

Topos Theory and the Connections between Category and Set Theory

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March 13, 2008

① Why Category Theory?

② Definition

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④ Relations to Set Theory

⑤ Topos Definition and Examples

⑥ Motivation and History of Topos theory

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Why Category Theory?

- “The fundamental idea of representing a function by an arrow first appeared in topology around 1940” (Maclane)
- Allows the abstraction of common structure. For example, Groups, sets, topological spaces all have associativity intrinsic to them.
- Unlike Set theory specifies the codomain of a function. This allows it to distinguish the difference between an identity map and an inclusion.
- Leads to a different foundation for mathematics via Topos Theory, which allows one to explicitly construct different types of logic.

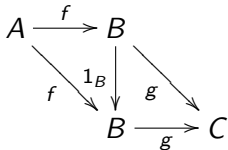
- In particular there exist “Intuitionist” logics that can be constructed that have all of the properties of classical logic except for the excluded middle (The statement $\alpha \vee \neg\alpha$ is not necessarily true).
 - Think about the quantum mechanical two slit experiment and try to evaluate the truth value of the statement, “ The electron went through the top slit.”
 - In these logics, proofs by contradiction do not hold: one must prove everything constructively.
 - Some physicists are using topos theory to construct a language to formulate physical theories with the hope of (among other things) identifying possible unification theories.
 - Apparently they can construct theories where the logical structure will demarcate which questions can be answered theoretically, which can be tested physically, and which are unanswerable!

If Category Theory generalizes set theory then all of the familiar objects and entities in set theory must be contained in Category theory somewhere. Which leads to the following questions:

- 1 How are Cartesian product, disjoint union, equivalence relations, inverse images, subsets, power sets, kernels, . . . represented in Category theory?
- 2 How are these notions and structures generalized by category theory?

A Category is comprised of

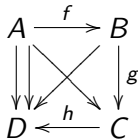
- 1 a collection of \mathcal{C} -objects and \mathcal{C} -arrows denoted $\mathcal{C}^{\rightarrow}$ with fixed domain and codomain.
- 2 Identity arrows for every \mathcal{C} -object in \mathcal{C} . That is, for any three \mathcal{C} -objects A, B, C s.t. , $A \xrightarrow{f} B \xrightarrow{g} C$ implies



- 3 $A \xrightarrow{f} B, B \xrightarrow{g} C \in \mathcal{C}^{\rightarrow} \implies A \xrightarrow{f \cdot g} C \in \mathcal{C}^{\rightarrow}$

- 4 Associativity holds: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \in \mathcal{C}^{\rightarrow}$

implies the commutativity of



$$\mathbf{1} \text{ vs. } \{\emptyset\} = 0$$

The category with only one object and one arrow.



No matter what we call the two objects it must have this structure. So we label the object 0 and the arrow $\langle 0, 0 \rangle = 1_0$, which gives the number zero by the diagram



2, 3, ...

Similarly, we can define the categories $\mathbf{2} : 0 \xrightarrow{\quad} 1$

and $\mathbf{3} : 0 \xrightarrow{\quad} 1 \xrightarrow{\quad} 2$

These are examples of partial orderings. Compare them against the set theoretic definitions of the natural numbers

$$0 = \emptyset, 1 = \{\emptyset\}, 2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, 3 = \{0, 1, 2\}, \dots$$

A **preorder category** is a category that between any two objects $a, b \in \mathcal{C}$ there exists at most one map $a \rightarrow b$. This allows one to define a reflexive and transitive relation. If this relation is also antisymmetric then it is called a **partial ordering** and the symbol \sqsubseteq will be used to denote the relation.

Posets and Skeletal Categories

Topos Theory
and the
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A **Poset** is a pair $\mathbf{P} = \langle P, \sqsubseteq \rangle$ where P is a set and \sqsubseteq is a partial ordering on P .

A **Skeletal** category is one in which “isomorphic” actually means equal (*i.e.* $a \cong b \implies a = b$).

Discrete Categories vs. Sets

If you took any set and added an identity arrow for each element, you would have a discrete category. In this sense discrete categories are sets.

$\{\text{board, flower, skateboard}\} =$


Board

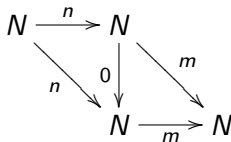
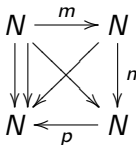
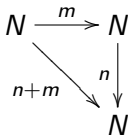

flower


skateboard

A monoid M is a category with one object. A monoid is determined by $\mathcal{C}^{\rightarrow}(M)$, the identity arrow 1_e and the composition rule \circ .

