

Notes for Math 261A  
 Lie groups and Lie algebras  
 June 28, 2006

**Contents**

Contents . . . . .	1
How these notes came to be . . . . .	3
Lecture 1 . . . . .	4
Lecture 2 . . . . .	6
Tangent Lie algebras to Lie groups, 6	
Lecture 3 . . . . .	9
Lecture 4 . . . . .	12
Lecture 5 . . . . .	16
Simply Connected Lie Groups, 16	
Lecture 6 - Hopf Algebras . . . . .	21
The universal enveloping algebra, 24	
Lecture 7 . . . . .	26
Universality of $U\mathfrak{g}$ , 26	
Gradation in $U\mathfrak{g}$ , 27	
Filtered spaces and algebras, 28	
Lecture 8 - PBW theorem, Deformations of Associative Algebras . . . . .	31
Deformations of associative algebras, 32	
Formal deformations of associative algebras, 34	
Formal deformations of Lie algebras, 35	
Lecture 9 . . . . .	36
Lie algebra cohomology, 37	
Lecture 10 . . . . .	41
Lie algebra cohomology, 41	
$H^2(\mathfrak{g}, \mathfrak{g})$ and Deformations of Lie algebras, 42	
Lecture 11 - Engel's Theorem and Lie's Theorem . . . . .	46
The radical, 51	
Lecture 12 - Cartan Criterion, Whitehead and Weyl Theorems . . . . .	52
Invariant forms and the Killing form, 52	
Lecture 13 - The root system of a semi-simple Lie algebra . . . . .	59
Irreducible finite dimensional representations of $\mathfrak{sl}(2)$ , 63	
Lecture 14 - More on Root Systems . . . . .	65
Abstract Root systems, 66	
The Weyl group, 68	
Lecture 15 - Dynkin diagrams, Classification of root systems . . . . .	71
Isomorphisms of small dimension, 76	

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Lecture 16 - Serre's Theorem . . . . .	78
Lecture 17 - Constructions of Exceptional simple Lie Algebras . . . . .	82
Lecture 18 - Representations . . . . .	87
Representations, 87	
Highest weights, 89	
Lecture 19 - The Weyl character formula . . . . .	93
The character formula, 94	
Lecture 20 . . . . .	100
Compact Groups, 102	
Lecture 21 . . . . .	105
What does a general Lie group look like?, 105	
Comparison of Lie groups and Lie algebras, 107	
Finite groups and Lie groups, 108	
Algebraic groups (over $\mathbb{R}$ ) and Lie groups, 109	
Lecture 22 - Clifford algebras and Spin groups . . . . .	111
Clifford Algebras and Spin Groups, 112	
Lecture 23 . . . . .	117
Clifford Groups, 118	
Lecture 24 . . . . .	123
Spin representations of Spin and Pin groups, 123	
Triality, 126	
More about Orthogonal groups, 127	
Lecture 25 - $E_8$ . . . . .	129
Lecture 26 . . . . .	134
Construction of $E_8$ , 136	
Lecture 27 . . . . .	140
Construction of the Lie group of $E_8$ , 143	
Real forms, 144	
Lecture 28 . . . . .	146
Working with simple Lie groups, 148	
Every possible simple Lie group, 150	
Lecture 29 . . . . .	152
Lecture 30 - Irreducible unitary representations of $SL_2(\mathbb{R})$ . . . . .	159
Finite dimensional representations, 159	
The Casimir operator, 162	
Lecture 31 - Unitary representations of $SL_2(\mathbb{R})$ . . . . .	165
Background about infinite-dimensional representations, 165	
The group $SL_2(\mathbb{R})$ , 166	
Solutions to (some) Exercises . . . . .	171
References . . . . .	176
Index . . . . .	178

## How these notes came to be

Among the Berkeley professors, there was once Alan Knutson, who would teach Math 261. But it happened that professor Knutson was on sabbatical elsewhere (I don't know where for sure), and eventually went there (or somewhere else) for good. During this turbulent time, Maths 261AB were cancelled two years in a row. The last of these four semesters (Spring 2006), some graduate students gathered together and asked Nicolai Reshetikhin to teach them Lie theory in a giant reading course. When the dust settled, there were two other professors willing to help in the instruction of Math 261A, Vera Serganova and Richard Borcherds. Thus Tag Team 261A was born.

After a few lectures, professor Reshetikhin suggested that the students write up the lecture notes for the benefit of future generations. The first four lectures were produced entirely by the “editors”. The remaining lectures were  $\text{\LaTeX}$ ed by Anton Geraschenko in class and then edited by the people in the following table. The columns are sorted by lecturer.

Nicolai Reshetikhin		Vera Serganova		Richard Borcherds	
1	Anton Geraschenko	11	Sevak Mkrtchyan	21	Hanh Duc Do
2	Anton Geraschenko	12	Jonah Blasiak	22	An Huang
3	Nathan George	13	Hannes Thiel	23	Santiago Canez
4	Hans Christianson	14	Anton Geraschenko	24	Lilit Martirosyan
5	Emily Peters	15	Lilit Martirosyan	25	Emily Peters
6	Sevak Mkrtchyan	16	Santiago Canez	26	Santiago Canez
7	Lilit Martirosyan	17	Katie Liesinger	27	Martin Vito-Cruz
8	David Cimasoni	18	Aaron McMillan	28	Martin Vito-Cruz
9	Emily Peters	19	Anton Geraschenko	29	Anton Geraschenko
10	Qingtau Chen	20	Hanh Duc Do	30	Lilit Martirosyan
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Richard Borcherds then edited the last third of the notes. The notes were further edited by Crystal Hoyt, Sevak Mkrtchyan, and Anton Geraschenko.

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## Lecture 1

**Definition 1.1.** A *Lie group* is a smooth manifold  $G$  with a group structure such that the multiplication  $\mu : G \times G \rightarrow G$  and inverse map  $\iota : G \rightarrow G$  are smooth maps.

► **Exercise 1.1.** If we assume only that  $\mu$  is smooth, does it follow that  $\iota$  is smooth?

**Example 1.2.** The group of invertible endomorphisms of  $\mathbb{C}^n$ ,  $GL_n(\mathbb{C})$ , is a Lie group. The automorphisms of determinant 1,  $SL_n(\mathbb{C})$ , is also a Lie group.

**Example 1.3.** If  $B$  is a bilinear form on  $\mathbb{C}^n$ , then we can consider the Lie group

$$\{A \in GL_n(\mathbb{C}) \mid B(Av, Aw) = B(v, w) \text{ for all } v, w \in \mathbb{C}^n\}.$$

If we take  $B$  to be the usual dot product, then we get the group  $O_n(\mathbb{C})$ . If we let  $n = 2m$  be even and set  $B(v, w) = v^T \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix} w$ , then we get  $Sp_{2m}(\mathbb{C})$ .

**Example 1.4.**  $SU_n \subseteq SL_n$  is a *real form* (look in lectures 27, 28, and 29 for more on real forms).

**Example 1.5.** We'd like to consider infinite matrices, but the multiplication wouldn't make sense, so we can think of  $GL_n \subseteq GL_{n+1}$  via  $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ , then define  $GL_\infty$  as  $\bigcup_n GL_n$ . That is, invertible infinite matrices which look like the identity almost everywhere.

Lie groups are hard objects to work with because they have global characteristics, but we'd like to know about representations of them. Fortunately, there are things called Lie algebras, which are easier to work with, and representations of Lie algebras tell us about representations of Lie groups.

**Definition 1.6.** A *Lie algebra* is a vector space  $V$  equipped with a *Lie bracket*  $[\cdot, \cdot] : V \times V \rightarrow V$ , which satisfies

1. Skew symmetry:  $[a, a] = 0$  for all  $a \in V$ , and
2. Jacobi identity:  $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$  for all  $a, b, c \in V$ .

A *Lie subalgebra* of a lie algebra  $V$  is a subspace  $W \subseteq V$  which is closed under the bracket:  $[W, W] \subseteq W$ .

**Example 1.7.** If  $A$  is a finite dimensional associative algebra, you can set  $[a, b] = ab - ba$ . If you start with  $A = \mathbb{M}_n$ , the algebra of  $n \times n$  matrices, then you get the Lie algebra  $\mathfrak{gl}_n$ . If you let  $A \subseteq \mathbb{M}_n$  be the algebra of matrices preserving a fixed flag  $V_0 \subset V_1 \subset \dots \subset V_k \subseteq \mathbb{C}^n$ , then you get *parabolic* Lie sub-algebras of  $\mathfrak{gl}_n$ .

**Example 1.8.** Consider the set of vector fields on  $\mathbb{R}^n$ ,  $\text{Vect}(\mathbb{R}^n) = \{\ell = \sum e^i(x) \frac{\partial}{\partial x_i} \mid [\ell_1, \ell_2] = \ell_1 \circ \ell_2 - \ell_2 \circ \ell_1\}$ .

► **Exercise 1.2.** Check that  $[\ell_1, \ell_2]$  is a first order differential operator.

**Example 1.9.** If  $A$  is an associative algebra, we say that  $\partial : A \rightarrow A$  is a derivation if  $\partial(ab) = (\partial a)b + a\partial b$ . Inner derivations are those of the form  $[d, \cdot]$  for some  $d \in A$ ; the others are called outer derivations. We denote the set of derivations of  $A$  by  $\mathcal{D}(A)$ , and you can verify that it is a Lie algebra. Note that  $\text{Vect}(\mathbb{R}^n)$  above is just  $\mathcal{D}(C^\infty(\mathbb{R}^n))$ .

The first Hochschild cohomology, denoted  $H^1(A, A)$ , is the quotient  $\mathcal{D}(A)/\{\text{inner derivations}\}$ .

