$a_{i,1F} = a_0\left(\frac{\tau_0}{\tau_{1F}}\right) + \left(\frac{\tau_{2u+e}}{\tau_{1F}}\right)\left(y_{i1,1} + y_{i2,1}\right) \tag{6}$$

$$\tau_{1F} = \frac{1}{\sigma_{1F}^2} = \tau_0 + 2\tau_{2u+e} \tag{7}$$

and in the case of diversification are given by:

$$a_i \sim \mathrm{N}\,(a_{i,1D}, \tau_{1D}^{-1}) \tag{8}$$

$$a_{i,1D} = a_0\left(\frac{\tau_0}{\tau_{1D}}\right) + \left(\frac{\tau_{u+e}}{\tau_{1D}}\right) y_{ij,1} \tag{9}$$

$$\tau_{1D} = \frac{1}{\sigma_{1D}^2} = \tau_0 + \tau_{u+e} \tag{10}$$

where $\tau_{2u+e} = (2\sigma_u^2 + \sigma_e^2)^{-1}$, $\tau_{u+e} = (\sigma_u^2 + \sigma_e^2)^{-1}$ *and project j is of type i in* (9).

To introduce risk-aversion, we assume that an agent has the following time-separable utility function:

$$U(c_1, c_2) = -\exp(-\gamma c_1) - \exp(-\gamma c_2) \tag{11}$$

where c_1 and c_2 are the consumption amounts in the two periods. γ is the Arrow-Pratt coefficient of risk aversion. We ignore intertemporal discounting (and assume that the interest rate is zero) to keep the exposition simple, but introducing it will not alter our results. Discounting will change the size of the relative benefit to diversification over focus, but not its sign. The assumption that the period utility function is exponential is made purely for clarity of exposition. Later, we show that the results hold for any time-separable lifetime utility formulation with a strictly concave period utility function, as well as for any lifetime utility specification which is a strictly concave function of total lifetime income.

The agent chooses c_1 and c_2 to maximize expected utility, assuming free transferability of consumption across periods. The following lemma incorporates optimal consumption choice into (11). Let $y_{1,t}$ and $y_{2,t}$ denote the period t income from projects 1 and 2 respectively, and let $Y_t = y_{1,t} + y_{2,t}$ denote the total period t income. $E_t[\cdot]$ is the expectations operator based on information at the end of period t.

Lemma 1.2 *When* $U(c_1, c_2)$ *is given by the expression in* (11),

$$\max_{c_1,c_2} \; E_0\,[\mathrm{U(c_1, c_2)}] \quad \text{s.t.} \quad c_1 + c_2 = Y_1 + Y_2$$
$$= -2E_0\left[\exp\left(-0.5\gamma\{Y_1 + E_1[Y_2|y_{1,1}, y_{2,1}] - 0.5\gamma Var_1[Y_2|y_{1,1}, y_{2,1}]\}\right)\right] \tag{12}$$

In Lemma 1.2 and in the following analysis, we use the well-known result that if a variable z is normally distributed with mean μ and variance σ^2, then,

$$E[\exp(-kz)] = \exp(-k\{\mu - 0.5k\sigma^2\}).$$

From Lemma 1.2, we see that learning affects the focus-diversification comparison through the conditional mean $E_1[Y_2|y_{1,1}, y_{2,1}]$ and the conditional variance $Var_1[Y_2|y_{1,1}, y_{2,1}]$. These can be calculated using Eqs. (5)–(10), which leads to the following lemma.

Lemma 1.3 *The expected utility under diversification, $E_0[U^D]$, and under focus, $E_0[U^F]$, when $U(c_1, c_2)$ is given by the expression in* (11) *are*

$$E_0[U^D] = -2 * \exp\left(-0.5\gamma\{4a_0 - 0.5\gamma S_{1D}^2 - 0.25\gamma d^2 S_{0D}^2\}\right) \qquad (13)$$
$$E_0[U^F] = -2 * \exp\left(-0.5\gamma\{4a_0 - 0.5\gamma S_{1F}^2 - 0.25\gamma f^2 S_{0F}^2\}\right) \qquad (14)$$

where

$$\begin{aligned} S_{0D}^2 &= Var_0[Y_1^D] = (2\sigma_0^2 + 2\sigma_u^2 + 2\sigma_e^2), \\ S_{1D}^2 &= Var_1[Y_2^D|y_{1,1}, y_{2,1}] = 2\sigma_{1D}^2 + 2\sigma_u^2 + 2\sigma_e^2, \\ S_{0F}^2 &= Var_0[Y_1^F] = (4\sigma_0^2 + 4\sigma_u^2 + 2\sigma_e^2), \\ S_{1F}^2 &= Var_1[Y_2^F|y_{1,1}, y_{2,1}] = 4\sigma_{1F}^2 + 4\sigma_u^2 + 2\sigma_e^2, \\ d &= \frac{\tau_0 + 2\tau_{u+e}}{\tau_0 + \tau_{u+e}} \quad \text{and} \quad f = \frac{\tau_0 + 4\tau_{2u+e}}{\tau_0 + 2\tau_{2u+e}} \end{aligned}$$

Lemma 1.3 states that expected utility is a function of expected total income and the variance of income. The (ex ante) expected total income from both periods is $4a_0$, from the law of iterated expectations. The variance of total income (i.e., aggregate risk) is split into two terms, S_{0D}^2 and S_{1D}^2, denoting the aggregate risk in periods 1 and 2 under diversification (S_{0F}^2 and S_{1F}^2 under focus). Under Bayesian learning with normal shocks, the evolution of the variance of beliefs from period to period is deterministic. Hence S_{1D}^2 and S_{1F}^2, which are the conditional variance terms in Lemma 1.2 under diversification and focus, respectively, are independent of the exact realizations of $y_{1,1}$ and $y_{2,1}$.

In Eqs. (13) and (14), d and f are the weights on the first period income in the agent's utility function under diversification and focus respectively, taking into account the first period income's direct effect on utility and its effect on beliefs regarding the second period income. A higher value of d or f implies that utility is more sensitive to first period income because of the information it contains about the second period income. Hence, a higher value of d or f increases the variance component of expected utility. Substituting for τ_{u+e} and τ_{2u+e} [from (7) and (10)] in the expressions for f and d, we get $f > d$. This is because faster learning under focus also implies that the effect of first period income on expected second period income is greater under focus.

Equations (13) and (14) give the expected utility under the diversification and focus strategies. Comparing the two expressions, we note that

$$E_0[U^F] \geq E_0[U^D] \quad \text{if and only if} \quad S_{1F}^2 + 0.5f^2 S_{0F}^2 \leq S_{1D}^2 + 0.5d^2 S_{0D}^2.$$

From (40) and (45) in the Appendix (in the proof of Lemma 1.3),

$$S_{1F}^2 - S_{1D}^2 = \left(\frac{4}{\tau_{1F}} - \frac{2}{\tau_{1D}}\right) + 2\sigma_u^2$$

It is easy to show that

$$\frac{4}{\tau_{1F}} > \frac{2}{\tau_{1D}} \tag{15}$$

Hence, $S_{1F}^2 > S_{1D}^2$. As noted before, from (7) and (10), it also follows that $f > d$. Moreover,

$$S_{0F}^2 = 4\sigma_0^2 + 4\sigma_u^2 + 2\sigma_e^2 \;>\; S_{0D}^2 = 2\sigma_0^2 + 2\sigma_u^2 + 2\sigma_e^2.$$

The above analysis leads to Proposition 1.

Proposition 1 *For the technology in* (1)–(2) *and the utility function in* (11)*, the expected utility under focus* $E_0[U^F]$ *is less than the expected utility under diversification* $E_0[U^D]$.

Proposition 1 follows from the linearity of the technology and the normality of the shocks. Under the assumption that the agent's period utility function is exponential, the expected lifetime utility is a function of the expectation and variance of income in each period. By the law of iterated expectations, expected income is identical under focus and diversification (ex ante). But the variance of total income in each period (given linearity and normality) is higher under focus, despite the more rapid decline in uncertainty caused by faster learning. This occurs because of the greater covariance of risks under focus, even in the absence of type-specific risk.[3] This is highlighted by inequality (15). Hence, focus is always inferior to diversification for this technology and learning process.

The linearity and normality assumptions for the technology imply that the ex ante distribution of aggregate lifetime income, $F_0(Y_1 + Y_2)$, under focus is a mean preserving spread of the distribution under diversification. Hence, it is clear that Proposition 1 is likely to hold for a wider class of utility functions. The following proposition generalizes Proposition 1.

Proposition 2 *For the technology in* (1)–(2) *and any utility function of the form* $u(c_1, c_2) = u(c_1) + u(c_2)$ *where* $u(.)$ *is increasing and strictly concave, the expected utility under focus is less than the expected utility under diversification.*

It is tempting to speculate that extending the model to more than two periods may change the results in Propositions 1 and 2, since this would allow the agent more time to learn the technology and thereby increase the benefits to learning. However, this reasoning is incorrect. Extending the time horizon increases learning under both focus and diversification, so that the net effect is not immediately obvious. However, in our Bayesian learning model, the greatest decline in uncertainty occurs in initial periods, with learning becoming progressively slower in later periods. Thus, it is in the initial periods that focus has a relative advantage

[3] The higher covariance under focus is the result of having a common prior for projects of the same type.

over diversification. Since focus is inferior to diversification even after just one period, it will certainly be inferior in subsequent periods.