Example: Consider the category N comprised of a single object N and an infinite collection of arrows labeled $0, 1, 2, 3, \dots$ which are the natural numbers.

- 1 Define composition law to be $n \circ m = n + m$, which means that the bottom left diagram commutes by definition.
- 2 Associative law holds (by associativity of addition) Hence middle diagram commutes.
- 3 Identity arrow 1_N is defined to be the number 0. Which implies commutativity of right diagram.



For any category \mathcal{C} and any $a \in \mathcal{C}^{ob}$, the set $\text{Hom}(a, a)$ is a monoid.

$$\text{Hom}(a, b) = \mathcal{C}(a, b) = \left\{ f : f \in \mathcal{C} \rightarrow \text{s.t. } a \xrightarrow{f} b \right\}, \text{ where } a, b \in \mathcal{C}^{ob}.$$

\mathcal{C} is a **Subcategory** of \mathcal{D} ($\mathcal{C} \subseteq \mathcal{D}$) if

- 1 $a \in \mathcal{C}^{ob} \implies a \in \mathcal{D}^{ob}$.
- 2 for any two $a, b \in \mathcal{C}^{ob} \implies \mathcal{C}(a, b) \subseteq \mathcal{D}(a, b)$.

\mathcal{C} is a **Full Subcategory** of \mathcal{D} if $\mathcal{C} \subseteq \mathcal{D}$ and for any two $a, b \in \mathcal{C}^{ob}$, $\mathcal{C}(a, b) = \mathcal{D}(a, b)$.

Monic vs. injectivity

Monic arrows are generalized from the notions of an injective function. We can get the proper definition in terms of arrows by realizing that injective functions are left cancelable *i.e.*

$$f(x) = f(y) \implies x = y$$

or

$$f \circ g(x) = f \circ h(x) \implies g(x) = h(x).$$

In arrows f is monic if for any two parallel functions $g, h : C \rightrightarrows A$ the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{g} & A \\ h \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B \end{array} .$$

That is $f \circ g = f \circ h \implies g = h.$

Epic vs. Surjectivity

Epic arrows are derived in the same way except these functions are right cancelable.

In arrows it amounts to the statement that f is an epic arrow if for any two parallel arrows $g, h : B \rightrightarrows C$ the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ f \downarrow & & \downarrow g \\ A & \xrightarrow{h} & B \end{array} .$$

That is $g \circ f = h \circ f \implies g = h$.

Iso arrow vs. Isomorphisms

The notion of an Iso arrow is generalized from isomorphic functions. These simply are the functions f where \exists a function g s.t. $f \circ g = 1 = g \circ f$.

In arrows it is the same: for $f \in \mathcal{C}^{\rightarrow}$ $f : a \rightarrow b$ is **iso** (or invertible), in \mathcal{C} if there is a $g \in \mathcal{C}^{\rightarrow}$, $g : b \rightarrow a$ s.t.
 $g \circ f = 1_a$ and $f \circ g = 1_b$.

$a, b \in \mathcal{C}^{ob}$ are **isomorphic** in \mathcal{C} ($a \cong b$) if there is a \mathcal{C} -arrow $f : a \rightarrow b$ that is iso in \mathcal{C} .

In the category \mathcal{Set} and in any Topos a monic and epic arrow is an iso arrow. However, this is not true in general!! Even though in any category **an iso arrow is monic and epic**.

Example: Take the category \mathbb{N} , every arrow is epic and monic since you can cancel on the left and right. And the only iso is $0 : \mathbb{N} \rightarrow \mathbb{N}$, since if m has an inverse n then $m \circ n = 1_N$ or $m + n = 0 \implies m = n = 0$.

Example: In a Poset category if $f : p \rightarrow q$ has an inverse $f^{-1} : q \rightarrow p$ then $p \sqsubseteq q$ and $q \sqsubseteq p \implies p = q \implies f = 1_p$. Thus every arrow in a Poset is monic and epic but the only iso's are the identities.

Initial objects vs. \emptyset

Initial objects were abstracted from $\mathcal{S}et$ by asking what properties characterize the null set \emptyset ? It turns out that for any set A there exists only one function $\emptyset \rightarrow A$. In $\mathcal{S}et$ the initial object is simply \emptyset and it happens to be unique.