**Definition 1.10.** A *Lie algebra homomorphism* is a linear map  $\phi : L \rightarrow L'$  that takes the bracket in  $L$  to the bracket in  $L'$ , i.e.  $\phi([a, b]_L) = [\phi(a), \phi(b)]_{L'}$ . A *Lie algebra isomorphism* is a morphism of Lie algebras that is a linear isomorphism.<sup>1</sup>

A very interesting question is to classify Lie algebras (up to isomorphism) of dimension  $n$  for a given  $n$ . For  $n = 2$ , there are only two: the trivial bracket  $[\cdot, \cdot] = 0$ , and  $[e_1, e_2] = e_2$ . For  $n = 3$ , it can be done without too much trouble. Maybe  $n = 4$  has been done, but in general, it is a very hard problem.

If  $\{e_i\}$  is a basis for  $V$ , with  $[e_i, e_j] = c_{ij}^k e_k$  (the  $c_{ij}^k$  are called the *structure constants* of  $V$ ), then the Jacobi identity is some quadratic relation on the  $c_{ij}^k$ , so the variety of Lie algebras is some quadratic surface in  $\mathbb{C}^{3n}$ .

Given a smooth real manifold  $M^n$  of dimension  $n$ , we can construct  $\text{Vect}(M^n)$ , the set of smooth vector fields on  $M^n$ . For  $X \in \text{Vect}(M^n)$ , we can define the *Lie derivative*  $L_X$  by  $(L_X \cdot f)(m) = X_m \cdot f$ , so  $L_X$  acts on  $C^\infty(M^n)$  as a derivation.

► **Exercise 1.3.** Verify that  $[L_X, L_Y] = L_X \circ L_Y - L_Y \circ L_X$  is of the form  $L_Z$  for a unique  $Z \in \text{Vect}(M^n)$ . Then we put a Lie algebra structure on  $\text{Vect}(M^n) = \mathcal{D}(C^\infty(M^n))$  by  $[X, Y] = Z$ .

There is a theorem (Ado's Theorem<sup>2</sup>) that any Lie algebra  $\mathfrak{g}$  is isomorphic to a Lie subalgebra of  $\mathfrak{gl}_n$ , so if you understand everything about  $\mathfrak{gl}_n$ , you're in pretty good shape.

<sup>1</sup>The reader may verify that this implies that the inverse is also a morphism of Lie algebras.

<sup>2</sup>Notice that if  $\mathfrak{g}$  has no center, then the adjoint representation  $ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  is already faithful. See Example 7.4 for more on the adjoint representation. For a proof of Ado's Theorem, see Appendix E of [FH91]

## Lecture 2

Last time we talked about Lie groups, Lie algebras, and gave examples. Recall that  $M \subseteq L$  is a Lie subalgebra if  $[M, M] \subseteq M$ . We say that  $M$  is a *Lie ideal* if  $[M, L] \subseteq M$ .

**Claim.** *If  $M$  is an ideal, then  $L/M$  has the structure of a Lie algebra such that the canonical projection is a morphism of Lie algebras.*

*Proof.* Take  $l_1, l_2 \in L$ , check that  $[l_1 + M, l_2 + M] \subseteq [l_1, l_2] + M$ .  $\square$

**Claim.** *For  $\phi : L_1 \rightarrow L_2$  a Lie algebra homomorphism,*

1.  $\ker \phi \subseteq L_1$  is an ideal,
2.  $\text{im } \phi \subseteq L_2$  is a Lie subalgebra,
3.  $L_1 / \ker \phi \cong \text{im } \phi$  as Lie algebras.

► **Exercise 2.1.** Prove this claim.

## Tangent Lie algebras to Lie groups

Let's recall some differential geometry. You can look at [Lee03] as a reference. If  $f : M \rightarrow N$  is a differentiable map, then  $df : TM \rightarrow TN$  is the derivative. If  $G$  is a group, then we have the maps  $l_g : x \mapsto gx$  and  $r_g : x \mapsto xg$ . Recall that a smooth vector field is a smooth section of the tangent bundle  $TM \rightarrow M$ .

**Definition 2.1.** A vector field  $X$  is *left invariant* if  $(dl_g) \circ X = X \circ l_g$  for all  $g \in G$ . The set of left invariant vector fields is called  $\text{Vect}_L(G)$ .

$$\begin{array}{ccc} TG & \xrightarrow{dl_g} & TG \\ X \uparrow & & \uparrow X \\ G & \xrightarrow{l_g} & G \end{array}$$

**Proposition 2.2.**  $\text{Vect}_L(G) \subseteq \text{Vect}(G)$  is a Lie subalgebra.

*Proof.* We get an induced map  $l_g^* : C^\infty(G) \rightarrow C^\infty(G)$ , and  $X$  is left invariant if and only if  $L_X$  commutes with  $l_g^*$ . Then

$X, Y$  left invariant  $\iff [L_X, L_Y]$  invariant  $\iff [X, Y]$  left invariant.  $\square$

All the same stuff works for right invariant vector fields  $\text{Vect}_R(G)$ .

**Definition 2.3.**  $\mathfrak{g} = \text{Vect}_L(G)$  is the tangent Lie algebra of  $G$ .

**Proposition 2.4.** *There are vector space isomorphisms  $\text{Vect}_L(G) \simeq T_e G$  and  $\text{Vect}_R(G) \simeq T_e G$ . Moreover, the Lie algebra structures on  $T_e G$  induced by these isomorphisms agree.*

Note that it follows that  $\dim \mathfrak{g} = \dim G$ .

*Proof.* Recall fibre bundles.  $dl_g : T_e G \xrightarrow{\sim} T_g G$ , so  $TG \simeq T_e \times G$ <sup>1</sup>.  $X$  is a section of  $TG$ , so it can be thought of as  $X : G \rightarrow T_e G$ , in which case the left invariant fields are exactly those which are constant maps, but the set of constants maps to  $T_e G$  is isomorphic to  $T_e G$ .  $\square$

If  $G$  is an  $n$ -dimensional  $C^\omega$  Lie group, then  $\mathfrak{g}$  is an  $n$ -dimensional Lie algebra. If we take local coordinates near  $e \in G$  to be  $x^1, \dots, x^n : U_e \rightarrow \mathbb{R}^n$  with  $m : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  the multiplication (defined near 0). We have a power series for  $m$  near 0,

$$m(x, y) = Ax + By + \alpha_2(x, y) + \alpha_3(x, y) + \dots$$

where  $A, B : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are linear,  $\alpha_i$  is degree  $i$ . Then we can consider the condition that  $m$  be associative (only to degree 3):  $m(x, m(y, z)) = m(m(x, y), z)$ .

$$\begin{aligned} m(x, m(y, z)) &= Ax + Bm(y, z) + \alpha_2(x, m(y, z)) + \alpha_3(x, m(y, z)) + \dots \\ &= Ax + B(Ay + Bz + \alpha_2(y, z) + \alpha_3(y, z)) \\ &\quad + \alpha_2(x, Ay + Bz + \alpha_2(y, z)) + \alpha_3(x, Ay + Bz) \\ m(m(x, y), z) &= \text{[[compute this stuff]]} \end{aligned}$$

Comparing first order terms (remember that  $A, B$  must be non-singular), we can get that  $A = B = I_n$ . From the second order term, we can get that  $\alpha_2$  is bilinear! Changing coordinates ( $\phi(x) = x + \phi_2(x) + \phi_3(x) + \dots$ , with  $\phi^{-1}(x) = x - \phi_2(x) + \phi_3(x) + \dots$ ), we use the fact that  $m_\phi(x, y) = \phi^{-1}m(\phi x, \phi y)$  is the new multiplication, we have

$$m_\phi(x, y) = x + y + \underbrace{(\phi_2(x) + \phi_2(y) + \phi_2(x + y))}_{\text{can be any symm form}} + \alpha_2(x, y) + \dots$$

so we can tweak the coordinates to make  $\alpha_2$  skew-symmetric. Looking at order 3, we have

$$\alpha_2(x, \alpha_2(y, z)) + \alpha_3(x, y + z) = \alpha_2(\alpha_2(x, y), z) + \alpha_3(x + y, z) \quad (*)$$

► **Exercise 2.2.** Prove that this implies the Jacobi identity for  $\alpha_2$ . (hint: skew-symmetrize  $(*)$ )

---

<sup>1</sup>is this true?

Remarkably, the Jacobi identity is the only obstruction to associativity; all other coefficients can be eliminated by coordinate changes.

**Example 2.5.** Let  $G$  be the set of matrices of the form  $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$  for  $a, b$  real,  $a > 0$ . Use coordinates  $x, y$  where  $e^x = a$ ,  $y = b$ , then

$$\begin{aligned} m((x, y), (x', y')) &= (x + x', e^x y' + y e^{-x'}) \\ &= (x + x', y + y' + \underbrace{(xy' - x'y)}_{skew} + \dots). \end{aligned}$$

The second order term is skew symmetric, so these are good coordinates. There are  $H, E \in T_e G$  corresponding to  $x$  and  $y$  respectively so that  $[H, E] = E^2$ .

► **Exercise 2.3.** Think about this. If  $a, b$  commute, then  $e^a e^b = e^{a+b}$ . If they do not commute, then  $e^a e^b = e^{f(a,b)}$ . Compute  $f(a, b)$  to order 3.

---

<sup>2</sup>what does this part mean?