In the model presented above, learning is passive—it reduces the uncertainty regarding the technological parameter, but has no effect on the productivity in the subsequent period. In the next section, we present a model where the technology is non-linear and learning is more "active". In this case, diversification may be inferior to learning.

4 Risk in a model of active learning

The target input model has been widely used to study learning-by-doing in economics. In this model, output from each project depends on a target level of input, which varies from period to period about a long run mean. Specifically,

$$\begin{aligned} z_{ij,t} &= I\left[1 - \left(y_{ij,t} - a_{ij,t}\right)^2\right], \text{ where,} \\ y_{ij,t} &= a_i + u_{i,t} + e_{ij,t}. \end{aligned} \tag{16}$$

$y_{ij,t}$ is the optimal or target input level for project $j \in \{1, 2\}$ of type $i \in \{P, R\}$ in period t. Output $z_{ij,t}$ is higher when the deviation between actual input $a_{ij,t}$ and the target input $y_{ij,t}$ is lower. I is a known parameter that denotes maximum output in any period. It also affects the sensitivity of output to errors in input choice. Following the literature, we assume that inputs are costless, so that output equals profits.

Equation (16) is analogous to (1), except that it pertains to the target input and not the output. The target input consists of three components: (i) a long run component that is stable over time, denoted by a_i; (ii) $u_{i,t}$, which is a type-specific shock that affects all projects of type i in period t; and (iii) $e_{ij,t}$, which is an idiosyncratic shock that varies from project to project. An agent does not know a_i, but begins with normal priors and updates her beliefs at the end of each period. Hence, the agent gets an increasingly precise estimate of the target input over time. These smaller errors in input choice result in increasing productivity. Thus, learning is "active", in that it affects the agent's actions and productivity over time.

We retain the distributional assumptions of the previous section [Eqs. (2) and (4)] regarding the period shocks. Hence the process of Bayesian updating is given by (5)–(10).[4] As in the previous section, we assume that there are two types of projects, P and R, and that the agent's utility function is given by (11).

4.1 The agent's problem

The agent chooses focus or diversification based on the lifetime expected utility from each at time $t = 0$. We assume that the period utility function $u(.)$ is strictly concave. As earlier, we also assume free transferability of consumption across periods. The expected utility in this case depends not only on the consumption-savings

[4] In the previous section, an agent could observe $y_{ij,t}$ directly. With the target input model, an agent can infer $y_{ij,t}$ from $z_{ij,t}$.

decision, but also on the choice of inputs in each period. Hence, given focus or diversification, the agent's problem is:

$$\max_{\tilde{a}_{1,1},\tilde{a}_{2,1}} E_0\left[\max_{c_1,\tilde{a}_{1,2},\tilde{a}_{2,2}} u(c_1) + E_1\left[u(c_2)|y_{1,1}, y_{2,1}\right]\right] \quad \text{s.t.} \quad w_2 = w_1 - c_1 + z_2 \tag{17}$$

where $z_t = z_{1,t} + z_{2,t}$ is the aggregate income in period t, w_t is the accumulated wealth prior to consumption at the end of period t with $w_0 = 0$, and $\tilde{a}_{i,t}$ is the input level for project i in period t. Note that while the target input $y_{ij,t}$ has a normal distribution, total income in any period involves the sum of two squared normal variates which are correlated under focus.

4.2 Optimal input choice

When the agent is risk-neutral, it is straightforward to show that the optimal input level in each period is equal to the expected target input. But when the agent is risk averse, it is not obvious that this is the case. This is especially so since income in each period contains information regarding the distribution of incomes in subsequent periods. The problem is particularly complex under focus because the project cash flows are correlated in that case.

We work backwards to solve this problem, beginning with the optimal input choice in the second period conditional on first period income.[5] Consumption in this period is equal to the sum of output and accumulated savings. Incorporating this into the utility function, we solve for the optimal input level in the second period. Then, moving to the first period, consumption is determined by equating expected marginal utilities in the two periods. Combining this first order condition with the condition for optimal input choice leads to a surprisingly simple solution. The following lemma gives the solution for a more general multi-period version of the optimization problem in (17). Let $\hat{y}_{t-1}$ denote the sequence of outcomes until period $(t-1)$, $\{(y_{1,1}, y_{2,1}), (y_{1,2}, y_{2,2}), \ldots, (y_{1,t-1}, y_{2,t-1})\}$.

Lemma 3.1 *The optimal input choice in period t for an agent maximizing expected utility as given in* (17) *under the target input model is*

$$\tilde{a}_t = E_{t-1}[a|\hat{y}_{t-1}]. \tag{18}$$

Lemma 3.1 states that there is no distortion in input choice from the risk-neutral level for any strictly concave utility function. It substantially simplifies the calculation of lifetime expected utility under focus and diversification.

4.3 Optimal consumption-savings choice

In the following analysis, we will assume that the agent's period utility function $u(.)$ is exponential, as given in (11). Applying Lemma 3.1, we find that the expected

[5] The problem is somewhat simplified by the absence of capital from the production technology, though Wieland (2000) demonstrates with a utility function which is linear in the squared input error that it can still be quite complex.

utility of second period income, $E_1[e^{-\gamma z_2}|y_{1,1}, y_{2,1}]$, depends on the prior period outcomes only through their impact on the conditional variance of beliefs, σ_{1F}^2 or σ_{1D}^2. However, under the Bayesian learning process with normal shocks, the evolution of the conditional variance of beliefs is deterministic and independent of past outcomes. Hence, expected utility of second period income is independent of first period outcomes. This substantially simplifies the consumption-savings problem at the end of the first period. We solve for the first period consumption function and incorporate that into (17) to get the expected first period utility. This analysis leads to the following proposition.

Proposition 3 *For the target input model, the expected utilities of the agent under focus and diversification are given by the following expressions*:

$$U^F = \frac{-2e^{-2\gamma I}}{F_1 F_2} \tag{19}$$

$$U^D = \frac{-2e^{-2\gamma I}}{D_1 D_2} \tag{20}$$

where F_1, F_2, D_1 and D_2 are defined below:

$$F_1 = (1 - \gamma I \sigma_e^2)^{1/2}(1 - \gamma I(\sigma_e^2 + 2\sigma_u^2 + 2\sigma_0^2))^{1/2} \tag{21}$$

$$F_2 = (1 - 2\gamma I \sigma_e^2)^{1/4}(1 - 2\gamma I(\sigma_e^2 + 2\sigma_u^2 + 2\sigma_{1F}^2))^{1/4} \tag{22}$$

$$D_1 = (1 - \gamma I(\sigma_e^2 + \sigma_u^2 + \sigma_0^2)) \tag{23}$$

$$D_2 = (1 - 2\gamma I(\sigma_e^2 + \sigma_u^2 + \sigma_{1D}^2))^{1/2} \tag{24}$$

From Proposition 3, it is clear that the comparison between focus and diversification hinges on the terms in the denominators of (19) and (20). Of these, the terms F_1 and D_1 pertain to the first period, and F_2 and D_2 pertain to the second period. Before analyzing these further, it is useful to recall that learning affects the focus-diversification trade-off through its effect on the variance of beliefs in the second period. Accordingly, F_2 is a function of σ_{1F}^2, while D_2 is a function of σ_{1D}^2. Recall from (7) and (10) that $\sigma_{1F}^2 < \sigma_{1D}^2$.

4.4 The focus-diversification choice

In Sect. 3, we found that even though $\sigma_{1F}^2 < \sigma_{1D}^2$, the fact that $2\sigma_{1F}^2 > \sigma_{1D}^2$ implied that the benefits of faster learning were outweighed by the costs of risk under focus. With the target input model however, the variance terms enter the expected utility function in a more complex manner, resulting in a slightly different trade-off between risk and learning. Therefore, it is possible for focus to be optimal under certain conditions. The following proposition is a formal statement of this existence result.

Proposition 4 *When the agent's utility function is given by* (11) *and the technology is given by* (16), *there exists a set of positive parameters* $\{\gamma, I, \sigma_0^2, \sigma_u^2, \sigma_e^2\}$ *such that the expected utility under focus* [*given in* (19)] *is greater than that under diversification* [*given in* (20)].

Proposition 4 contrasts with Proposition 1 which ruled out the possibility that focus might be optimal. The difference between the two models is in the technology. Since our purpose here is to highlight the trade-off between learning and risk, we will not dwell on the specific technological differences but rather, concentrate on the target input model and study the role of the nature of risk in the choice between focus and diversification.

We begin by noting that since U^F and U^D are negative, $U^F > U^D$ if $F_1 F_2 > D_1 D_2$. Therefore, we will examine how changes in σ_0^2, σ_e^2, and σ_u^2 affect the $\frac{F_1 F_2}{D_1 D_2}$ ratio.