An object 0 is **initial** in category \mathcal{C} if for every

$$a \in \mathcal{C}^{ob}, \quad \exists! f \in \mathcal{C}^{\rightarrow} \quad \text{s.t.} \quad 0 \xrightarrow{f} a$$

Any two initial \mathcal{C} -objects must be isomorphic in \mathcal{C}

Proof: Suppose $\exists 0, 0'$ that are initial in \mathcal{C} . then

$$\exists! f, g \in \mathcal{C}^{\rightarrow} \quad \text{s.t.} \quad 0 \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} 0' . \quad \text{This means that}$$

$1_0 = g \circ f : 0 \rightarrow 0$ and $1_{0'} = f \circ g : 0' \rightarrow 0'$. Hence f has an inverse arrow and $0 \cong 0'$.

Example: of a category that has two initial objects that are isomorphic but not equal.



Example: In $\mathcal{G}rp$ and $\mathcal{M}on$ the initial objects are any one element algebra $M = \{e\}$. Each of these categories has infinitely many objects (think products).

Terminal objects vs. singletons

By reversing the direction of the arrows in the definition of initial object we get the idea of a terminal object. In $\mathcal{S}et$ terminal objects are the singleton sets (the sets with only one element).

An object 1 is **terminal** in a category \mathcal{C} if for every $a \in \mathcal{C}^{ob}$ $\exists ! f \in \mathcal{C}^{\rightarrow}$ s.t. $a \xrightarrow{f} 1$.

Duality

Duality is the process “reverse all arrows” with the following identifications.

Statement Σ

$$f : a \rightarrow b$$

$$a = \text{dom} f$$

$$i = 1_a$$

Isomorphic

$$h = g \circ f$$

f is monic

u is a right inverse of h

f is invertible

t is a terminal object

$S, T : C \rightarrow B$ are functors

T is full

T is faithful

Dual Statement Σ^*

$$f : b \rightarrow a$$

$$a = \text{cod} f$$

$$i = 1_a$$

Isomorphic

$$h = f \circ g$$

f is epi

u is a left inverse of h

f is invertible

t is an initial object

$S, T : C \rightarrow B$ are functors

T is full

T is faithful

The **Dual** category \mathcal{C}^{op} is comprised of the same objects that are in \mathcal{C} with all of the \mathcal{C} -arrows reversed.

Example: For any \mathcal{C} : $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

Example: If \mathcal{C} is discrete then $\mathcal{C}^{op} = \mathcal{C}$

Example: If \mathcal{C} is a pre-prder (P, R) , with $R \subseteq P \times P$, then \mathcal{C}^{op} is the pre-order (P, R^{-1}) , where $pR^{-1}q \iff qRp$. That is R^{-1} is the inverse relation to R .

Example: The dual statement of the theorem “any two initial \mathcal{C} objects are isomorphic” is the theorem “any two terminal \mathcal{C} objects are isomorphic”

Products vs. Cartesian Product

Set Definition: $A \times B = \{(x, y) : x \in A, y \in B\}$

What structure of this definition should we use to form our definition in category theory (since we can't use elements)?

If we define the projection maps

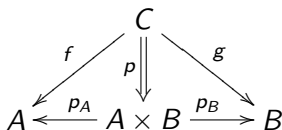
$$p_A : A \times B \rightarrow A \qquad p_A(x, y) = x$$

$$p_B : A \times B \rightarrow B \qquad p_B(x, y) = y$$

and we are given another set C with a pair of maps $f : C \rightarrow A$ and $g : C \rightarrow B$ define $p : C \rightarrow A \times B$ by the rule

$$p(x) = \langle f(x), g(x) \rangle.$$

The diagram



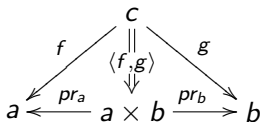
commutes. Furthermore, the map p is the only arrow that can make the diagram commute.

Proof: Suppose that $p(x) = \langle h, k \rangle$. Since we know that the diagram commutes $p_A \circ p(x) = f(x)$ and $p_A(p(x)) = p_A(\langle h(x), k(x) \rangle) = h(x) \implies h(x) = f(x)$. Similarly for k .

The uniqueness of the map p is what is generalized in category theory.

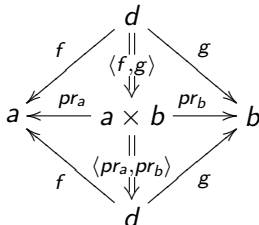
Category theory product

A **product** in a category \mathcal{C} of two objects a and b is an object $a \times b \in \mathcal{C}^{ob}$ together with a pair of arrows $(pr_a : a \times b \rightarrow a, pr_b : a \times b \rightarrow b)$ s.t. for any two arrows of the form $f : c \rightarrow a$ and $g : c \rightarrow b \exists! \langle f, g \rangle : c \rightarrow a \times b$ makes the following commute.



