Elastic Instability of Growing Cylindrical Vessels Immersed in Fluid

K. Simon T. Huynh*1,2, Thomas G. Fai**1,2,3

1Department of Mathematics, Brandeis University, Waltham MA 02453
2Materials Research Science and Engineering Center, Brandeis University, Waltham MA 02453
3Volen National Center for Complex Systems, Brandeis University, Waltham MA 02453

Motivation and Background

• Tortuous arteries and veins are commonly seen, but the underlying buckling mechanism is not well understood [2].

• Being induced to grow, pre-existing blood vessels can buckle and deform into spatially periodic structures [4].

Mathematical Model

Neo-Hookean Elastic Energy

• Treat the vessel as a 2D cylindrical shell embedded in a 3D space of fluid.

• The strain-energy density function $W$ is

$$W = \frac{\mu}{2} (I_1 - 3) + \frac{k}{2} (J - 1)^2,$$

where $\mu$ is the shear modulus, $k$ is the bulk modulus.

• Neo-Hookean energy $E_{\text{elastic}} = \int W \, dA$.

Bending Rigidity

• Bending energy on the shell is given by

$$E_{\text{bend}} = \frac{k_b}{2} \int_H H^2 dA,$$

where $k_b$ is the bending modulus, related to the vessel’s thickness $h$ by

$$k_b = \frac{Y h^3}{12 (1 - \nu^2)}.$$

$Y$ is the Young’s modulus, $\nu$ is the Poisson’s ratio.

• Total energy is $E = E_{\text{elastic}} + E_{\text{bend}}$.

Numerical Simulations

Discretization

• Surface triangulation of the 2D neo-Hookean shell.

• The elastic and bending energies on the shell are discretized following the discretization in [5].

• Solve the Fluid-structure Interactions with the Immersed Boundary Method [3].

Buckling Mechanism: Growth in Confined Space

• The vessel wants to grow and increase its surface area but is being restricted from lengthening.

• The shell is under compression, altering its original geometry [4].

Numerical Results and Discussion

• We recover the bending and barreling modes of buckling described in [1,4] and a qualitative result in [1]: as the vessels get thinner, it will transition from bending to barreling instability.

Outlooks:

• Analyze quantitatively the transition between the bending and barreling modes of buckling and compare it with the transition curves in [1].

• Introduce flow within the shell and study how deformation affects the fluid flow.

• Use a more realistic elastic model for the blood vessel’s wall such as the exponential Fung model.

References and Acknowledgments


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*stuynh@brandeis.edu, **tfai@brandeis.edu