To simplify notation, let $\gamma I \sigma_0^2 = g$, $\frac{\sigma_e^2}{\sigma_0^2} = x$, and $\frac{\sigma_u^2}{\sigma_0^2} = h$. Then, $\left(\frac{F_1}{D_1}\right)$ and $\left(\frac{F_2}{D_2}\right)$ can be written as:

$$\frac{F_1}{D_1} = \frac{(1-gx)^{\frac{1}{2}}(1-g(x+2+2h))^{\frac{1}{2}}}{(1-g(x+1+h))} \tag{25}$$

$$\frac{F_2}{D_2} = \frac{(1-2gx)^{\frac{1}{4}}(1-2g(x+2h+\frac{2(x+2h)}{(2+x+2h)}))^{\frac{1}{4}}}{(1-2g(x+h+\frac{(x+h)}{(1+x+h)}))^{\frac{1}{2}}} \tag{26}$$

$\left(\frac{F_1}{D_1}\right)$ captures the utility of focus relative to diversification in the first period. The term can be rewritten as

$$\frac{F_1}{D_1} = \left[\frac{(1-gx-g(1+h))^2 - g^2(1+h)^2}{(1-gx-g(1+h))^2}\right]^{\frac{1}{2}}$$

Clearly, $\left(\frac{F_1}{D_1}\right) < 1$, highlighting the fact that focus is always inferior in the first period. Therefore, focus will be better than diversification overall only if the second period ratio $\left(\frac{F_2}{D_2}\right)$ is sufficiently greater than 1. From the expression for $\left(\frac{F_2}{D_2}\right)$, we note that the role of learning is reflected in the relative sizes of the variances of beliefs in the second period. These are given by

$$\frac{\sigma_{1F}^2}{\sigma_0^2} = \frac{(x+2h)}{(2+x+2h)} \quad \text{and} \quad \frac{\sigma_{1D}^2}{\sigma_0^2} = \frac{(x+h)}{(1+x+h)}. \tag{27}$$

4.4.1 Prior uncertainty and idiosyncratic risk

We study the effect of prior uncertainty and idiosyncratic risk on the focus-diversification choice with the help of three examples, which will also highlight the importance of the speed of learning in the choice. We abstract from type-specific risk by assuming that $h = 0$.

Example 1 Let $x = \varepsilon$, a very small number. Then, $\frac{x+2h}{2+x+2h} \approx \varepsilon \approx \frac{(x+h)}{1+x+h}$. Hence, for $g \leq \frac{1}{2}$ (so that F_1 is real), we have $g\varepsilon \approx 0$ and

$$\frac{F_1 F_2}{D_1 D_2} \approx \frac{(1-2g)^{\frac{1}{2}}}{(1-g)} < 1.$$

Thus, focus is inferior to diversification in this case. This is because learning is too fast when the noise level is very small. Hence the agent is able to learn the technology quickly even under diversification, and finds it unnecessary to focus in order to speed up learning.

Example 2 Next, consider the opposite case in which x is very large. Then, $\frac{x+2h}{2+x+2h} \approx 1 \approx \frac{(x+h)}{1+x+h}$. Hence,

$$\frac{F_1 F_2}{D_1 D_2} \approx \left[\frac{(1-gx-g)^2 - g^2}{(1-gx-g)^2}\right]^{\frac{1}{2}} \left[\frac{(1-2gx-2g)^2 - 4g^2}{(1-2gx-2g)^2}\right]^{\frac{1}{4}} < 1.$$

Again, diversification is better than focus, albeit for a different underlying reason. In contrast to Example 1, learning is too slow in this case. Thus, there is hardly any change in the variance of beliefs after one period. Given this, the agent is better off ignoring learning altogether and investing in different types of projects.

The above two examples highlight the fact that for focus to be optimal, the speed of learning should be such that the difference between the variance of beliefs in the second period under the two strategies (σ_{1F}^2 and σ_{1D}^2) should be significant. When learning is very fast or very slow, σ_{1D}^2 is too close to σ_{1F}^2, and the trade-off between focus and diversification is dominated by the first period terms F_1 and D_1 with $F_1 < D_1$. Next we will consider a case where the speed of learning is intermediate, and examine the conditions under which focus is optimal.

Example 3 Let $x = 1$. Then, from (27), $\sigma_{1D}^2 = \frac{\sigma_0^2}{2}$ and $\sigma_{1F}^2 = \frac{\sigma_0^2}{3}$. Hence,

$$\frac{F_1 F_2}{D_1 D_2} = \frac{(1-g)^{\frac{1}{2}}(1-3g)^{\frac{1}{2}}}{(1-2g)} \frac{(1-2g)^{\frac{1}{4}}(1-\frac{10}{3}g)^{\frac{1}{4}}}{(1-3g)^{\frac{1}{2}}} = \frac{(1-g)^{\frac{1}{2}}(1-\frac{10}{3}g)^{\frac{1}{4}}}{(1-2g)^{\frac{3}{4}}}$$

Simplifying we get,

$$\frac{F_1 F_2}{D_1 D_2} > 1 \Leftrightarrow \left(\frac{F_1 F_2}{D_1 D_2}\right)^4 > 1 \quad \Leftrightarrow \quad g(14g^2 - 13g + 2) > 0.$$

Solving the quadratic equation, we find that $\frac{F_1 F_2}{D_1 D_2} > 1$ if $g \in \left(0, \frac{13-\sqrt{57}}{28} \approx 0.1946\right)$. Thus, focus becomes the preferred strategy when γI is sufficiently small and σ_0^2 is of comparable magnitude to σ_e^2 so that learning is neither too fast nor too slow.

This example also highlights the marginal effect of the prior variance of beliefs on the focus-diversification choice. Differentiating $\left(\frac{F_1 F_2}{D_1 D_2}\right)^4$ with respect to g, we get

$$\frac{d\left(\frac{F_1 F_2}{D_1 D_2}\right)^4}{dg} = \frac{2(1-g)(1-8g)}{3(1-2g)^4},$$

which is positive if $g \in (0, 0.125)$. Thus, for $g < 0.125$, focus is superior to diversification and becomes increasingly preferred as the prior variance, σ_0^2, increases.[6] This occurs because as the prior variance increases, there is more to be learnt. Hence, it pays to focus in order to learn the technology faster. Thus, under certain conditions, an increase in the prior uncertainty regarding the technology leads to an increase in the expected utility under focus relative to that under diversification, i.e., in $U^F - U^D$.

Finally, comparing Example 1 to Example 3, we see that an increase in idiosyncratic risk (σ_e^2) may also lead to focus becoming more dominant relative to diversification. This is because learning slows down as σ_e^2 increases, so that it enters the moderate range necessary for focus to be superior.

The positive effects of prior variance and idiosyncratic risk on the relative attractiveness of focus are, however, limited. Each of these two aspects of risk affects the focus-diversification choice in two ways, through its impact on the speed of learning and through its impact on aggregate risk. While the benefits of learning are limited, the costs of risk are not. Hence, there is a concave relationship between these two types of risks and $U^F - U^D$. This is shown in Fig. 1, in which $U^F - U^D$ is graphed against the variance of prior beliefs (σ_0^2) and idiosyncratic risk (σ_e^2). The values of the other parameters used in this numerical simulation are: $\sigma_u^2 = 0.8; \gamma = 1,$ and $I = 0.025$. The figure illustrates the concave relationship between σ_0^2 and $U^F - U^D$ for a fixed value of σ_e^2, and between σ_e^2 and $U^F - U^D$ for fixed σ_0^2.[7]

4.4.2 Type-specific risk

In order to examine the effect of type-specific risk, we rewrite $\frac{F_1}{D_1}$ and $\frac{F_2}{D_2}$ as follows:

$$\frac{F_1}{D_1} = \left(1 - \frac{g(1+h)}{1 - g(1+x+h)}\right)^{1/2} \left(1 - \frac{g(1+h)}{1-gx}\right)^{-1/2}$$

$$\frac{F_2}{D_2} = \left(1 - \frac{2g\left(h + \frac{x+h}{1+x+h} - \frac{2x}{(2+x+2h)(1+x+h)}\right)}{1 - 2g\left(x + h + \frac{x+h}{1+x+h}\right)}\right)^{1/4} \left(1 - \frac{2g\left(h + \frac{x+h}{1+x+h}\right)}{1-2gx}\right)^{-1/4}$$

From the above expressions, we can easily see that for a fixed value of g and x, an increase in h leads to a decrease in both $\frac{F_1}{D_1}$ and $\frac{F_2}{D_2}$. The effect of an increase in type-specific risk is thus unambiguous – it makes focus less attractive relative to diversification. This is in contrast to idiosyncratic risk, which may have a positive effect on $U^F - U^D$. The reason for this is that unlike idiosyncratic risk, an increase in type-specific risk slows down learning under focus to a greater extent than under diversification. This is easily verifiable from (27). In addition, an increase in type-specific risk also increases aggregate risk in each period by more under focus than under diversification. The combined effect therefore is to reduce $U^F - U^D$.