Products are not unique but are isomorphic

Suppose that d is also a product of $a \times b$ then consider



Hence

$$\langle pr_a, pr_b \rangle \circ \langle f, g \rangle = 1_d.$$

To complete the isomorphism of $d \cong a \times b$ just interchange the roles of d and $a \times b$ to get

$$\langle f, g \rangle \circ \langle pr_a, pr_b \rangle = 1_{a \times b}.$$

Co-products vs. Disjoint Union

A **Co-product** of $a, b \in \mathcal{C}$ is an object $a + b \in \mathcal{C}$ together with a pair of \mathcal{C} arrows (i_a, i_b) (where $i_a : a \rightarrow a + b$ and $i_b : b \rightarrow a + b$) s.t. for any \mathcal{C} arrows of the form $f : a \rightarrow c$ and $g : b \rightarrow c$ there is a unique arrow $[f, g] : a + b \rightarrow c$ making the following commute

$$\begin{array}{ccccc} a & \xrightarrow{i_a} & a + b & \xleftarrow{i_b} & b \\ & \searrow f & \downarrow [f, g] & \swarrow g & \\ & & c & & \end{array}$$

In $\mathcal{S}et$ the co-product of A and B is their disjoint union

$$A + B := (A \times \{0\}) \cup (B \times \{1\}).$$

The injections are given by

$$i_A(a) = (a, 0) \qquad i_B(b) = (b, 1)$$

In this case you need to define the function

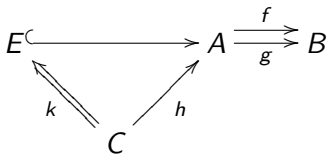
$$[f, g](x, d) = f(x)(1 - d) + dg(x)$$

We get to use elements since we are in $\mathcal{S}et$.

Equalisers vs. Equalisers

Set Definition: Given a pair of parallel functions $A \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} B$ define $E = \{x : x \in A, f(x) = g(x)\}$. E equalizes f and g by defining the inclusion $E \hookrightarrow A$ we get $f \circ i = g \circ i$.

It turns out that i is a canonical equaliser of f and g . That is if $h : C \rightarrow A$ is any other equaliser of f and g (i.e. $f \circ h = g \circ h$) then there exists a unique $k : C \rightarrow E$ s.t. $i \circ k = h$ (h 'factors uniquely through' i).

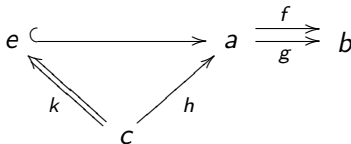


Uniqueness: If $i \circ k = h$ then $i(k(c)) = k(c) = h(c) \implies f(h(c)) = g(h(c)) \implies h(c) \in E$.

Category Definition:

An arrow $i : e \rightarrow a$ in \mathcal{C} is an **equaliser** (in \mathcal{C}) of parallel $\mathcal{C} \rightarrow f, g : a \rightarrow b$ if:

- ① $f \circ i = g \circ i$
- ② If $h : c \rightarrow a$ satisfies $f \circ h = g \circ h$ then $\exists ! k \in \mathcal{C} \rightarrow k : c \rightarrow e$ s.t. $i \circ k = h$



- Every equaliser is monic
- In any category, an epic equaliser is iso.
- In Set (and any Topos) monics are equalisers

Why are the statements of category theory so complicated?

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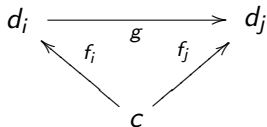
Acknowledgements

They don't have to be.

Limits and Co-limits vs.

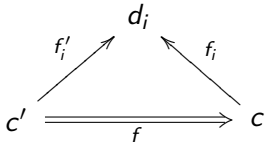
A **Diagram D** is simply a Subcategory of some other category (i.e. a collection of \mathcal{C} objects and some \mathcal{C} -arrows between these objects).

A **D -Cone** (denoted $\{f_i : c \rightarrow d_i\}$) of a diagram D consists of a $c \in \mathcal{C}^{ob}$ with a \mathcal{C} arrow $f_i : c \rightarrow d_i$, $\forall d_i \in D$ s.t.



commutes whenever g is an arrow in the diagram D .

A **D -limit** of a diagram D is a D -cone $\{f_i : c \rightarrow d_i\}$ with the property that for any other D -cone $\{f'_i : c' \rightarrow d_i\}$, $\exists!$ arrow $f : c' \rightarrow c$ s.t.



commutes for every object $d_i \in D$.