## Lecture 3

Last time we saw how to get a Lie algebra  $\text{Lie}(G)$  from a Lie group  $G$ .

$$\text{Lie}(G) = \text{Vect}_L(G) \simeq \text{Vect}_R(G).$$

Let  $x^1, \dots, x^n$  be local coordinates near  $e \in G$ , and let  $m(x, y)^i$  be the  $i^{\text{th}}$  coordinate of  $(x, y) \mapsto m(x, y)$ . In this local coordinate system,  $m(x, y)^i = x^i + y^i + \frac{1}{2} \sum c_{jk}^i x^j y^k + \dots$ . If  $e_1, \dots, e_n \in T_e G$  is the basis induced by  $x^1, \dots, x^n$ , ( $e_i \sim \partial_i$ ), then

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

**Example 3.1.** Let  $G$  be  $GL_n$ , and let  $(g_{ij})$  be coordinates. Let  $X : GL_n \rightarrow TGL_n$  be a vector field.

$$\begin{aligned} L_X(f)(g) &= \sum_{i,j} X_{ij}(g) \frac{\partial f(g)}{\partial g_{ij}} \\ &\quad , \text{ where } L_X(l_h^*(f))(g) = \left\{ \begin{array}{l} l_h : g \mapsto hg \\ l_h^*(f)(g) = f(h^{-1}g) \end{array} \right\} \\ &= \sum_{i,j} X_{ij}(g) \frac{\partial f(h^{-1}g)}{\partial g_{ij}} = \sum_{i,j} X_{ij}(g) \frac{\partial (h^{-1}g)_{kl}}{\partial g_{ij}} \frac{\partial f(x)}{\partial x_k l} \Big|_{x=h^{-1}g} \\ &= \left\langle \frac{\partial (h^{-1}g)_{kl}}{\partial g_{ij}} = \sum_m (h^{-1})_{km} \underbrace{\frac{\partial g_{ml}}{\partial g_{ij}}}_{=\delta_{im}\delta_{lj}} \right\rangle = (h^{-1})_{ki} \delta_{jl} \\ &= \sum_{i,j,k} X_{ij}(g) (h^{-1})_{ki} \frac{\partial f}{\partial x_{kj}} \Big|_{x=h^{-1}g} \\ &= \sum_{j,k} \left( \sum_i (h^{-1})_{ki} X_{ij}(g) \right) \frac{\partial f}{\partial x_{kj}} \Big|_{x=h^{-1}g} \end{aligned}$$

If we want  $X$  to be left invariant,  $\sum_i (h^{-1})_{ki} X_{ij}(g) = X_{kj}(h^{-1}g)$ , then  $L_X(l_h^*(f)) = l_h^*(L_X(f))$ , (left invariance of  $X$ ).

**Example 3.2.** All solutions are  $X_{ij}(g) = (g \cdot M)_{ij}$ ,  $M$ -constant  $n \times n$  matrix. gives that left invariant vector fields on  $GL_n \approx n \times n$  matrices =  $\mathfrak{gl}_n$ . The “Natural Basis” is  $e_{ij} = (E_{ij})$ ,  $L_{ij} = \sum_m (g)_{mj} \frac{\partial}{\partial g_{mi}}$ .

**Example 3.3.** Commutation relations between  $L_{ij}$  are the same as commutation relations between  $e_{ij}$ .

Take  $\tau \in T_e G$ . Define the vector field:  $v_\tau : G \rightarrow TG$  by  $v_\tau(g) = dl_g(\tau)$ , where  $l_g : G \rightarrow G$  is left multiplication.  $v_\tau$  is a left invariant vector field by construction.

Consider  $\phi : I \rightarrow G$ ,  $\frac{d\phi(t)}{dt} = v_\tau(\phi(t))$ ,  $\phi(0) = e$ .

**Proposition 3.4.**

1.  $\phi(t+s) = \phi(t)\phi(s)$
2.  $\phi$  extends to a smooth map  $\phi : \mathbb{R} \rightarrow G$ .

*Proof.* 1. Fix  $s$  and  $\alpha(t) = \phi(s)\phi(t)$ ,  $\beta(t) = \phi(s+t)$ .

- $\alpha(0) = \phi(s) = \beta(0)$
- $\frac{d\beta(t)}{dt} = \frac{d\phi(s+t)}{dt} = v_\tau(\beta(t))$
- $\frac{d\alpha(t)}{dt} = \frac{d}{dt}(\phi(s)\phi(t)) = dl_{\phi(s)} \cdot v_\tau(\phi(t)) = v_\tau(\phi(s)\phi(t)) = v_\tau(\alpha(t))$ ,  
where the second equality is because  $v_\tau$  is linear.

$\implies \alpha$  satisfies same equation as  $\beta$  and same initial conditions, so by uniqueness, they coincide for  $|t| < \epsilon$ .

2. Now we have (1) for  $|t+s| < \epsilon$ ,  $|t| < \epsilon$ ,  $|s| < \epsilon$ . Then extend  $\phi$  to  $|t| < 2\epsilon$ . Continue this to cover all of  $\mathbb{R}$ .

□

This shows that for all  $\tau \in T_e G$ , we have a mapping  $\mathbb{R} \rightarrow G$  and its image is a 1-parameter (1-dimensional) Lie subgroup in  $G$ .

$$\begin{aligned} \exp : \mathfrak{g} = T_e G &\rightarrow G \\ \tau &\mapsto \phi_\tau(1) = \exp(\tau) \end{aligned}$$

Notice that  $\lambda\tau \mapsto \exp(\lambda\tau) = \phi_{\lambda\tau}(1) = \phi_\tau(\lambda)$

**Example 3.5.**  $GL_n$ ,  $\tau \in \mathfrak{gl}_n = T_e GL_n$ ,  $\frac{d\phi(t)}{dt} = v_\tau(\phi(t)) \in T_{\phi(t)} GL_n \simeq \mathfrak{gl}_n$ .

$$v_\tau(\phi(t)) = \phi(t) \cdot \tau, \quad \frac{d\phi(t)}{dt} = \phi(t) \cdot \tau, \quad \phi(0) = I,$$

$$\phi(t) = \exp(tI) = \sum_{n=0}^{\infty} \frac{t^n \tau^n}{n!}$$

$\exp : \mathfrak{gl}_n \rightarrow GL_n$

$$\left[ L_{\gamma(0)=g}(f)(g) = \frac{d}{dt} f(\gamma(t))|_{t=0} \right]$$

**Baker-Campbell-Hausdorff formula:**

$$e^X \cdot e^Y = e^{H(X,Y)}$$

$$H(X, Y) = \underbrace{X + Y}_{\text{sym}} + \underbrace{\frac{1}{2}[X, Y]}_{\text{skew}} + \underbrace{\frac{1}{12}([X[X, Y]] + [Y[Y, X]])}_{\text{symmetric}} + \dots$$

**Proposition 3.6.** 1. Let  $f : G \rightarrow H$  be a Lie group homomorphism,

then the diagram  $G \xrightarrow{f} H$  is commutative.

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \exp \uparrow & & \uparrow \exp \\ \text{Lie}(G) & \xrightarrow{df_e} & \text{Lie}(H) \end{array}$$

2. If  $G$  is connected, then  $(df)_e$  defines the Lie group homomorphism  $f$  uniquely.

*Proof.* Next time. □

**Proposition 3.7.**  $G, H$  Lie groups,  $G$  simply connected, then  $\alpha : \text{Lie}(G) \rightarrow \text{Lie}(H)$  is a Lie algebra homomorphism if and only if there is a Lie group homomorphism  $A : G \rightarrow H$  lifting  $\alpha$ .

*Proof.* Next time. □

$\{\text{Lie algebras}\} \xrightarrow{\text{exp}} \{\text{Lie groups (connected, simply connected)}\}$  is an equivalence of categories.

## Lecture 4

**Theorem 4.1.** *Suppose  $G$  is a topological group. Let  $U \subset G$  be an open neighbourhood of  $e \in G$ . If  $G$  is connected, then*

$$G = \bigcup_{n \geq 1} U^n.$$

*Proof.* Choose a non-empty open set  $V \subset U$  such that  $V = V^{-1}$ , for example  $V = U \cap U^{-1}$ . Define  $H = \bigcup_{n \geq 1} V^n$ , and observe  $H$  is an abstract subgroup, since  $V^n V^m \subseteq V^{n+m}$ .  $H$  is open since it is the union of open sets. If  $\sigma \notin H$ , then  $\sigma H \not\subseteq H$ , since otherwise if  $h_1, h_2 \in H$  satisfy  $\sigma h_1 = h_2$ , then  $\sigma = h_2 h_1^{-1} \in H$ . Thus  $H$  is a complement of the union of all cosets not containing  $H$ . Hence  $H$  is closed. Since  $G$  is connected,  $H = G$ .  $\square$

**Theorem 4.2.** *Let  $f : G \rightarrow H$  be a Lie group homomorphism. Then the following diagram commutes:*

$$\begin{array}{ccc} T_e & \xrightarrow{(df)_e} & T_e H \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{f} & H \end{array}$$

*Further, if  $G$  is connected,  $(df)_e$  determines  $f$  uniquely.*

*Proof.* 1) Commutative diagram. Fix  $\tau \in T_e G$  and set  $\eta = df_e \tau \in T_e H$ . Recall we defined the vector field  $V_\tau(g) = (dl_g)(\tau)$ , then if  $\phi(t)$  solves

$$\frac{d\phi}{dt} = V_\tau(\phi(t)) \in T_{\phi(t)} G,$$

we have  $\exp(\tau) = \phi(1)$ . Let  $\psi$  solve

$$\frac{d\psi}{dt} = V_\eta(\psi(t)),$$

so that  $\exp(\eta) = \psi(1)$ . Observe  $\tilde{\psi}(t) = f(\phi(t))$  satisfies

$$\frac{d\tilde{\psi}}{dt} = (df) \left( \frac{d\phi}{dt} \right) = V_\eta(\tilde{\psi}),$$

so by uniqueness of solutions to ordinary differential equations,  $\psi = \tilde{\psi}$ .

2) Uniqueness of  $f$ . The exponential map is an isomorphism of a neighborhood of  $0 \in \mathfrak{g}$  and a neighborhood of  $e \in G$ . But if  $G$  is connected,  $G = \bigcup_{n \geq 1} (\text{nbhd } e)^n$ .  $\square$

**Theorem 4.3.** *Suppose  $G$  is a topological group, with  $G^0 \subset G$  the connected component of  $e$ . Then 1)  $G^0$  is normal and 2)  $G/G^0$  is discrete.*

*Proof.* 2)  $G^0 \subset G$  is open implies  $\text{pr}^{-1}([e]) = eG^0$  is open in  $G$ , which in turn implies  $\text{pr}^{-1}([g]) \in G/G^0$  is open for every  $g \in G$ . Thus each coset is both open and closed, hence  $G/G^0$  is discrete.