[6] Since $g = \gamma I \sigma_0^2$, a change in g could be due to a change in γ, I or σ_0^2. However, since γ and I also enter the numerator of the expected utility expressions, the same argument does not apply to those two parameters.

[7] The regions with a negative relationship between idiosyncratic risk and $U^F - U^D$ are more clearly seen in Fig. 3.

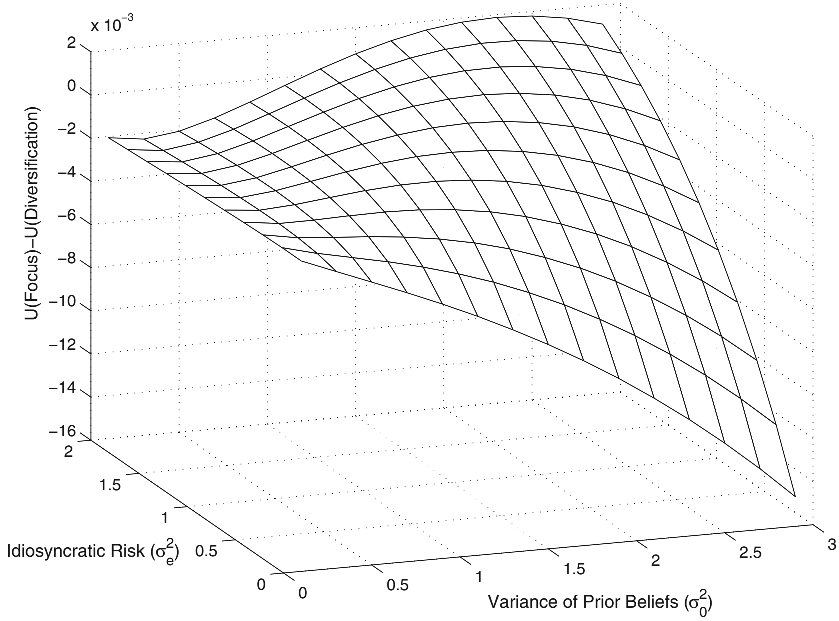

Fig. 1 Impact of variance of prior beliefs and idiosyncratic risk with constant type-specific risk. U(Focus) and U(Diversification) are the discounted lifetime utilities under focus and diversification, respectively

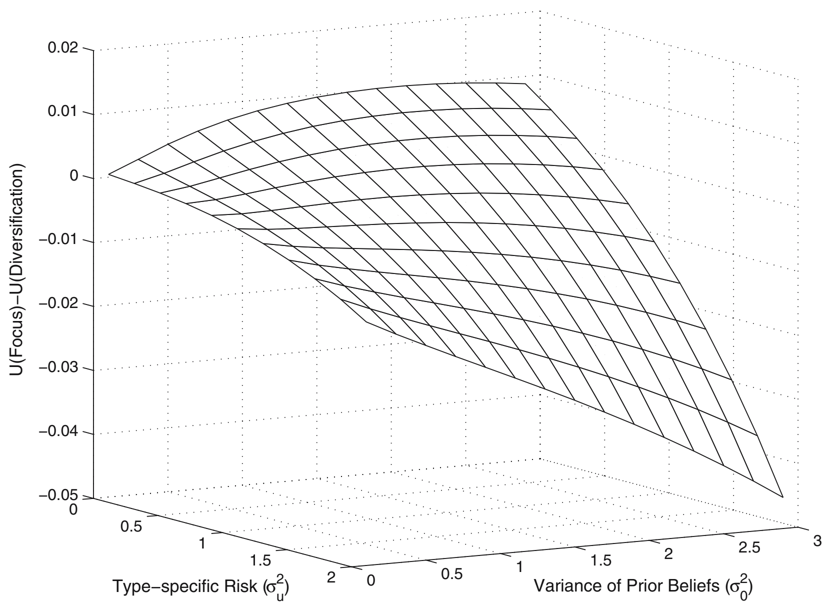

Fig. 2 Impact of variance of prior beliefs and type-specific risk with constant idiosyncratic risk. U(Focus) and U(Diversification) are the discounted lifetime utilities under focus and diversification, respectively

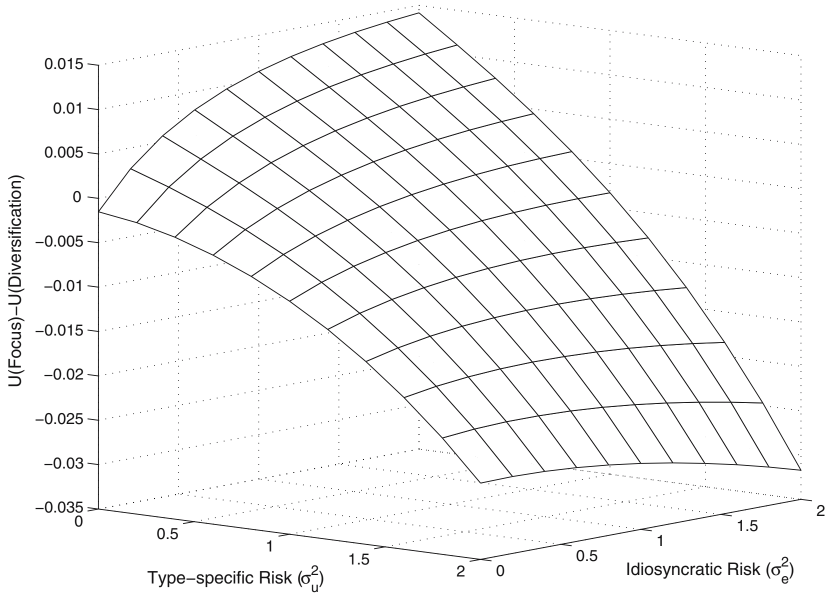

Fig. 3 Impact of idiosyncratic and type-specific risk with constant variance of prior beliefs. U(Focus) and U(Diversification) are the discounted lifetime utilities under focus and diversification, respectively

Figures 2 and 3 present surface plots of $U^F - U^D$ against $\left(\sigma_u^2, \sigma_e^2\right)$ and $\left(\sigma_u^2, \sigma_0^2\right)$, respectively. The figures illustrate the negative effect of type-specific risk on focus. Figure 4 presents the comparative statics for the three elements of risk in the form of univariate graphs of $U^F - U^D$ against each of them. Comparing the solid line with any of the other two lines in each panel of Fig. 4 enables us to study the second order effects of each type of risk.[8] For example, comparing the solid line in panel 1 to the dashed line, we see that an increase in type-specific risk with constant idiosyncratic risk not only decreases $U^F - U^D$ uniformly, but also reduces the slope of $U^F - U^D$ with respect to σ_0^2. The same effect is observed with respect to σ_e^2 in panel 3. These comparisons show that the first and second order effects of type-specific risk on $U^F - U^D$ are both negative. While we do not present an analytical proof of this hypothesis here, we note from the numerical computations above that the negative first and second order effects of type-specific risk, together with the limited positive effects of σ_0^2 and σ_e^2 on $U^F - U^D$, suggest that there is a level of type-specific risk (for each value of γI) beyond which focus can never be better than diversification.

To summarize, focus is optimal when the type-specific risk is low relative to the prior variance of beliefs, but not so low that learning is substantially complete within one period. For a given speed of learning, a shift in risk from idiosyncratic to type-specific risk leads to focus becoming less attractive relative to diversification.

[8] By first order effect, we refer to $\frac{d(U^F - U^D)}{d\sigma_k^2}$ where $k \in \{0, u, e\}$. By second order effect, we refer to $\frac{d^2(U^F - U^D)}{d\sigma_k^2 d\sigma_j^2}$ where $k, j \in \{0, u, e\},\ k \neq j$.