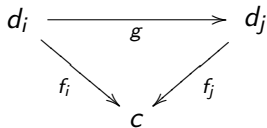
Example: Consider the arrowless diagram of two \mathcal{C} objects a and b . A D -cone is an object c with two arrows f and g that form $a \xleftarrow{f} c \xrightarrow{g} b$. What is another name for a D -limit of this diagram? Example: Let D be the diagram $a \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} b$. We

can write a D -cone in this case as the commutative

$c \xrightarrow{h} a \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} b$. What is another name for the D -limit of this diagram?

Co-cones and Co-limits

By duality a **Co-cone** $\{f_i : d_i \rightarrow c\}$ of a diagram D consists of an object c and arrows $f_i : d_i \rightarrow c$ for each object $d_i \in D$. A co-limit is defined analogously to limits. That is, a **limit co-cone** is a co-cone with the co-universal property that for any other co-cone $\{f'_i : d_i \rightarrow c'\}$, $\exists!$ arrow $f : c \rightarrow c'$ s.t. the following commutes for every $d_i \in D$.



Pop QUIZ !!!!!

- What is a cone of an empty diagram?

Pop QUIZ !!!!!

- What is a cone of an empty diagram?

C

Pop QUIZ !!!!!

- What is a cone of an empty diagram?

C

- What is the limit of the empty diagram?

Pop QUIZ !!!!!

- What is a cone of an empty diagram?

c

- What is the limit of the empty diagram?

$$c' \Longrightarrow c$$

Pop QUIZ !!!!!

- What is a cone of an empty diagram?

c

- What is the limit of the empty diagram?

$$c' \Longrightarrow c$$

- What is another name for c ?

Pop QUIZ !!!!!

- What is a cone of an empty diagram?

c

- What is the limit of the empty diagram?

$$c' \Longrightarrow c$$

- What is another name for c ?

The limit of an empty diagram is a terminal object.

- What is a cone (and limit) for the following diagram?

a

b

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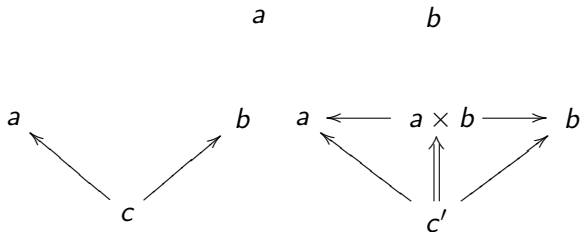
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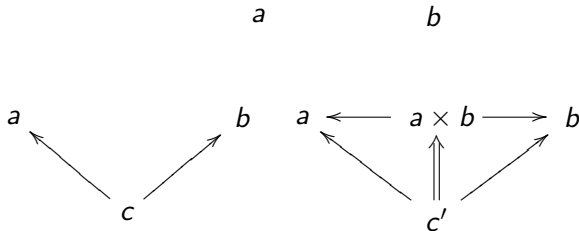
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- What is a cone (and limit) for the following diagram?



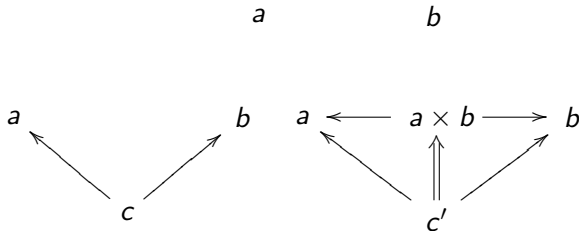
- What is a cone (and limit) for the following diagram?



Dually:

- What is the co-limit of an empty diagram?

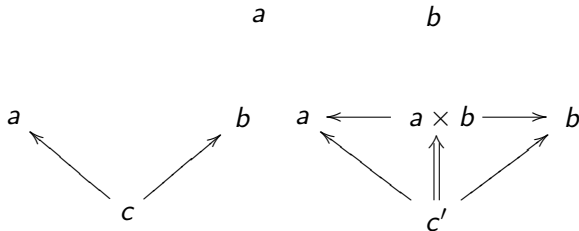
- What is a cone (and limit) for the following diagram?



Dually:

- What is the co-limit of an empty diagram?
An initial object.