1) Fix  $g \in G$  and consider the map  $G \rightarrow G$  defined by  $x \mapsto gxg^{-1}$ . This map fixes  $e$  and is continuous, which implies it maps  $G^0$  into  $G^0$ . In other words,  $gG^0g^{-1} \subset G^0$ , or  $G^0$  is normal.  $\square$

We recall some basic notions of algebraic topology. Suppose  $M$  is a connected topological space. Let  $x, y \in M$ , and suppose  $\gamma(t) : [0, 1] \rightarrow M$  is a path from  $x$  to  $y$  in  $M$ . We say  $\tilde{\gamma}(t)$  is *homotopic* to  $\gamma$  if there is a continuous map  $h(s, t) : [0, 1]^2 \rightarrow M$  satisfying

$$\begin{aligned} \bullet h(s, 0) &= x, \quad h(s, 1) = y \\ \bullet h(0, t) &= \gamma(t), \quad h(1, t) = \tilde{\gamma}(t). \end{aligned}$$

We call  $h$  the *homotopy*. On a smooth manifold, we may replace  $h$  with a smooth homotopy. Now fix  $x_0 \in M$ . We define the first fundamental group of  $M$

$$\pi_1(M, x_0) = \{ \text{homotopy classes of loops based at } x_0 \}.$$

It is clear that this is a group with group multiplication composition of paths. It is also a fact that the definition does not depend on the base point  $x_0$ :

$$\pi_1(M, x_0) \simeq \pi_1(M, x'_0).$$

By  $\pi_1(M)$  we denote the isomorphism class of  $\pi_1(M, \cdot)$ . Lastly, we say  $M$  is *simply connected* if  $\pi_1(M) = \{e\}$ , that is if all closed paths can be deformed to the trivial one.

**Theorem 4.4.** *Suppose  $G$  and  $H$  are Lie groups with Lie algebras  $\mathfrak{g}, \mathfrak{h}$  respectively. If  $G$  is simply connected, then any Lie algebra homomorphism  $\rho : \mathfrak{g} \rightarrow \mathfrak{h}$  lifts to a Lie group homomorphism  $R : G \rightarrow H$ .*

In order to prove this theorem, we will need the following lemma.

**Lemma 4.5.** *Let  $\xi : \mathbb{R} \rightarrow \mathfrak{g}$  be a smooth mapping. Then*

$$\frac{dg}{dt} = (dl_g)(\xi(t))$$

*has a unique solution on all of  $\mathbb{R}$  with  $g(t_0) = g_0$ .*

For convenience, we will write  $g\xi := (dl_g)(\xi)$ .

*Proof.* Since  $\mathfrak{g}$  is a vector space, we identify it with  $\mathbb{R}^n$  and for sufficiently small  $r > 0$ , we identify  $B_r(0) \subset \mathfrak{g}$  with a small neighbourhood of  $e$ ,  $U_e(r) \subset G$ , under the exponential map. Here  $B_r(0)$  is measured with the usual Euclidean norm  $\|\cdot\|$ . Note for any  $g \in U_e(r)$  and  $|t - t_0|$  sufficiently small, we have  $\|g\xi(t)\| \leq C$ . Now according to Exercise 4.1, the solution with  $g(t_0) = e$  exists for sufficiently small  $|t - t_0|$  and

$$g(t) \in U_e(r) \quad \forall |t - t_0| < \frac{r}{C'}.$$

Now define  $h(t) = g(t)g_0$  so that  $h(t) \in U_{g_0}(r)$  for  $|t - t_0| < r/C'$ . That is,  $r$  and  $C'$  do not depend on the choice of initial conditions, and we can cover  $\mathbb{R}$  by intervals of length, say  $r/C'$ .  $\square$

► **Exercise 4.1.** Verify that there is a constant  $C'$  such that if  $|t - t_0|$  is sufficiently small, we have

$$\|g(t)\| \leq C'|t - t_0|.$$

*Proof of Theorem 4.4.* We will construct  $R : G \rightarrow H$ . Beginning with  $g(t) : [0, 1] \rightarrow G$  satisfying  $g(0) = e$ ,  $g(1) = g$ , define  $\xi(t) \in \mathfrak{g}$  for each  $t$  by

$$g(t)\xi(t) = \frac{d}{dt}g(t).$$

Let  $\eta(t) = \rho(\xi(t))$ , and let  $h(t) : [0, 1] \rightarrow H$  satisfy

$$\frac{d}{dt}h(t) = h(t)\eta(t), \quad h(0) = e.$$

Define  $R(g) = h(1)$ .

**Claim:**  $h(1)$  does not depend on the path  $g(t)$ , only on  $g$ .

*Proof of Claim.* Suppose  $g_1(t)$  and  $g_2(t)$  are two different paths connecting  $e$  to  $g$ . Then there is a smooth homotopy  $g(t, s)$  satisfying  $g(t, 0) = g_1(t)$ ,  $g(t, 1) = g_2(t)$ . Define  $\xi(t, s)$  and  $\eta(t, s)$  by

$$\begin{aligned} \frac{\partial g}{\partial t} &= g(t, s)\xi(t, s); \\ \frac{\partial g}{\partial s} &= g(t, s)\eta(t, s). \end{aligned}$$

Observe

$$\frac{\partial^2 g}{\partial s \partial t} = g\eta \circ \xi + g \frac{\partial \xi}{\partial t} \quad \text{and} \quad (1)$$

$$\frac{\partial^2 g}{\partial t \partial s} = g\xi \circ \eta + g \frac{\partial \eta}{\partial s}, \quad (2)$$

and (1) is equal to (2) since  $g$  is smooth. Consequently

$$\frac{\partial \eta}{\partial t} - \frac{\partial \xi}{\partial s} = [\eta, \xi].$$

Now define an  $s$  dependent family of solutions  $h(\cdot, s)$  to the equations

$$\frac{\partial h}{\partial t}(t, s) = h(t, s)\rho(\xi(t, s)), \quad h(0, s) = e.$$

Define  $\theta(t, s)$  by

$$\begin{cases} \frac{\partial \theta}{\partial t} - \frac{\partial \rho(\xi)}{\partial s} = [\rho(\xi), \theta], \\ \theta(0, s) = 0. \end{cases} \quad (3)$$

Observe  $\tilde{\theta}(t, s) = \rho(\eta(t, s))$  also satisfies equation (3), so that  $\theta = \tilde{\theta}$  by uniqueness of solutions to ODEs. Finally,

$$g\eta(1, s) = \frac{\partial g}{\partial s}(1, s) = 0 \implies \theta(1, s) = 0 \implies \frac{\partial h}{\partial s}(1, s) = 0,$$

justifying the claim.

We need only show  $R : G \rightarrow H$  is a homomorphism. Let  $g_1, g_2 \in G$  and set  $g = g_1 g_2$ . Let  $\tilde{g}_i(t)$  be a path from  $e$  to  $g_i$  in  $G$  for each  $i = 1, 2$ . Then the path  $\tilde{g}(t)$  defined by

$$\tilde{g}(t) = \begin{cases} \tilde{g}_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ g_1 \tilde{g}_2(2t - 1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

goes from  $e$  to  $g$ . Let  $\tilde{h}_i$  for  $i = 1, 2$  and  $\tilde{h}$  be the paths in  $H$  corresponding to  $\tilde{g}_1, \tilde{g}_2$ , and  $\tilde{g}$  respectively and calculate

$$R(g_1 g_2) = R(g) = \tilde{h}(1) = \tilde{h}_1(1) \tilde{h}_2(1) = R(g_1) R(g_2).$$

□

## Lecture 5

Last time we talked about connectedness, and proved the following things:

- Any connected topological group  $G$  has the property that  $G = \bigcup_n V^n$ , where  $V$  is any neighborhood of  $e \in G$ .
- If  $G$  is a connected Lie group, with  $\alpha : \text{Lie}(G) \rightarrow \text{Lie}(H)$  a Lie algebra homomorphism, then if there exists  $f : G \rightarrow H$  with  $df_e = \alpha$ , it is unique.
- If  $G$  is connected, simply connected, with  $\alpha : \text{Lie}(G) \rightarrow \text{Lie}(H)$  a Lie algebra homomorphism, then there is a unique  $f : G \rightarrow H$  such that  $df_e = \alpha$ .

### Simply Connected Lie Groups

The map  $p$  in  $Z \rightarrow X \xrightarrow{p} Y$  is a *covering map* if it is a locally trivial fiber bundle with discrete fiber  $Z$ . Locally trivial means that for any  $y \in Y$  there is a neighborhood  $U$  such that if  $f : U \times Z \rightarrow Z$  is the map defined by  $f(u, z) = u$ , then the following diagram commutes:

$$\begin{array}{ccc}
 p^{-1}U & \simeq & U \times Z \\
 \downarrow & \swarrow f & \\
 Y \supseteq U & & 
 \end{array}$$

The exact sequence defined below is an important tool. Suppose we have a locally trivial fiber bundle with fiber  $Z$  (not necessarily discrete), with  $X, Y$  connected. Choose  $x_0 \in X, z_0 \in Z, y_0 \in Y$  such that  $p(x_0) = y_0$ , and  $i : Z \rightarrow p^{-1}(y_0)$  is an isomorphism such that  $i(z_0) = x_0$ :

$$\begin{array}{ccc}
 x_0 \in \pi^{-1}(y_0) & \xleftarrow{i} & Z \\
 \downarrow & & \\
 y_0 & & 
 \end{array}$$

We can define  $p_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$  in the obvious way ( $\pi_1$  is a functor). Also define  $i_* : \pi_1(Z, z_0) \rightarrow \pi_1(X, x_0)$ . Then we can define  $\partial : \pi_1(Y, y_0) \rightarrow \pi_0(Z) = \{\text{connected components of } Z\}$  by taking a loop  $\gamma$  based at  $y_0$  and lifting it to some path  $\tilde{\gamma}$ . This path is not unique, but up to fiber-preserving homotopy it is. The new path  $\tilde{\gamma}$  starts at  $x_0$  and ends at  $x'_0$ . Then we define  $\partial$  to be the map associating the connected component of  $x'_0$  to the homotopy class of  $\gamma$ .