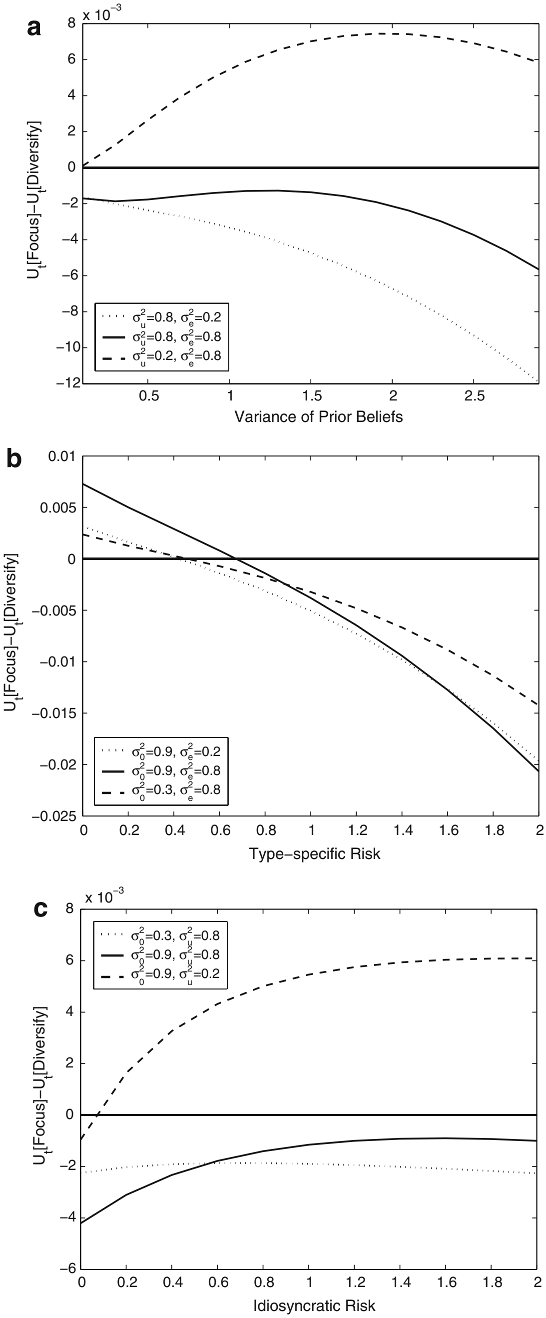

Fig. 4 Effect of learning and risk on the focus-diversification choice. U(Focus) and U(Diversification) are the discounted lifetime utilities under focus and diversification, respectively. Comparison of the *solid line* with one of *broken lines* in each panel provides an understanding of second order effects. For example, a comparison the *solid line* with the *dashed line* in the *top-most panel* shows that an increase in type-specific risk decreases the marginal impact of variance of prior beliefs on the relative benefits to focus

Finally, it may be noted that the dependence of the focus-diversification choice on the speed of learning also implies that the agent's investment horizon should neither be too long nor too short for focus to be optimal.

4.5 Capital market imperfections

We have assumed so far that the agent is able to costlessly borrow and lend, so that consumption can be freely transferred from one period to the other. This may not be feasible in certain economic contexts. For example, poor rural households in developing countries have limited access to consumption loans. In this case, focus becomes even more risky since it trades off greater risk in initial periods in return for lower risk in later periods. It is therefore necessary to examine whether it is still possible for focus to be better than diversification with capital market imperfections. To do so, we assume that the agent is unable to transfer consumption across periods. Hence, consumption equals income in each period.

It is relatively easy to show that the optimal input choice is unaffected by the inability to transfer consumption across periods. Hence, following an argument similar to that presented in the proof of Lemma 3.1 and Proposition 3, it can be shown that the agent's two-period utility in this case is

$$E_0[U^F] = -2e^{-2\gamma I}\left(\frac{1}{F_1'} + \frac{1}{F_2^2}\right) \tag{28}$$

$$E_0[U^F] = -2e^{-2\gamma I}\left(\frac{1}{D_1'} + \frac{1}{D_2^2}\right) \tag{29}$$

where F_2 and D_2 are defined in Proposition 3, and F_1' and D_1' are defined below:

$$F_1' = (1 - 2\gamma I\sigma_e^2)^{1/2}(1 - 2\gamma I(\sigma_e^2 + 2\sigma_u^2 + 2\sigma_0^2))^{1/2} \tag{30}$$

$$D_1' = (1 - 2\gamma I(\sigma_e^2 + \sigma_u^2 + \sigma_0^2)). \tag{31}$$

Applying the logic of the previous section, it is possible to show that focus may be better than diversification and that prior uncertainty may have a positive effect on focus in this case also. The proofs are omitted to conserve space. We use Fig. 5 to illustrate the point. Figure 5 is the analogue to Fig. 3 with no borrowing or lending. Figures 5 and 3 are seen to be quite similar, except that $U^F - U^D$ is uniformly lower in Fig. 5 due to the constrained nature of the optimization. This is to be expected, since focus trades off higher risk in the first period for lower second period risk. Hence any constraint on intertemporal transfer of consumption is likely to affect utility under focus to a greater extent than under diversification.

5 Conclusion

We have studied the issue of project choice when a risk-averse agent must choose whether to invest in two projects of the same type (*focus*) or of different types (*diversification*). Investing in projects of the same type is more risky within each

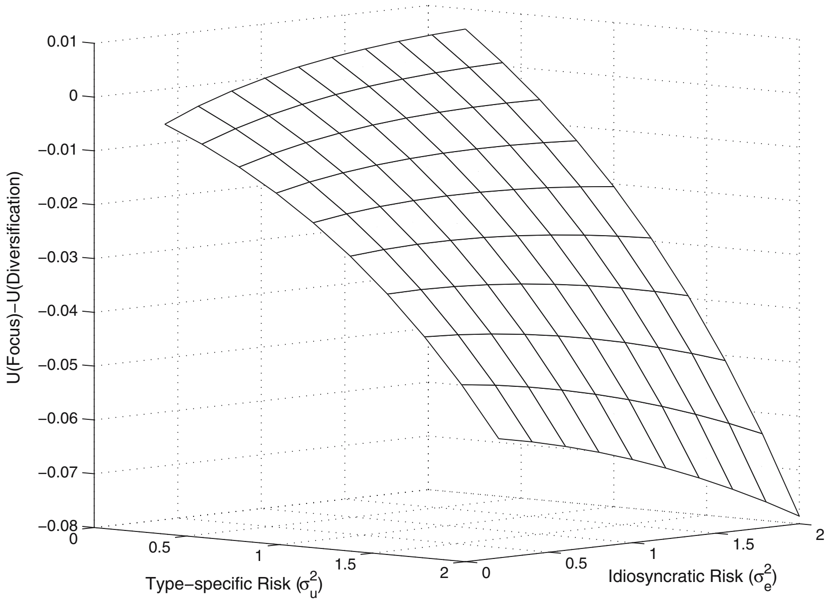

Fig. 5 Impact of idiosyncratic and type-specific risk with constant variance of prior beliefs. Consumption is constrained to equal income in each period. U(Focus) and U(Diversification) are the discounted lifetime utilities under focus and diversification, respectively

period, but enables faster learning across periods. Optimal project choice involves balancing these two considerations. We study Bayesian learning-by-doing with normal shocks in two models. In the first model, the technology is linear in an unknown parameter, and diversification is always better than learning, even without type-specific risk. Next, we consider the target input model and show that focus may be preferred to diversification for intermediate learning speeds. We show that, contrary to intuition, an increase in the prior uncertainty regarding the technology may lead to a decrease in diversification, even when the agent is risk-averse. We also show that the effect of type-specific risk is always negative. Both the prior variance and the type-specific risk are aspects of technology-specific uncertainty. But an increase in the first might lead to greater focus while an increase in the second will always lead to lesser focus. Thus, what matters for the focus-diversification choice is not only the level of risk, but also the nature of risk, i.e., whether it is permanent or it can be reduced through learning.

The trade-off between learning and insurance motives is likely to occur in several economic settings. As mentioned in the introduction, occupational choice within households in less developed countries which are subject to large weather shocks is likely to take into account the potential benefits of diversifying across trades. Similarly, when members of the same group choose projects under group-lending, they may trade off risk-sharing against learning. It has also been pointed out to us by a referee that a modification of the target-input model to the case of a monopolist with uncertain production and facing a linear demand function could contribute to the literature on experimentation, for example, by multinational companies.

A corporate manager who holds a lot of firm-specific wealth may also face a similar trade-off. While diversification across industries would reduce the industry-specific that he faces, this may impose a cost in terms of slower accumulation of skills and knowledge. Prior work on corporate diversification has tended to treat insurance and learning aspects separately. For example, Amihud and Lev (1981) argue that risk-reduction is the motive behind conglomerate mergers. However, as Jovanovich (1993) notes, firms "tend to diversify into technologically related industries, thereby exposing themselves to common technological shocks and hence more risk."[9] We find that when both motives are taken into account, it is not just the *level* of risk, but also the *nature* of risk that is important for the diversification decision. If the risk is basic technological uncertainty that can be "learnt away", then greater risk might actually imply focus rather than diversification. Thus, we might expect to see greater focus in industries on the technological frontier and greater diversification in the more established cyclical industries.

In prior work, learning and the associated risks have usually been dealt with separately. The analysis on the learning side has mainly tackled issues such as when to switch to a new technology based on relative profitability, and the analysis on the risk side has concentrated on risk-shifting behavior (which occurs when risk-averse managers do not invest in some positive NPV projects). This paper combines both the costs and benefits of learning in a unified model with risk-averse agents, and thereby contributes to the literature.