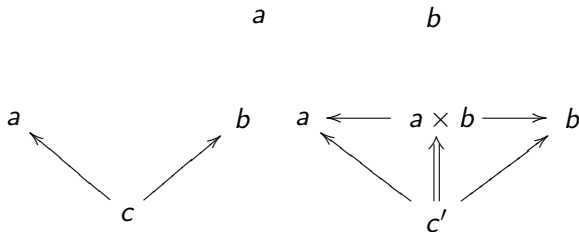
- What is a cone (and limit) for the following diagram?



Dually:

- What is the co-limit of an empty diagram?
An initial object.
- What diagram is the coproduct a limit of?

- What is a cone (and limit) for the following diagram?



Dually:

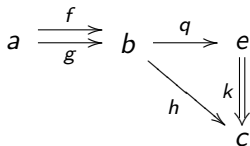
- What is the co-limit of an empty diagram?
An initial object.
- What diagram is the coproduct a limit of?

a b

Co-equalisers vs. Equiv. relations

The co-equaliser of a pair of parallel arrows f, g \mathcal{C} -arrows is a co-limit for the diagram $a \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} b$. Or equivalently, it can be defined as a \mathcal{C} -arrow $q : b \rightarrow c$ s.t.

- 1 $q \circ f = q \circ g$
- 2 If $h : b \rightarrow c$ satisfies $h \circ f = h \circ g$ in \mathcal{C} then $\exists!$ arrow $k : e \rightarrow c$ s.t.



Definition of k

Suppose that we have a k s.t. $k \circ f_R = h$. Then

$$k([a]) = k(f_R(a)) = h(a)$$

So define $k([a]) = h(a)$. This is well defined since if $[a] = [b]$ then

$$f_R(a) = f_R(b) \implies h(a) = h(b).$$

This is a natural map (*i.e.* the only way to define this map) and hence unique.

Pullback vs. Inverse Image

A **Pullback** of a diagram $a \xrightarrow{f} c \xleftarrow{g} b$ is a limit in \mathcal{C} for the diagram

$$\begin{array}{ccc} & & b \\ & & \downarrow g \\ a & \xrightarrow{f} & c \end{array}$$

A cone and universal cone for this diagram is

$$\begin{array}{ccc} d & \xrightarrow{f'} & b \\ & \searrow h & \downarrow g \\ g' \downarrow & & c \\ a & \xrightarrow{f} & c \end{array}$$

In $\mathcal{S}et$, given a function $f : A \rightarrow B$, the inverse image of a set $C \subset B$ is simply

$$f^{-1}(C) = \{x : x \in A, f(x) \in C\}.$$

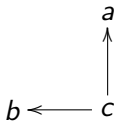
Categorically this is represented by the pullback diagram below

$$\begin{array}{ccc}
 f^{-1}(C) & \xrightarrow{f^*} & C \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{f} & B
 \end{array}$$

where f^* is the restriction of f to $f^{-1}(C)$.

Pushouts

Are constructed by dualizing Pullbacks. They are the co-limit of this diagram.



Completeness

- A category \mathcal{C} is complete if every diagram in \mathcal{C} has a limit in \mathcal{C} .
- Dually a category \mathcal{C} is co-complete if every diagram in \mathcal{C} has a co-limit in \mathcal{C}
- A category \mathcal{C} is bi-complete if it is complete and co-complete.

Exponentiation (all products exist)

In Set theory: We are familiar with

$$B^A = \left\{ f : A \xrightarrow{f} B \right\}.$$

This set is related to products by the evaluation function.

$$ev : B^A \times A \rightarrow B \quad \text{s.t.} \quad ev(\langle f, x \rangle) = f(x)$$

Exponentiation (all products exist)

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ev has a universal property among functions of the form

$$C \times A \xrightarrow{g} B. \quad \forall g \text{ of the above form } \exists! \hat{g} : C \rightarrow B^A \text{ s.t.}$$

$$\begin{array}{ccc}
 B^A \times A & & \\
 \uparrow \hat{g} \times id_A & \searrow ev & \\
 C \times A & & B \\
 & \nearrow g & \\
 & &
 \end{array}$$

The only choice of $\hat{g} : C \rightarrow B^A$
making this commute is
defined:

$$g_c(a) = g(c, a) \quad \forall a \in A$$

$$\text{Then } \hat{g}(a) = g_c(a) \quad \forall c \in C$$

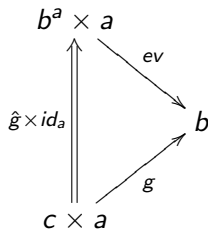
\mathcal{C} has **exponentiation** if $a, b \in \mathcal{C}^{ob} \implies$

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② $\exists b^a \in \mathcal{C}^{ob}$ and an **evaluation arrow** $ev : b^a \times a \rightarrow b$

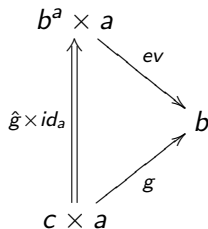
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This assignment of \hat{g} to g establishes a iso
 $\mathcal{C}(c \times a, b) \cong \mathcal{C}(c, b^a)$.