**Claim.** *The following sequence is exact:*

$$\pi_1(Z, z_0) \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{p_*} \pi_1(Y, y_0) \xrightarrow{\partial} \pi_0(Z) \longrightarrow \{0\}$$

1.  $\text{im } i_* = \ker p_*$
2.  $\{\text{fibers of } \partial\} \simeq \pi_1(Y, y_0) / \text{im } p_*$
3.  $\partial$  is surjective.

*Proof.*

1.  $\ker p_*$  is the set of all loops which map to contractible loops in  $Y$ , which are loops which are homotopic to a loop in  $\pi^{-1}(y_0)$  based at  $x_0$ . These are exactly the loops of  $\text{im } i_*$ .
2. The fiber of  $\partial$  over the connected component  $Z_z \subseteq Z$  is the set of all (homotopy classes of) loops in  $Y$  based at  $y_0$  which lift to a path connecting  $x_0$  to a point in the connected component of  $\pi^{-1}(y_0)$  containing  $i(Z_z)$ . If two loops  $\beta, \gamma$  based at  $y_0$  are in the same fiber, homotope them so that they have the same endpoint. Then  $\tilde{\gamma}\tilde{\beta}^{-1}$  is a loop based at  $x_0$ . So fibers of  $\partial$  are in one to one correspondence with loops in  $Y$  based at  $y_0$ , modulo images of loops in  $X$  based at  $x_0$ , which is just  $\pi_1(Y, y_0) / \text{im } p_*$ .
3. This is obvious, since  $X$  is connected. □

Now assume we have a covering space with discrete fiber, i.e. maps

$$\begin{array}{ccc} X & \longleftarrow & Z \\ & & \downarrow p \\ & & Y \end{array}$$

such that  $\pi_1(Z, z_0) = \{e\}$  and  $\pi_0(Z) = Z$ . Then we get the sequence

$$\{e\} \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{p_*} \pi_1(Y, y_0) \xrightarrow{\partial} Z \longrightarrow \{0\}$$

and since  $p_*$  is injective,  $Z = \pi_1(Y) / \pi_1(X)$ .

Classifying all covering spaces of  $Y$  is therefore the same as describing all subgroups of  $\pi_1(Y)$ . The *universal cover* of  $Y$  is the space  $\tilde{Y}$  such that  $\pi_1(\tilde{Y}) = \{e\}$ , and for any other covering  $X$ , we get a factorization of covering maps  $\tilde{Y} \xrightarrow{f} X \xrightarrow{p} Y$ .

We construct  $\tilde{X}$ , the universal cover, in the following way: fix  $x_0 \in X$ , and define  $\tilde{X}_{x_0}$  to be the set of basepoint-fixing homotopy classes of paths

connecting  $x_0$  to some  $x \in X$ . We have a natural projection  $[\gamma_{x_0, x}] \mapsto x$ , and the fiber of this projection (over  $x_0$ ) can be identified with  $\pi_1(X, x_0)$ . It is clear for any two basepoints  $x_0$  and  $x'_0$ ,  $\tilde{X}_{x_0} \simeq \tilde{X}_{x'_0}$  via any path  $\gamma_{x_0, x'_0}$ . So we have

$$\begin{array}{ccc} \tilde{X}_{x_0} & \longleftarrow & \pi_1(X) \\ \downarrow p & & \\ X & & \end{array}$$

**Claim.**  $\tilde{X}_{x_0}$  is simply connected.

*Proof.* We need to prove that  $\pi_1(\tilde{X}_{x_0})$  is trivial, but we know that the fibers of  $p$  can be identified with both  $\pi_1(X)$  and  $\pi_1(X)/\pi_1(\tilde{X}_{x_0})$ , so we're done.  $\square$

Let  $G$  be a connected Lie group. We would like to produce a simply connected Lie group which also has the Lie algebra  $\text{Lie}(G)$ . It turns out that the obvious candidate,  $\tilde{G}_e$ , is just what we are looking for. It is not hard to see that  $\tilde{G}_e$  is a smooth manifold (typist's note: it is not that easy either. See [Hat02], pp. 64-65, for a description of the topology on  $\tilde{G}_e$ . Once we have a topology and a covering space map, the smooth manifold structure of  $G$  lifts to  $\tilde{G}_e$ . – Emily). We show it is a group as follows.

Write  $\gamma_g$  for  $\gamma : [0, 1] \rightarrow G$  with endpoints  $e$  and  $g$ . Define multiplication by  $[\gamma_g][\gamma'_h] := [\{\gamma_g(t)\gamma'_h(t)\}_{t \in [0, 1]}]$ . The unit element is the homotopy class of a contractible loop, and the inverse is given by  $[\{\gamma(t)^{-1}\}_{t \in [0, 1]}]$ .

**Claim.**

1.  $\tilde{G} = \tilde{G}_e$  is a group.
2.  $p : \tilde{G} \rightarrow G$  is a group homomorphism.
3.  $\pi_1(G) \subseteq \tilde{G}$  is a normal subgroup.
4.  $\text{Lie}(\tilde{G}) = \text{Lie}(G)$ .
5.  $\tilde{G} \rightarrow G$  is the universal cover (i.e.  $\pi_1(G)$  is discrete).

*Proof.* 1. Associativity is inherited from associativity in  $G$ , composition with the identity does not change the homotopy class of a path, and the product of an element and its inverse is the identity.

2. This is clear, since  $p([\gamma_g][\tilde{\gamma}_h]) = gh$ .

3. We know  $\pi_1(G) = \ker p$ , and kernels of homomorphisms are normal.

4. The topology on  $\tilde{G}$  is induced by the topology of  $G$  in the following way: If  $\mathcal{U}$  is a basis for the topology on  $G$  then fix a path  $\gamma_{e,g}$  for all  $g \in G$ . Then  $\tilde{\mathcal{U}} = \{\tilde{U}_{\gamma_{e,g}}\}$  is a basis for the topology on  $\tilde{G}$  with  $\tilde{U}_{\gamma_{e,g}}$  defined to be the set of paths of the form  $\gamma_{e,g}^{-1}\beta\gamma_{e,g}$  with  $\beta$  a loop based at  $g$  contained entirely in  $U$ .

Now take  $U$  a connected, simply connected neighborhood of  $e \in G$ . Since all paths in  $U$  from  $e$  to a fixed  $g \in G$  are homotopic, we have that  $U$  and  $\tilde{U}$  are diffeomorphic and isomorphic, hence Lie isomorphic. Thus  $\text{Lie}(\tilde{G}) = \text{Lie}(G)$ .

5. As established in (4),  $G$  and  $\tilde{G}$  are diffeomorphic in a neighborhood of the identity. Thus all points  $x \in p^{-1}(e)$  have a neighborhood which does not contain any other inverse images of  $e$ , so  $p^{-1}(e)$  is discrete; and  $p^{-1}(e)$  and  $\pi_1(G)$  are isomorphic. □

We have that for any Lie group  $G$  with a given Lie algebra  $\text{Lie}(G) = \mathfrak{g}$ , there exists a simply connected Lie group  $\tilde{G}$  with the same Lie algebra, and  $\tilde{G}$  is the universal cover of  $G$ .

**Lemma 5.1.** *A discrete normal subgroup  $H \subseteq G$  of a connected topological group  $G$  is always central.*

*Proof.* Consider the map  $\phi_h : G \rightarrow H, g \mapsto ghg^{-1}h^{-1}$ , which is continuous. Since  $G$  is connected, the image is also connected, but  $H$  is discrete, so the image must be a point. In fact, it must be  $e$  because  $\phi_h(h) = e$ . So  $H$  is central. □

**Corollary 5.2.**  *$\pi_1(G)$  is central, because it is normal and discrete. In particular,  $\pi_1(G)$  is commutative.*

**Corollary 5.3.**  *$G \simeq \tilde{G}/\pi_1(G)$ , with  $\pi_1(G)$  discrete central.*

The following corollary describes all (connected) Lie groups with a given Lie algebra.

**Corollary 5.4.** *Given a Lie algebra  $\mathfrak{g}$ , take  $\tilde{G}$  with Lie algebra  $\mathfrak{g}$ . Then any other connected  $G$  with Lie algebra  $\mathfrak{g}$  is a quotient of  $\tilde{G}$  by a discrete central subgroup of  $\tilde{G}$ .*

Suppose  $G$  is a topological group and  $G^0$  is a connected component of  $e$ .

**Claim.**  *$G^0 \subseteq G$  is a normal subgroup, and  $G/G^0$  is a discrete group.*

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If we look at  $\{\text{Lie groups}\} \rightarrow \{\text{Lie algebras}\}$ , we have an “inverse” given by exponential:  $\exp(\mathfrak{g}) \subseteq G$ . Then  $G^0 = \bigcup_n (\exp \mathfrak{g})^n$ . So for a given Lie algebra, we can construct a well-defined isomorphism class of connected, simply connected Lie groups. When we say “take a Lie group with this Lie algebra”, we mean to take the connected, simply connected one.

**Coming Attractions:** We will talk about  $U\mathfrak{g}$ , the universal enveloping algebra,  $C(G)$ , the Hopf algebra, and then we’ll do classification of Lie algebras.

## Lecture 6 - Hopf Algebras

Last time: We showed that a finite dimensional Lie algebra  $\mathfrak{g}$  uniquely determines a connected simply connected Lie group. We also have a “map” in the other direction (taking tangent spaces). So we have a nice correspondence between Lie algebras and connected simply connected Lie groups.