Appendix

Proof of Lemma 1.2

First consider the problem of the agent at the end of the first period, when the first period income Y_1 is known, but the second period income is unknown. The maximand of the agent's expected utility maximization problem at this point is

$$\begin{aligned} \mathrm{E}_1[\mathrm{U}] &= -\exp(-\gamma c_1) - E_1\left[\exp(-\gamma\{Y_2 + Y_1 - c_1\}|y_{1,1}, y_{2,1}\right] \\ &= -\exp(-\gamma c_1) - \exp(-\gamma\{Y_1 - c_1\}) * E_1\left[\exp(-\gamma Y_2)|y_{1,1}, y_{2,1}\right] \end{aligned} \tag{32}$$

It is easily seen from (1), (5), and (8) that the distribution of Y_2 conditional on $\{y_{1,1}, y_{2,1}\}$ is normal. Further, for a random variable z that is normally distributed, the following result can also be easily derived:

$$E\left[-\exp(-\gamma z)\right] = -\exp\left(-\gamma\{E[z] - 0.5\gamma \mathrm{Var}[z]\}\right) \tag{33}$$

Applying these results to (32), we have

$$\begin{aligned} \mathrm{E}_1[\mathrm{U}] = -\exp(-\gamma c_1) - \exp(-\gamma\{&Y_1 - c_1 + E_1[Y_2|y_{1,1}, y_{2,1}] \\ &-0.5\gamma \mathrm{Var}[Y_2|y_{1,1}, y_{2,1}]\}) \end{aligned} \tag{34}$$

The first order condition for maximizing $\mathrm{E}_1[\mathrm{U}]$ is seen to be

$$\begin{aligned} \gamma\exp(-\gamma c_1) - \gamma\exp(-\gamma\{&Y_1 - c_1 + E_1[Y_2|y_{1,1}, y_{2,1}] \\ &-0.5\gamma \mathrm{Var}[Y_2|y_{1,1}, y_{2,1}]\}) = 0, \end{aligned}$$

[9] Jovanovich (1993), pp. 203–204.

which simplifies to

$$c_1 = 0.5 * \left(Y_1 + E_1[Y_2|y_{1,1}, y_{2,1}] - 0.5\gamma \mathrm{Var}[Y_2|y_{1,1}, y_{2,1}]\right) \quad (35)$$

Substituting for c_1 in (34) gives Lemma 1.2.

Proof of Lemma 1.3

We deal with diversification first and focus next.

Diversification

When the two projects are of different types, then their outputs in each period are independent. Therefore, assuming without loss of generality that project 1 is of type P and project 2 of type R,

$$y_{1,2} + y_{2,2} = a_P + u_{P2} + e_{1,2} + a_R + u_{R2} + e_{2,2} \quad (36)$$

From Lemma 1.1 and Eqs. (8)–(10),

$$\begin{aligned} E_1[y_{1,2} + y_{2,2}|y_{1,1}, y_{2,1}] &= E_1[a_P|y_{1,1}] + E_1[a_R|y_{2,1}] \\ &= 2a_0\left(\frac{\tau_0}{\tau_{1D}}\right) + (y_{1,1} + y_{2,1})\left(\frac{\tau_{u+e}}{\tau_{1D}}\right) \end{aligned} \quad (37)$$

Hence,

$$y_{1,1}+y_{2,1}+E_1[y_{1,2}+y_{2,2}|y_{1,1}, y_{2,1}] = 2a_0\left(\frac{\tau_0}{\tau_{1D}}\right)+(y_{1,1}+y_{2,1})\left(\frac{\tau_{1D} + \tau_{u+e}}{\tau_{1D}}\right) \quad (38)$$

Let $d = \frac{\tau_{1D}+\tau_{u+e}}{\tau_{1D}} = \frac{\tau_0+2\tau_{u+e}}{\tau_0+\tau_{u+e}}$. Then, we have

$$y_{1,1} + y_{2,1} + E_1[y_{1,2} + y_{2,2}|y_{1,1}, y_{2,1}] = 2a_0(2 - d) + d(y_{1,1} + y_{2,1}) \quad (39)$$

Since the types are independent and $Var_1[a_P|y_{1,1}] = Var_1[a_R|y_{2,1}] = \tau_1^{-1}$,

$$\begin{aligned} S_{1D}^2 &= Var_1[y_{1,2} + y_{2,2}|y_{1,1}, y_{2,1}] = Var_1[y_{1,2}|y_{1,1}] + Var_1[y_{2,2}|y_{2,1}] \\ &= 2\tau_{1D}^{-1} + 2\sigma_u^2+2\sigma_e^2=2\tau_{1D}^{-1}+2\tau_{u+e}^{-1} \end{aligned} \quad (40)$$

Substituting (39) and (40) in the maximand in (12),

$$\begin{aligned} E_0[U^D] = -2 * \exp\left(-0.5\gamma\{2a_0(2 - d) - 0.5\gamma S_{1D}^2\}\right) \\ *E_0\left[\exp\left(-0.5\gamma d(y_{1,1} + y_{2,1})\right)\right] \end{aligned} \quad (41)$$

Since $y_{1,1}$ and $y_{2,1}$ are normally distributed and independent, it follows that

$$\begin{aligned} &E_0\left[\exp\left(-0.5\gamma(y_{1,1}+y_{2,1})d\right)\right] \\ &\quad= \exp\left(-0.5\gamma d\{E_0[y_{1,1}+y_{2,1}] - 0.25\gamma d Var_0[y_{1,1}+y_{2,1}]\}\right) \\ &\quad= \exp\left(-0.5\gamma d\{2a_0 - 0.25\gamma d(2\sigma_0^2 + 2\sigma_u^2 + 2\sigma_e^2)\}\right) \end{aligned} \tag{42}$$

Let $S_{0D}^2 = Var_0[y_{1,1}+y_{2,1}] = (2\sigma_0^2 + 2\sigma_u^2 + 2\sigma_e^2)$. From (41) and (42), we have (13) of Lemma 1.3.

Focus

In this case, the output from the two projects in each period are correlated. The agent starts with the same prior regarding the project quality of either project. After the first period, she updates her beliefs twice based on the output from each project and begins the next period with the same updated prior for both projects. Therefore, suppressing type subscripts, we have

$$y_{1,2} + y_{2,2} = a + u + e_1 + a + u + e_2 = 2a + 2u + e_1 + e_2 \tag{43}$$

Hence, applying Lemma 1.1 to Eqs. (5)–(7), we have

$$\begin{aligned} E_1[y_{1,2}+y_{2,2}|y_{1,1},y_{2,1}] &= 2 * E_1[a|y_{1,1},y_{2,1}] \\ &= 2a_0\left(\frac{\tau_0}{\tau_{1F}}\right) + 2(y_{1,1}+y_{2,1})\left(\frac{\tau_{2u+e}}{\tau_{1F}}\right), \end{aligned} \tag{44}$$

and

$$\begin{aligned} S_{1F}^2 = Var_1[y_{1,2}+y_{2,2}|y_{1,1},y_{2,1}] &= 4 * Var_1[a|y_{1,1},y_{2,1}] + 4\sigma_u^2 + 2\sigma_e^2 \\ &= 4\tau_{1F}^{-1} + 4\sigma_u^2 + 2\sigma_e^2 \\ &= 4\tau_{1F}^{-1} + 2\tau_{2u+e}^{-1} \end{aligned} \tag{45}$$

Comparing (37) with (44) and (40) with (45), the impact of faster learning under focus is seen in the greater precision of beliefs, $\tau_{1F} > \tau_{1D}$. However, the sign of $S_{1F}^2 - S_{1D}^2$ is not immediately obvious, and the net effect benefit of focus over diversification is, as yet, ambiguous.

From (44),

$$\begin{aligned} &y_{1,1} + y_{2,1} + E_1[y_{1,2}+y_{2,2}|y_{1,1},y_{2,1}] \\ &\quad= 2a_0\left(\frac{\tau_0}{\tau_{1F}}\right) + (y_{1,1}+y_{2,1})\left(\frac{\tau_{1F} + 2\tau_{2u+e}}{\tau_{1F}}\right). \end{aligned}$$

Let $f = \frac{\tau_{1F}+2\tau_{2u+e}}{\tau_{1F}} = \frac{\tau_0+4\tau_{2u+e}}{\tau_0+2\tau_{2u+e}}$. Then, we have

$$y_{1,1} + y_{2,1} + E_1[y_{1,2}+y_{2,2}|y_{1,1},y_{2,1}] = 2a_0(2-f) + f(y_{1,1}+y_{2,1}). \tag{46}$$

Substituting (45) and (46) in the maximand in (12), we have

$$\begin{aligned} E_0[U^F] = -2 * \exp&\left(-0.5\gamma\{2a_0(2-f) - 0.5\gamma S_{1F}^2\}\right) \\ &* E_0\left[\exp\left(-0.5\gamma f(y_{1,1}+y_{2,1})\right)\right] \end{aligned} \tag{47}$$

Since $(y_{1,1} + y_{2,1})$ is normally distributed, it follows that

$$\begin{aligned}
&E_0\left[\exp\left(-0.5\gamma f(y_{1,1} + y_{2,1})\right)\right] \\
&\quad = \exp\left(-0.5\gamma f\{E_0[y_{1,1} + y_{2,1}] - 0.25\gamma f Var_0[y_{1,1} + y_{2,1}]\}\right) \\
&\quad = \exp\left(-0.5\gamma f\{2a_0 - 0.25\gamma f(4\sigma_0^2 + 4\sigma_u^2 + 2\sigma_e^2)\}\right) \qquad (48)
\end{aligned}$$

Let $S_{0F}^2 = Var_0[y_{1,1} + y_{2,1}] = (4\sigma_0^2 + 4\sigma_u^2 + 2\sigma_e^2)$. From (47) and (48), we have (14) of Lemma 1.3.