Subobjects vs. Subsets

- In category theory we don't have access to the elements directly since we only have objects and arrows.

$$A \subset B \quad \text{iff} \quad x \in A \implies x \in B$$

- How does one get at the notion of a subset without referring to the elements contained within each set?

Subobjects vs. Subsets

- In category theory we don't have access to the elements directly since we only have objects and arrows.

$$A \subset B \quad \text{iff} \quad x \in A \implies x \in B$$

- How does one get at the notion of a subset without referring to the elements contained within each set?
- The following logic begins to paint a neat picture:
 - If $A \subset B$ then $A \hookrightarrow B$ is injective, hence monic.
 - On the other hand any monic function $C \xrightarrow{f} B$ determines a subset of B .
 - In fact it creates a bijection $\text{Im}(f) \cong C$.
 - Hence the domain (of a monic arrow), up to isomorphism, is a subset of the codomain.
 - Should a subobject of an arrow g be any monic function that has the same codomain as g ?

- How do we make this analogy with subset stronger?
- We need to define subobject, the category equivalent of subset. But it might be the case that the subset of some set A is not even an object in our category!@!

Outline

Why Category
Theory?

Definition

Basic
Examples

Relations to
Set Theory

Topos
Definition and
Examples

Motivation
and History of
Topos theory

Acknowledgements

- How do we make this analogy with subset stronger?
- We need to define subobject, the category equivalent of subset. But it might be the case that the subset of some set A is not even an object in our category!@!
- Which pushes us in the direction of defining the equivalent notion of subset on the arrows.
- OK. Then we need to know (for $g \in \mathcal{C}^{\rightarrow}$) how to determine which $f \in \mathcal{C}^{\rightarrow}$ satisfy $f \supseteq g$ or $f \subseteq g$?
- And check that \subseteq makes $(\text{Sub}(D), \subseteq)$ a poset.

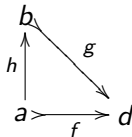
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Some interesting structure.

- The relation of set inclusion is a partial ordering on the power set $\mathcal{P}(D)$ of a set D .
- Hence $(\mathcal{P}(D), \subseteq)$ is a poset which is a category where there exists an arrow $A \rightarrow B$ iff $A \subseteq B$.
- When $A \subseteq B$, \exists a vertical arrow that makes the diagram commute

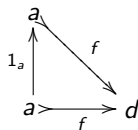
Subobject

- Suppose that we defined subobject of $d \in \mathcal{C}^{ob}$ to be a monic arrow with codomain d .
- Then consider the following definition of 'inclusion' relation between subobjects.
- Given $a \xrightarrow{f} d$, $b \xrightarrow{g} d$. $f \subseteq g$ iff $\exists h \in \mathcal{C}^{\rightarrow}$ s.t.

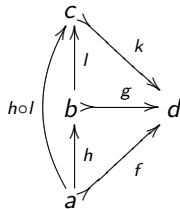


- We need to check that this definition is:

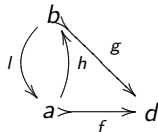
- Reflexive ($f \subseteq f$)



- Transitive ($f \subseteq g, g \subseteq k \implies f \subseteq k$)



- Antisymmetric ($f \cong g \implies f = g$)



Actual definition of Subobject

- The antisymmetric diagram does not guarantee that $f = g$ when it commutes only that $f \cong g$. Easy check: it might be the case that $a \neq b$.
- Modify the definition to make equality hold:

Actual definition of Subobject

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- Modify the definition to make equality hold:

A **subobject** of d is the *equivalence class* of all monic arrows with codomain d .

- This definition makes $(\text{Sub}(D), \subseteq)$ a poset.
- This definition corresponds to the usual subobjects (defined via elements) in the categories $\mathcal{R}ng, \mathcal{G}rp, \mathcal{A}b$ and $\mathcal{R} - \mathcal{M}od$ but apparently not in $\mathcal{T}op$.

- To each subset there is an associated characteristic function.
- Is the same true for every subobject?
- If so how do you define it categorially?

Properties of Power Set

- $\mathcal{P}(D) \cong 2^D$. There is a bijective correspondence between the functions $D \rightarrow 2$ and the subsets of D .
- You can get this by using the characteristic function χ_A of a subset A .