There is another nice kind of structure: Associative algebras. How do these relate to Lie algebras and groups?

Let  $\Gamma$  be a finite group and let  $\mathbb{C}[\Gamma] := \{\sum_g c_g g | g \in \Gamma, c_g \in \mathbb{C}\}$  be the  $\mathbb{C}$  vector space with basis  $\Gamma$ . We can make  $\mathbb{C}[\Gamma]$  into an associative algebra by taking multiplication to be the multiplication in  $\Gamma$  for basis elements and linearly extending this to the rest of  $\mathbb{C}[\Gamma]$ .<sup>1</sup>

*Remark 6.1.* Recall that the tensor product  $V$  and  $W$  is the linear span of elements of the form  $v \otimes w$ , modulo some linearity relations. If  $V$  and  $W$  are infinite dimensional, we will look at the *algebraic tensor product* of  $V$  and  $W$ , i.e. we only allow finite sums of the form  $\sum a_i \otimes b_i$ .

We have the following maps

Comultiplication:  $\Delta : \mathbb{C}[\Gamma] \rightarrow \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma]$ , given by  $\Delta(\sum x_g g) = \sum x_g g \otimes g$

Counit:  $\varepsilon : \mathbb{C}[\Gamma] \rightarrow \mathbb{C}$ , given by  $\varepsilon(\sum x_g g) = \sum x_g$ .

Antipode:  $S : \mathbb{C}[\Gamma] \rightarrow \mathbb{C}[\Gamma]$  given by  $S(\sum x_g g) = \sum x_g g^{-1}$ .

You can check that

- $\Delta(xy) = \Delta(x)\Delta(y)$  (i.e.  $\Delta$  is an algebra homomorphism),
- $(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta$ . (follows from the associativity of  $\otimes$ ),
- $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$  (i.e.  $\varepsilon$  is an algebra homomorphism),
- $S(xy) = S(y)S(x)$  (i.e.  $S$  is an algebra antihomomorphism).

Consider

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta} \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \xrightarrow{S \otimes \text{Id}, \text{Id} \otimes S} \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \xrightarrow{m} \mathbb{C}[\Gamma].$$

You get

$$m(S \otimes \text{Id})\Delta(g) = m(g^{-1} \otimes g) = e$$

so the composition sends  $\sum x_g g$  to  $(\sum_g x_g)e = \varepsilon(x)1_A$ .

So we have

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<sup>1</sup>“If somebody speaks Danish, I would be happy to take lessons.”

1.  $A = \mathbb{C}[\Gamma]$  an associative algebra with  $1_A$
2.  $\Delta : A \rightarrow A \otimes A$  which is coassociative and is a homomorphism of algebras
3.  $\varepsilon : A \rightarrow \mathbb{C}$  an algebra homomorphism, with  $(\varepsilon \otimes \text{Id})\Delta = (\text{Id} \otimes \varepsilon)\Delta = \text{Id}$ .

**Definition 6.2.** Such an  $A$  is called a *bialgebra*, with comultiplication  $\Delta$  and counit  $\varepsilon$ .

We also have  $S$ , the antipode, which is an algebra anti-automorphism, so it is a linear isomorphism with  $S(ab) = S(b)S(a)$ , such that

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow[\text{Id} \otimes S]{S \otimes \text{Id}} & A \otimes A \\
 \Delta \uparrow & & \downarrow m \\
 A & \xrightarrow{\varepsilon} \mathbb{C} \xrightarrow{1_A} & A
 \end{array}$$

**Definition 6.3.** A bialgebra with an antipode is a *Hopf algebra*.

If  $A$  is finite dimensional, let  $A^*$  be the dual vector space. Define the multiplication,  $\Delta_*$ ,  $S_*$ ,  $\varepsilon_*$ ,  $1_{A^*}$  on  $A^*$  in the following way:

- $lm(a) := (l \otimes m)(\Delta a)$  for all  $l, m \in A^*$
- $\Delta_*(l)(a \otimes b) := l(ab)$
- $S_*(l)(a) := l(S(a))$
- $\varepsilon_*(l) := l(1_A)$
- $1_{A^*}(a) := \varepsilon(a)$

**Theorem 6.4.**  $A^*$  is a Hopf algebra with this structure, and we say it is dual to  $A$ . If  $A$  is finite-dimensional, then  $A^{**} = A$ .

► **Exercise 6.1.** Prove it.

We have an example of a Hopf algebra ( $\mathbb{C}[\Gamma]$ ), what is the dual Hopf algebra?<sup>2</sup> Let's compute  $A^* = \mathbb{C}[\Gamma]^*$ .

Well,  $\mathbb{C}[\Gamma]$  has a basis  $\{g \in \Gamma\}$ . Let  $\{\delta_g\}$  be the dual basis, so  $\delta_g(h) = 0$  if  $g \neq h$  and 1 if  $g = h$ . Let's look at how we multiply such things

- $\delta_{g_1} \delta_{g_2}(h) = (\delta_{g_1} \otimes \delta_{g_2})(h \otimes h) = \delta_{g_1}(h) \delta_{g_2}(h)$ .
- $\Delta_*(\delta_g)(h_1 \otimes h_2) = \delta_g(h_1 h_2)$

<sup>2</sup> If you want to read more, look at S. Montgomery's *Hopf algebras*, AMS, early 1990s. [Mon93]

- $S_*(\delta_g)(h) = \delta_g(h^{-1})$
- $\varepsilon_*(\delta_g) = \delta_g(e) = \delta_{g,e}$
- $1_{A^*}(h) = 1.$

It is natural to think of  $A^*$  as the set of functions  $\Gamma \rightarrow \mathbb{C}$ , where  $(\sum x_g \delta_g)(h) = \sum x_g \delta_g(h)$ . Then we can think about functions

- $(f_1 f_2)(h) = f_1(h) f_2(h)$
- $\Delta_*(f)(h_1 \times h_2) = f(h_1 h_2)$
- $S_*(f)(h) = f(h^{-1})$
- $\varepsilon_*(f) = f(e)$
- $1_{A^*} = 1$  constant.

So this is the Hopf algebra  $C(\Gamma)$ , the space of functions on  $\Gamma$ . If  $\Gamma$  is any affine algebraic group, then  $C(\Gamma)$  is the space of polynomial functions on  $\Gamma$ , and all this works. The only concern is that we need  $C(\Gamma \times \Gamma) \cong C(\Gamma) \otimes C(\Gamma)$ , which we only have in the finite dimensional case; you have to take completions of tensor products otherwise.

So we have the notion of a bialgebra (and duals), and the notion of a Hopf algebra (and duals). We have two examples:  $A = \mathbb{C}[\Gamma]$  and  $A^* = C(\Gamma)$ . A natural question is, “what if  $\Gamma$  is an infinite group or a Lie group?” and “what are some other examples of Hopf algebras?”

Let’s look at some infinite dimensional examples. If  $A$  is an infinite dimensional Hopf algebra, and  $A \otimes A$  is the algebraic tensor product (finite linear combinations of formal  $a \otimes b$  s). Then the comultiplication should be  $\Delta : A \rightarrow A \otimes A$ . You can consider cases where you have to take some completion of the tensor product with respect to some topology, but we won’t deal with this kind of stuff. In this case,  $A^*$  is too big, so instead of the notion of the dual Hopf algebra, we have dual pairs.

**Definition 6.5.** A *dual pairing* of Hopf algebras  $A$  and  $H$  is a pair with a bilinear map  $\langle \cdot, \cdot \rangle : A \otimes H \rightarrow \mathbb{C}$  which is nondegenerate such that

- (1)  $\langle \Delta a, l \otimes m \rangle = \langle a, lm \rangle$
- (2)  $\langle ab, l \rangle = \langle a \otimes b, \Delta_* l \rangle$
- (3)  $\langle Sa, l \rangle = \langle a, S_* l \rangle$
- (4)  $\varepsilon(a) = \langle a, 1_H \rangle, \varepsilon_H(l) = \langle 1_A, l \rangle$

Exmaple:  $A = \mathbb{C}[x]$ , then what is  $A^*$ ? You can evaluate a polynomial at 0, or you can differentiate some number of times before you evaluate at 0.  $A^*$  = span of linear functionals on polynomial functions of  $\mathbb{C}$  of the form

$$l_n(f) = \left( \frac{d}{dx} \right)^n f(x)|_{x=0}.$$

A basis for  $\mathbb{C}[x]$  is  $1, x^n$  with  $n \geq 1$ , and we have

$$l_n(x^m) = \begin{cases} m! & , n = m \\ 0 & , n \neq m \end{cases}$$

What is the Hopf algebra structure on  $A$ ? We already have an algebra with identity. Define  $\Delta(x) = x \otimes 1 + 1 \otimes x$  and extend it to an algebra homomorphism, then it is clearly coassociative. Define  $\varepsilon(1) = 1$  and  $\varepsilon(x^n) = 0$  for all  $n \geq 1$ . Define  $S(x) = -x$ , and extend to an algebra homomorphism. It is easy to check that this is a Hopf algebra.

Let's compute the Hopf algebra structure on  $A^*$ . We have

$$\begin{aligned} l_n l_m(x^N) &= (l_n \otimes l_m)(\Delta(x^N)) \\ &= (l_n \otimes l_m)\left(\sum \binom{N}{k} x^{N-k} \otimes x^k\right) \end{aligned}$$

► **Exercise 6.2.** Compute this out. The answer is that  $A^* = \mathbb{C}[y = \frac{d}{dx}]$ , and the Hopf algebra structure is the same as  $A$ .

This is an example of a dual pair:  $A = \mathbb{C}[x], H = \mathbb{C}[y]$ , with  $\langle x^n, y^m \rangle = \delta_{n,m} m!$ .