Hence the proof.

Proof of Proposition 2

We use the following two results for random variables x_1 and x_2:

$$\begin{aligned}
&\text{If } (x_1, x_2) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho), \text{ then } (x_1 + x_2) \\
&\quad \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2). \qquad (49) \\
&\text{If } x_1 \sim N(\mu_1, \sigma_1^2) \text{ and } x_2|x_1 \sim N(b\mu_1 + (1-b)x_1, \sigma_2^2), \text{ then } (x_1 + x_2) \\
&\quad \sim N(2\mu_1, (2-b)^2\sigma_1^2 + \sigma_2^2). \qquad (50)
\end{aligned}$$

Under focus, the incomes from the 2 projects in each period are jointly bivariate normal. Hence, $Y_1^F = y_{1,1}^F + y_{2,1}^F \sim N(2a_0, S_{0F}^2)$. Further, from (6)–(7), $Y_2^F|Y_1^F \sim N(2a_0 b + Y_1^F(1-b), S_{1F}^2)$, where $b = \frac{\tau_0}{\tau_{1F}}$. Applying result (50), we get the distribution of total lifetime income to be

$$Y_1^F + Y_2^F \sim N\left(4a_0, \left(2 - \frac{\tau_0}{\tau_{1F}}\right)^2 S_{0F}^2 + S_{1F}^2\right) \qquad (51)$$

Under diversification, from (9)–(10), applying result (50) to *each project*, we get

$$Y_1^D + Y_2^D \sim N\left(4a_0, \left(2 - \frac{\tau_0}{\tau_{1D}}\right)^2 S_{0D}^2 + S_{1D}^2\right) \qquad (52)$$

Comparing (51) and (52), we see that $\tau_{1F} > \tau_{1D}$, $S_{0F}^2 > S_{0D}^2$ and from (15), $S_{1F}^2 > S_{1D}^2$. From this, it is clear that $(Y_1^F + Y_2^F)$ is a mean-preserving spread of $(Y_1^D + Y_2^D)$. Hence, for any utility function of the form $u(c_1, c_2) = u(c_1) + u(c_2)$ where $u(.)$ is increasing and strictly concave, $E_0^F[u(c_1) + u(Y_1^F + Y_2^F - c_1)] < E_0^D[u(c_1) + u(Y_1^D + Y_2^D - c_1)]$ for any fixed c_1. Since the free transferrability of consumption across periods implies that any first period consumption c_1 under focus is also feasible under diversification, we have

$$\begin{aligned}
&E_0^F[\max_{c_1} u(c_1) + E_1^F[u(Y_1^F + Y_2^F - c_1)]] \\
&\quad < E_0^D[\max_{c_1} u(c_1) + E_1^D[u(Y_1^D + Y_2^D - c_1)]].
\end{aligned}$$

Hence the proof.

Proof of Lemma 3.1

In a T-period version of (17), let $\hat{y}_{t-1}$ denote the sequence of past outcomes from the two projects until period $(t-1)$, i.e., $\{(y_{1,1}, y_{2,1}), (y_{1,2}, y_{2,2}), \ldots, (y_{1,t-1}, y_{2,t-1})\}$. Under focus, the agent's problem may recursively be written as

$$V_t(w_t) = \max_{\tilde{a}_{t+1}, c_t} u(c_t) + E_t\left[V_{t+1}(w_{t+1})|\hat{y}_t\right] \quad \text{s.t. } w_{t+1} = w_t - c_t + z_{t+1} \tag{53}$$

where z_t is the total income (from both projects combined) in period t and $w_0 = 0$. Note that the input choice is the same for both projects under focus. Choosing different input levels requires at least one of them to be different from the single project optimal level. This will be optimal only if the loss of output caused by the deviation from the single project optimum is compensated by higher output in later periods through faster learning. However, we have already noted that the change in precision over time is deterministic in the Bayesian learning model with normal shocks. Hence, the optimal choice must be the same for both projects in each period under focus. The first order conditions for this problem are

$$E_t\left[V'_{t+1}(w_{t+1})|\hat{y}_t\right] = u'(c_t) \tag{54}$$

$$E_t\left[V'_{t+1}(w_{t+1}) * \left(\frac{dw_{t+1}}{d\tilde{a}_{t+1}}\right)|\hat{y}_t\right] = 0 \tag{55}$$

Since $z_{t+1} = z_{1,t+1} + z_{2,t+1} = 2I - I\left((y_{1,t+1} - \tilde{a}_{t+1})^2 + (y_{2,t+1} - \tilde{a}_{t+1})^2)\right)$,

$$\frac{dw_{t+1}}{d\tilde{a}_{t+1}} = \frac{dz_{t+1}}{d\tilde{a}_{t+1}} = 2I\left[(y_{1,t+1} - \tilde{a}_{t+1}) + (y_{2,t+1} - \tilde{a}_{t+1})\right]$$

From (16), $(y_{1,t+1}, y_{2,t+1})$ is bivariate normal under focus, with the marginal means equal to $E_t[a|\hat{y}_t]$. Hence, if $\tilde{a}_{t+1} = E_t[a|\hat{y}_t]$, then the marginal distributions of $(y_{1,t+1} - \tilde{a}_{t+1})$ and $(y_{2,t+1} - \tilde{a}_{t+1})$ as well as the conditional distribution $(y_{2,t+1} - \tilde{a}_{t+1})|(y_{1,t+1} - \tilde{a}_{t+1})$ are symmetric about 0. Further, $V'_{t+1}(w_{t+1})$ is also symmetric about $(y_{1,t+1} - \tilde{a}_{t+1})$ and $(y_{2,t+1} - \tilde{a}_{t+1})$. Therefore, $\tilde{a}_{t+1} = E_t[a|\hat{y}_t]$ is a solution to (55).

Next, we prove uniqueness by showing that $V'_{t+1}(w_{t+1}) * \left(\frac{dw_{t+1}}{d\tilde{a}_{t+1}}\right)$ is a monotonic function of $\tilde{a}_{t+1}$. Differentiating $V'_{t+1}(w_{t+1}) * \left(\frac{dw_{t+1}}{d\tilde{a}_{t+1}}\right)$ with respect to $\tilde{a}_{t+1}$, we get

$$V''_{t+1}(w_{t+1}) * \left(\frac{dw_{t+1}}{d\tilde{a}_{t+1}}\right)^2 + V'_{t+1}(w_{t+1}) * \left(\frac{d^2 w_{t+1}}{d\tilde{a}^2_{t+1}}\right)$$

which is negative if $V'_{t+1}(w_{t+1}) \geq 0$ and $V''_{t+1}(w_{t+1}) < 0$, since it is easily seen that $\left(\frac{d^2 w_{t+1}}{d\tilde{a}^2_{t+1}}\right) < 0$. Therefore, an increasing and strictly concave period utility function $u(.)$ is sufficient for $\frac{d\left(V'_{t+1}(w_{t+1})\left(\frac{dw_{t+1}}{d\tilde{a}_{t+1}}\right)\right)}{d\tilde{a}_{t+1}}$ to be negative, as the value function $V(.)$ inherits the properties of monotonicity and strict concavity from the period utility function (see Fama 1970; Fernandez-Corugedo 2002).

This proves Lemma 3.1 for the case of focus. The proof for the case of diversification is analogous.

At this point, a note is in order regarding the state variables in the dynamic optimization program above. In our problem, at the end of every period, the agent has some stock of wealth and information, the latter captured by the moments of the posterior distribution of the target input. In this sense, both wealth and the moments ought to be state variables. Given our assumption of normality, there are only two potentially relevant moments, the mean and the variance. Under Bayesian learning, the posterior variance evolves exogenously from period to period [as seen from Eqs. (7) and (10)]. Hence, it is not relevant for determining the optimal action. This leaves the posterior mean, given by $E_t[a|\hat{y}_t]$. Under our candidate solution, $\tilde{a}_{t+1} = E_t[a|\hat{y}_t]$. As stated above, conditional on this rule, the marginal distributions of $(y_{1,t+1} - \tilde{a}_{t+1})$ and $(y_{2,t+1} - \tilde{a}_{t+1})$ as well as the conditional distribution $(y_{2,t+1} - \tilde{a}_{t+1})|(y_{1,t+1} - \tilde{a}_{t+1})$ are symmetric about 0. Hence, the actual realizations $y_{1,t}$ and $y_{2,t}$ in period t do not affect the posterior distribution of output in subsequent periods. In the terminology of Datta et al. (2000), there is no signal-dependence, i.e., the signals are not pay-off relevant. This implies that even the posterior mean is not relevant beyond the current period. Hence, we are justified in excluding the moments of posterior distribution from the arguments to the value function. (We are grateful to a referee for suggesting this.)

Proof of Proposition 3

The case of focus is more intricate than that of diversification, since the project cash flows are correlated under focus. Hence, we prove the proposition for focus, drawing parallels for diversification where appropriate. We first prove the following two lemmas, which, together with Lemma 3.1, lead to Proposition 3.