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

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$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

- The characteristic function tells us if a particular element $x \in X$ is also an element in A .
- The characteristic functions 'see' the elements.
- Is there a universal property that would allow us to define a similar function for category theory?

- It turns out that there is a universal property.
- Let $A_f = \{x : x \in D, f(x) = 1\} = f^{-1}(\{1\})$.
Then the set A_f arises by the pullback square below.

$$\begin{array}{ccc} A_f & \hookrightarrow & D \\ \downarrow & & \downarrow f \\ \{1\} & \hookrightarrow & 2 \end{array}$$

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- For a set $A \subset D$,

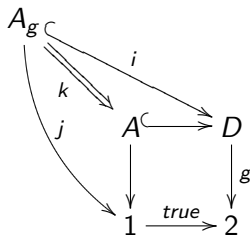
$$\begin{array}{ccc} A \hookrightarrow & D & \\ \downarrow & \chi_A \downarrow & \\ \{1\} \hookrightarrow & 2 & \end{array}$$

is a pullback square.

- The universal property is that χ_A is the only function that can make this a pullback square for the set A .

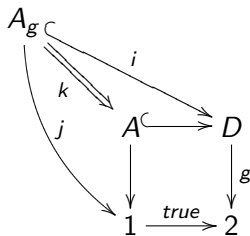
Proof:

- Suppose there was some other function g that would make the previous diagram a pullback. Then we get the following diagram



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- For $x \in A$, $g(x) = 1 \implies x \in A_g$. Thus, $A \subseteq A_g$.
- Since the outer square commutes and i and j are inclusions then so is k . Hence, $A_g \subseteq A$.
- But then f is the characteristic function of A therefore $f = \chi_A$.

Subobject Classifier

Given a category \mathcal{C} with initial object 1 . A **subobject classifier** for \mathcal{C} is $\Omega \in \mathcal{C}^{ob}$ together with a \mathcal{C} -arrow $true : 1 \rightarrow \Omega$ that satisfies the Ω -axiom below.

Ω -axiom: For each monic $a \xrightarrow{f} d$ there is one and only one $\chi_f : d \rightarrow \Omega$ s.t. the following is a pullback square.

$$\begin{array}{ccc}
 a & \xrightarrow{f} & d \\
 \downarrow & & \downarrow \chi_f \\
 1 & \xrightarrow{true} & \Omega
 \end{array}$$

An **Elementary Topos** is a category \mathcal{E} s.t.

- 1 \mathcal{E} is finitely complete.
- 2 \mathcal{E} is finitely co-complete.
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Also (1), (3), and (4) imply (2). So another way of defining a topos \mathcal{E} is to say that \mathcal{E} is cartesian closed category with a subobject classifier.

Topoi Examples

- *Set* - the prime example and the motivation for the concept in the first place
- *Finset* is a topos with limits, exponentials and $\top : 1 \rightarrow \Omega$ exactly as in *Set*.
- Set^2
- Set^{\rightarrow} the category of functions.

Some true statements about Topoi

- ~ 1963 Lawvere figured out new foundations for Mathematics based on category theory. (*i.e.* what is so great about sets?)
- Even earlier than this the concept of tracking structure of systems via category theory became important in algebraic geometry (via work by Grothendieck on the Weil conjectures).

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- So Topos theory was invented from a merger of ideas from geometry and logic.
- Topos is a category with some extra properties that make it look like Set .
- Many people study a space by studying the sheaves on that space. Apparently the main utility of a topos arises in situations in math where topological intuition is very effective but the space does not allow a topology. It is sometimes possible to formalize the intuition via a topos.

Some hand wavy connections with logic

- The exponential Ω^a is the topos analogue of 2^A in $\mathcal{S}et$.
- Some questions:
 - $2^A \cong \mathcal{P}(A)$ is the same true for Ω^a ? Is it the “power set” of a ?

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It turns out that it does behave similarly.

- 2^A logically represents all the mappings from A to the two point set $\{0, 1\}$, which can be thought of true and false. What does Ω^a represent?
- What if $\Omega = \{0, \frac{1}{2}, 1\}$?

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At a very basic level this means that if we regard a as ‘containing’ statements of some sort then there are more options than just true and false.

(*i.e.* the law of excluded middle does not hold!!)

Acknowledgements

I would like to thank:

- YOU !
- and the people that came to the last talk but found it too boring to come to this one.
- and the fact that there exists a truck banana category, without which I don't think that I could sleep at night.