Summary: If  $A$  is finite dimensional, you get a dual, but in the infinite dimensional case, you have to use dual pairs.

## The universal enveloping algebra

The idea is to construct a map from Lie algebras to associative algebras so that the representation theory of the associative algebra is equivalent to the representation theory of the Lie algebra.

1) let  $V$  be a vector space, then we can form the free associative algebra (or tensor algebra) of  $V$ :  $T(V) = \mathbb{C} \oplus (\oplus_{n \geq 1} V^{\otimes n})$ . The multiplication is given by concatenation:  $(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$ . It is graded:  $T_n(V)T_m(V) \subseteq T_{n+m}(V)$ . It is also a Hopf algebra, with  $\Delta(x) = x \otimes 1 + 1 \otimes x$ ,  $S(x) = -x$ ,  $\varepsilon(1) = 1$  and  $\varepsilon(x) = 0$ . If you choose a basis  $e_1, \dots, e_n$  of  $V$ , then  $T(V)$  is the free associative algebra  $\langle e_1, \dots, e_n \rangle$ . This algebra is  $\mathbb{Z}_+$ -graded:  $T(V) = \oplus_{n \geq 0} T_n(V)$ , where the degree of 1 is zero and the degree of each  $e_i$  is 1. It is also a  $\mathbb{Z}$ -graded bialgebra:  $\Delta(T_n(V)) \subseteq \oplus (T_i \otimes T_{n-i}), S(T_n(V)) \subseteq T_n(V), \varepsilon : T(V) \rightarrow \mathbb{C}$  is a mapping of graded spaces ( $(\mathbb{C})_n = \{0\}$ ).



**Definition 6.6.** Let  $A$  be a Hopf algebra. Then a two-sided ideal  $I \subseteq A$  is a *Hopf ideal* if  $\Delta(I) \subseteq A \otimes I + I \otimes A$ ,  $S(I) = I$ , and  $\varepsilon(I) = 0$ .

You can check that the quotient of a Hopf algebra by a Hopf ideal is a Hopf algebra (and that the kernel of a map of Hopf algebras is always a Hopf ideal).

► **Exercise 6.3.** Show that  $I_0 = \langle v \otimes w - w \otimes v \mid v, w \in V = T_1(V) \subseteq T(V) \rangle$  is a homogeneous Hopf ideal.

**Corollary 6.7.**  $S(V) = T(V)/I_0$  is a graded Hopf algebra.

Choose a basis  $e_1, \dots, e_n$  in  $V$ , so that  $T(V) = \langle e_1, \dots, e_n \rangle$  and  $S(V) = \langle e_1, \dots, e_n \rangle / \langle e_i e_j - e_j e_i \rangle$

► **Exercise 6.4.** Prove that the Hopf algebra  $S(V)$  is isomorphic to  $\mathbb{C}[e_1] \otimes \dots \otimes \mathbb{C}[e_n]$ .

*Remark 6.8.* From the discussion of  $\mathbb{C}[x]$ , we know that  $S(V)$  and  $S(V^*)$  are dual.

► **Exercise 6.5.** Describe the Hopf algebra structure on  $T(V^*)$  that is determined by the pairing  $\langle v_1 \otimes \dots \otimes v_n, l_1 \otimes \dots \otimes l_m \rangle = \delta_{m,n} l_1(v_1) \dots l_n(v_n)$ . (free coalgebra of  $V^*$ )

Now assume that  $\mathfrak{g}$  is a Lie algebra.

**Definition 6.9.** The universal enveloping algebra of  $\mathfrak{g}$  is  $U(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - y \otimes x - [x, y] \rangle$ .

Exercise: prove that  $\langle x \otimes y - y \otimes x - [x, y] \rangle$  is a Hopf ideal.

**Corollary 6.10.**  $U\mathfrak{g}$  is a Hopf algebra.

If  $e_1, \dots, e_n$  is a basis for  $V$ .  $U\mathfrak{g} = \langle e_1, \dots, e_n \mid e_i e_j - e_j e_i = \sum_k c_{ij}^k e_k \rangle$ , where  $c_{ij}^k$  are the structure constants of  $[\ , \ ]$ .

*Remark 6.11.* The ideal  $\langle e_i e_j - e_j e_i \rangle$  is homogeneous, but  $\langle x \otimes y - y \otimes x - [x, y] \rangle$  is not, so  $U\mathfrak{g}$  isn't graded, but it is *filtered*.

## Lecture 7

Last time we talked about Hopf algebras. Our basic examples were  $\mathbb{C}[\Gamma]$  and  $C(\Gamma) = \mathbb{C}[\Gamma]^*$ . Also, for a vector space  $V$ ,  $T(V)$  is a Hopf algebra. Then  $S(V) = T(V)/\langle x \otimes y - y \otimes x \mid x, y \in V \rangle$ . And we also have  $U\mathfrak{g} = T\mathfrak{g}/\langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$ .

Today we'll talk about the universal enveloping algebra. Later, we'll talk about deformations of associative algebras because that is where recent progress in representation theory has been.

### Universality of $U\mathfrak{g}$

We have that  $\mathfrak{g} \hookrightarrow T\mathfrak{g} \rightarrow U\mathfrak{g}$ . And  $\sigma : \mathfrak{g} \hookrightarrow U\mathfrak{g}$  canonical embedding (of vector spaces and Lie algebras). Let  $A$  be an associative algebra with  $\tau : \mathfrak{g} \rightarrow L(A) = \{A \mid [a, b] = ab - ba\}$  a Lie algebra homomorphism such that  $\tau([x, y]) = \tau(x)\tau(y) - \tau(y)\tau(x)$ .

**Proposition 7.1.** *For any such  $\tau$ , there is a unique  $\tau' : U\mathfrak{g} \rightarrow A$  homomorphism of associative algebras which extends  $\tau$ :*

$$\begin{array}{ccc} U\mathfrak{g} & \xrightarrow{\tau'} & A \\ \uparrow \sigma & \nearrow \tau & \\ \mathfrak{g} & & \end{array}$$

*Proof.* Because  $T(V)$  is generated (freely) by 1 and  $V$ ,  $U\mathfrak{g}$  is generated by 1 and the elements of  $\mathfrak{g}$ . Choose a basis  $e_1, \dots, e_n$  of  $\mathfrak{g}$ . Then we have that  $\tau(e_i)\tau(e_j) - \tau(e_j)\tau(e_i) = \sum_k c_{ij}^k \tau(e_k)$ . The elements  $e_{i_1} \cdots e_{i_k}$  (this is a product) span  $U\mathfrak{g}$  for indices  $i_j$ . From the commutativity of the diagram,  $\tau'(e_i) = \tau(e_i)$ . Since  $\tau'$  is a homomorphism of associative algebras, we have that  $\tau'(e_{i_1} \cdots e_{i_k}) = \tau'(e_{i_1}) \cdots \tau'(e_{i_k})$ , so  $\tau'$  is determined by  $\tau$  uniquely:  $\tau'(e_{i_1} \cdots e_{i_k}) = \tau(e_{i_1}) \cdots \tau(e_{i_k})$ . We have to check that the ideal we mod out by is in the kernel. But that ideal is in the kernel because  $\tau$  is a mapping of Lie algebras.  $\square$

**Definition 7.2.** A linear representation of  $\mathfrak{g}$  in  $V$  is a pair  $(V, \phi : \mathfrak{g} \rightarrow \text{End}(V))$ , where  $\phi$  is a Lie algebra homomorphism. If  $A$  is an associative algebra, then  $(V, \phi : A \rightarrow \text{End}(V))$  a linear representation of  $A$  in  $V$ .

**Corollary 7.3.** *There is a bijection between representations of  $\mathfrak{g}$  (as a Lie algebra) and representations of  $U\mathfrak{g}$  (as an associative algebra).*

*Proof.*  $(\Rightarrow)$  By the universality,  $A = \text{End}(V)$ ,  $\tau = \phi$ .  $(\Leftarrow)$   $\mathfrak{g} \subset L(U\mathfrak{g})$  is a Lie subalgebra.  $\square$

**Example 7.4** (Adjoint representation).  $ad : \mathfrak{g} \rightarrow \text{End } \mathfrak{g}$  given by  $x : y \mapsto [x, y]$ . This is also a representation of  $U\mathfrak{g}$ . Let  $e_1, \dots, e_n$  be a basis in  $\mathfrak{g}$ . Then we have that  $ad_{e_i}(e_j) = [e_i, e_j] = \sum_k c_{ij}^k e_k$ , so the matrix representing the adjoint action of the element  $e_i$  is the matrix  $(ad_{e_i})_{jk} = (c_{ij}^k)$  of structural constants. You can check that  $ad_{[e_i, e_j]} = (ad_{e_i})(ad_{e_j}) - (ad_{e_j})(ad_{e_i})$  is same as the Jacobi identity for the  $c_{ij}^k$ . We get  $ad : U\mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  by defining it on the monomials  $e_{i_1} \cdots e_{i_k}$  as  $ad_{e_{i_1} \cdots e_{i_k}} = (ad_{e_{i_1}}) \cdots (ad_{e_{i_k}})$  (the product of matrices).

Let's look at some other properties of  $U\mathfrak{g}$ .

### Gradation in $U\mathfrak{g}$

Recall that  $V$  is a  $\mathbb{Z}_+$ -graded vector space if  $V = \bigoplus_{n=0}^{\infty} V_n$ . A linear map  $f : V \rightarrow W$  between graded vector spaces is *grading-preserving* if  $f(V_n) \subseteq W_n$ . If we have a tensor product  $V \otimes W$  of graded vector spaces, it has a natural grading given by  $(V \otimes W)_n = \bigoplus_{i=0}^n V_i \otimes W_{n-i}$ . The “geometric meaning” of this is that there is a linear action of  $\mathbb{C}$  on  $V$  such that  $V_n = \{x | t(x) = t^n \cdot x \text{ for all } t \in \mathbb{C}\}$ . A graded morphism is a linear map respecting this action, and the tensor product has the diagonal action of  $\mathbb{C}$ , given by  $t(x \otimes y) = t(x) \otimes t(y)$ .