Lemma 3.2 *Under focus, the expected utility of second period income conditional on the first period targets $y_{1,1}$ and $y_{2,1}$, denoted by U_2^F, is given by*

$$U_2^F = E_1^F[e^{-\gamma z_2}|y_{1,1}, y_{2,1}, \tilde{a}_2 = E_2[a|\hat{y}_t]$$
$$= \frac{e^{-2\gamma I}}{(1 - 2\gamma I\sigma_e^2)^{\frac{1}{2}}(1 - 2\gamma I(\sigma_e^2 + 2\sigma_u^2 + 2\sigma_{1F}^2))^{\frac{1}{2}}} \tag{56}$$

Lemma 3.3 *The optimal first period consumption choice, c_1^F, and the expected lifetime utility under the optimal consumption policy are*

$$c_1^F = 0.5z_1 - \frac{\ln U_2^F}{2\gamma} \quad \textit{and} \quad E_0[U^F(c_1^F)] = -2\sqrt{U_2^F}E_0^F[e^{-0.5\gamma z_1}]. \tag{57}$$

Proof of Lemma 3.2

Denote the expected target $E_1[a|y_{1,1}, y_{2,1}]$ by $\overline{a}$. Following Lemma 3.1, $\overline{a}$ is the optimal input choice for both projects in the second period. The actual target inputs in the second period for the two projects are given by

$$y_{1,2} = a + u_2 + e_{1,2} \quad \text{and} \quad y_{2,2} = a + u_2 + e_{2,2}.$$

Let $x = a + u_2 - \overline{a}$. Note that $x \sim N(0, \sigma_{1F}^2 + \sigma_u^2)$. The income in the second period from the first and second projects are therefore

$$z_{1,2} = I[1 - (x + e_{1,2})^2] \quad \text{and} \quad z_{2,2} = I[1 - (x + e_{2,2})^2].$$

Substituting for $z_{1,2}$ and $z_{2,2}$ in the expression for U_2^F, we get

$$U_2^F = e^{-2\gamma I} \int_{-\infty}^{\infty} d\Phi(x) \int_{-\infty}^{\infty} e^{\gamma I(x+e_{2,2})^2} d\Phi(e_{2,2}) \int_{-\infty}^{\infty} e^{\gamma I(x+e_{1,2})^2} d\Phi(e_{1,2}) \quad (58)$$

where $\Phi(.)$ denotes the normal distribution function corresponding to that particular variable. By completion of squares, we can show that if $\nu \sim N(0, \sigma_\nu^2)$ and k and b are constants with $1 - 2b\sigma_\nu^2 > 0$,

$$\int_{-\infty}^{\infty} e^{b(k+\nu)^2} \mathrm{d}\Phi(\nu) = \frac{e^{\frac{bk^2}{(1-2b\sigma_\nu^2)}}}{\sqrt{1 - 2b\sigma_\nu^2}}$$

Substituting in (58),

$$\begin{aligned}
U_2^F &= \frac{e^{-2\gamma I}}{\sqrt{1 - 2\gamma I \sigma_e^2}} \int_{-\infty}^{\infty} exp\left\{\frac{\gamma I x^2}{(1 - 2\gamma I \sigma_e^2)}\right\} d\Phi(x) \\
&\quad \int_{-\infty}^{\infty} exp\{\gamma I (x + e_{2,2})^2\} d\Phi(e_{2,2}) \\
&= \frac{e^{-2\gamma I}}{(1 - 2\gamma I \sigma_e^2)} \int_{-\infty}^{\infty} exp\left\{\frac{2\gamma I}{(1 - 2\gamma I \sigma_e^2)} x^2\right\} d\Phi(x) && (59) \\
&= \frac{e^{-2\gamma I}}{(1 - 2\gamma I \sigma_e^2)} \left(1 - \frac{4\gamma I \sigma_x^2}{(1 - 2\gamma I \sigma_e^2)}\right)^{-1/2} && (60) \\
&= \frac{e^{-2\gamma I}}{(1 - 2\gamma I \sigma_e^2)^{\frac{1}{2}} (1 - 2\gamma I (\sigma_e^2 + 2\sigma_u^2 + 2\sigma_{1F}^2))^{\frac{1}{2}}} && (61)
\end{aligned}$$

This ends the proof of Lemma 3.2.

Proof of Lemma 3.3

Lemma 3.2 shows that $E_1^F[e^{-\gamma z_2} | y_{1,1}, y_{2,1}]$ is independent of $y_{1,1}$ and $y_{2,1}$. (Similarly, it can be established that $E_1^D[e^{-\gamma z_2} | y_{1,1}, y_{2,1}]$ is also independent of $y_{1,1}$ and $y_{2,1}$.)

The agent's lifetime utility under the optimal input choice under focus is therefore

$$E_0[U^F(c_1)] = -E_0[e^{-\gamma c_1}] - E_0[U_2^F . e^{-\gamma(Y_1 - c_1)}] \tag{62}$$

Choosing c_1 to maximize this expression leads to the following solution:

$$c_1^F = \frac{z_1}{2} - \frac{\ln(U_2^F)}{2\gamma} \quad \text{and} \quad E_0[U^F(c_1^F)] = -2E_0^F[e^{-\gamma c_1^F}]$$
$$= -2\sqrt{U_2^F}\, E_0^F[e^{-0.5\gamma z_1}]. \tag{63}$$

Hence the proof. (Analogous expressions may be derived for the case of diversification.)

In (63), $E_0^F[e^{-0.5\gamma z_1}]$ is evaluated by integration—the steps are similar to those used for calculating U_2^F in the proof of Lemma 3.2. This leads to the expression for U^F in Proposition 3. The expression for U^D is similarly derived.

This ends the proof of Proposition 3.

Proof of Proposition 4

Since $E_0[U^F]$ and $E_0[U^D]$ are negative, $E_0[U^F] > E_0[U^D]$ if $F_1F2 > D_1D_2$. We show below that there exists a set of parameters such that $\frac{F_1F_2}{D_1D_2} > 1$ with $\{F_1, F_2, D_1, D_2\} \in \Re_+^4$. We first prove the proposition for the case where there is no type-specific risk. The proof is extended to the case of positive type-specific risk by continuity of $\left(\frac{F_1F_2}{D_1D_2}\right)$ w.r.t. $\left(\frac{\sigma_u^2}{\sigma_0^2}\right)$.

To simplify notation, let $\gamma I\sigma_0^2 = g$, $\left(\frac{\sigma_e^2}{\sigma_0^2}\right) = x$, and $\left(\frac{\sigma_u^2}{\sigma_0^2}\right) = h$. If $\sigma_u^2 = 0$, then, from (7) and (10), we have

$$\sigma_{1D}^2 = \frac{1}{\tau_0 + \tau_{u+e}} = \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_u^2+\sigma_e^2}} = \frac{\sigma_0^2 x}{(1+x)} \tag{64}$$

and

$$\sigma_{1F}^2 = \frac{1}{\tau_0 + 2\tau_{2u+e}} = \frac{1}{\frac{1}{\sigma_0^2} + \frac{2}{2\sigma_u^2+\sigma_e^2}} = \frac{\sigma_0^2 x}{(2+x)}. \tag{65}$$

Substituting the above expressions in $\frac{F_1F_2}{D_1D_2}$, we get

$$\frac{F_1F_2}{D_1D_2} = \left(\frac{[1-gx]^{1/2}[1-g(2+x)]^{1/2}}{[1-g(1+x)]}\right)\left(\frac{[1-2gx]^{1/4}[1-2g(x+\frac{2x}{2+x})]^{1/4}}{[1-2g(1+\frac{x}{1+x})]^{1/2}}\right) \tag{66}$$

Examining (66), we see that one condition that will ensure that $\{F_1, F_2, D_1, D_2\} \in \Re^4_+$ is $1 - 2gx - 4g \geq 0$. For any given g, choose x such that $1 - 2gx - 4g = 0$. Hence,

$$x = \frac{1}{2g} - 2 \tag{67}$$

We will restrict g to (0, 0.25) so that x is a finite, positive number. Substituting for x in (66), we get

$$\frac{F_1 F_2}{D_1 D_2} = \frac{2g^{1/4}[(1+4g)(1-2g)]^{1/2}}{(1+2g)} \tag{68}$$

In order to show that the RHS of the expression in (68) is greater than one for some $g \in (0, 0.25)$, we solve the equation

$$\frac{2g^{1/4}[(1+4g)(1-2g)]^{1/2}}{(1+2g)} = 1$$

which simplifies to

$$1024g^5 - 528g^4 - 224g^3 + 40g^2 + 8g - 1 = 0 \tag{69}$$

Equation (69) has two real roots between 0 and 0.25. Denote these by r_1 and r_2 with $r_2 > r_1$. ($r_1 \approx 0.1092$ and $r_2 \approx 0.1892$.) It can be verified that the LHS of the equation is positive for $r_1 < g < r_2$.

Hence, the proof.

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