**Example 7.5.** If  $V = \mathbb{C}[x]$ ,  $\frac{d}{dx}$  is not grading preserving,  $x \frac{d}{dx}$  is.

We say that  $(V, [ , ])$  is a  $\mathbb{Z}_+$ -graded Lie algebra if  $[ , ] : V \otimes V \rightarrow V$  is grading-preserving.

**Example 7.6.** Let  $V$  be the space of polynomial vector fields on  $\mathbb{C} = \text{Span}(z^n \frac{d}{dz})_{n \geq 0}$ . Then  $V_n = \mathbb{C}z^n \frac{d}{dz}$ .

An associative algebra  $(V, m : V \otimes V \rightarrow V)$  is  $\mathbb{Z}_+$ -graded if  $m$  is grading-preserving.

### Example 7.7.

- (1)  $V = \mathbb{C}[x]$ , where the action of  $\mathbb{C}$  is given by  $x \mapsto tx$ .
- (2)  $V = \mathbb{C}[x_1, \dots, x_n]$  where the degree of each variable is 1 ... this is the  $n$ -th tensor power of the previous example.
- (3) Lie algebra:  $\text{Vect}(\mathbb{C}) = \{\sum_{n \geq 0} a_n x^{n+1} \frac{d}{dx}\}$  with  $\text{Vect}_n(\mathbb{C}) = \mathbb{C}x^{n+1} \frac{d}{dx}$ ,  $\text{deg}(x) = 1$ . You can embed  $\text{Vect}(\mathbb{C})$  into polynomial vector fields on  $S^1$  (Virasoro algebra).
- (4)  $T(V)$  is a  $\mathbb{Z}_+$ -graded associative algebra, as is  $S(V)$ . However,  $U\mathfrak{g}$  is not because we have modded out by a non-homogeneous ideal. But the ideal is not so bad.  $U\mathfrak{g}$  is a  $\mathbb{Z}_+$ -filtered algebra:

## Filtered spaces and algebras

**Definition 7.8.**  $V$  is a *filtered space* if it has an increasing filtration

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$$

such that  $V = \bigcup V_i$ , and  $V_n$  is a subspace of dimension less than or equal to  $n$ .  $f : V \rightarrow W$  is a *morphism of filtered vector spaces* if  $f(V_n) \subseteq W_n$ .

We can define filtered lie algebras and associative algebras as such that the bracket/multiplication are filtered maps.

There is a functor from filtered vector spaces to graded associative algebras  $Gr : V \rightarrow Gr(V)$ , where  $Gr(V) = V_0 \oplus V_1/V_0 \oplus V_2/V_1 \cdots$ . If  $f : V \rightarrow W$  is filtration preserving, it induces a map  $Gr(f) : Gr(V) \rightarrow Gr(W)$  functorially such that this diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ Gr \downarrow & & \downarrow Gr \\ Gr(V) & \xrightarrow{Gr(f)} & Gr(W) \end{array}$$

Let  $A$  be an associative filtered algebra (i.e.  $A_i A_j \subseteq A_{i+j}$ ) such that for all  $a \in A_i, b \in A_j, ab - ba \in A_{i+j-1}$ .

**Proposition 7.9.** For such an  $A$ ,

- (1)  $Gr(A)$  has a natural structure of an associative, commutative algebra (that is, the multiplication in  $A$  defines an associative, commutative multiplication in  $Gr(A)$ ).
- (2) For  $a \in A_{i+1}, b \in A_{j+1}$ , the operation  $\{aA_i, bA_j\} = aA_i bA_j - bA_j aA_i \pmod{A_{i+j}}$  is a lie bracket on  $Gr(A)$ .
- (3)  $\{x, yz\} = \{x, y\}z + y\{x, z\}$ .

*Proof.* Exercise<sub>1</sub>. You need to show that the given bracket is well defined, and then do a little dance, keeping track of which graded component you are in.  $\square$

**Definition 7.10.** A commutative associative algebra  $B$  is called a *Poisson algebra* if  $B$  is also a lie algebra with lie bracket  $\{ , \}$  (called a Poisson bracket) such that  $\{x, yz\} = \{x, y\}z + y\{x, z\}$  (the bracket is a derivation).

**Example 7.11.** Let  $(M, \omega)$  be a symplectic manifold (i.e.  $\omega$  is a closed non-degenerate 2-form on  $M$ ), then functions on  $M$  form a Poisson algebra. We could have  $M = \mathbb{R}^{2n}$  with coordinates  $p_1, \dots, p_n, q_1, \dots, q_n$ , and  $\omega = \sum_i dp_i \wedge dq_i$ . Then the multiplication and addition on  $C^\infty(M)$  is the usual one, and we can define  $\{f, g\} = \sum_{ij} p^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$ , where  $\omega = \sum \omega_{ij} dx^i \wedge dx^j$  and  $(p^{ij})$  is the inverse matrix to  $(\omega_{ij})$ . You can check that this is a Poisson bracket.

Let's look at  $U\mathfrak{g} = \langle 1, e_i | e_i e_j - e_j e_i = \sum_k c_{ij}^k e_k \rangle$ . Then  $U\mathfrak{g}$  is filtered, with  $(U\mathfrak{g})_n = \text{Span}\{e_{i_1} \cdots e_{i_k} | k \leq n\}$ . We have the obvious inclusion  $(U\mathfrak{g})_n \subseteq (U\mathfrak{g})_{n+1}$  and  $(U\mathfrak{g})_0 = \mathbb{C} \cdot 1$ .

**Proposition 7.12.**

(1)  $U\mathfrak{g}$  is a filtered algebra (i.e.  $(U\mathfrak{g})_r (U\mathfrak{g})_s \subseteq (U\mathfrak{g})_{r+s}$ )

(2)  $[(U\mathfrak{g})_r, (U\mathfrak{g})_s] \subseteq (U\mathfrak{g})_{r+s-1}$ .

*Proof.* 1) obvious. 2) Exercise<sub>2</sub> (almost obvious). □

Now we can consider  $Gr(U\mathfrak{g}) = \mathbb{C} \cdot 1 \oplus (\bigoplus_{r \geq 1} (U\mathfrak{g})_r / (U\mathfrak{g})_{r-1})$

**Claim.**  $(U\mathfrak{g})_r / (U\mathfrak{g})_{r-1} \simeq S^r(\mathfrak{g}) = \text{symmetric elements of } (\mathbb{C}[e_1, \dots, e_n])_r$ .

*Proof.* Exercise<sub>3</sub>. □

So  $Gr(U\mathfrak{g}) \simeq S(\mathfrak{g})$  as a commutative algebra.

$S(\mathfrak{g}) \cong \text{Polynomial functions on } \mathfrak{g}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$ .

Consider  $C^\infty(M)$ . How can we construct a bracket  $\{, \}$  which satisfies Leibniz (i.e.  $\{f, g_1 g_2\} = \{f, g_1\} g_2 + \{f, g_2\} g_1$ ). We expect that  $\{f, g\}(x) = p^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \langle p(x), df(x) \wedge dg(x) \rangle$ . Such a  $p$  is called a bivector field (it is a section of the bundle  $TM \wedge TM \rightarrow M$ ). So a Poisson structure on  $C^\infty(M)$  is the same as a bivector field  $p$  on  $M$  satisfying the Jacobi identity. You can check that the Jacobi identity is some bilinear identity on  $p^{ij}$  which follows from the Jacobi identity on  $\{, \}$ . This is equivalent to the Schouten identity, which says that the Schouten bracket of some things vanishes [There should be a reference here]. This is more general than the symplectic case because  $p^{ij}$  can be degenerate.

Let  $\mathfrak{g}$  have the basis  $e_1, \dots, e_n$  and corresponding coordinate functions  $x^1, \dots, x^n$ . On  $\mathfrak{g}^*$ , we have that dual basis  $e^1, \dots, e^n$  (you can identify these with the coordinates  $x^1, \dots, x^n$ ), and coordinates  $x_1, \dots, x_n$  (which you can identify with the  $e_i$ ). The bracket on polynomial functions on  $\mathfrak{g}^*$  is given by

$$\{p, q\} = \sum c_{ij}^k x_k \frac{\partial p}{\partial x_i} \frac{\partial q}{\partial x_j}.$$

This is a Lie bracket and clearly acts by derivations.

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Next we will study the following. If you have polynomials  $p, q$  on  $\mathfrak{g}^*$ , you can try to construct an associative product  $p *_t q = pq + tm_1(p, q) + \dots$ . We will discuss deformations of commutative algebras. The main example will be the universal enveloping algebra as a deformation of polynomial functions on  $\mathfrak{g}^*$ .

## Lecture 8 - PBW theorem, Deformations of Associative Algebras

Last time, we introduced the universal enveloping algebra  $U\mathfrak{g}$  of a Lie algebra  $\mathfrak{g}$ , with its universality property. We discussed graded and filtered spaces and algebras. We showed that under some condition on a filtered algebra  $A$ , the graded algebra  $Gr(A)$  is a Poisson algebra. We also checked that  $U\mathfrak{g}$  satisfies this condition, and that  $Gr(U\mathfrak{g}) \simeq S(\mathfrak{g})$  as graded commutative algebras. The latter space can be understood as the space  $Pol(\mathfrak{g}^*)$  of polynomial functions on  $\mathfrak{g}^*$ . It turns out that the Poisson bracket on  $Gr(U\mathfrak{g})$ , expressed in  $Pol(\mathfrak{g}^*)$ , is given by

$$\{f, g\}(x) = x([df_x, dg_x])$$

for  $f, g \in Pol(\mathfrak{g}^*)$  and  $x \in \mathfrak{g}^*$ . Note that  $f$  is a function on  $\mathfrak{g}^*$  and  $x$  an element of  $\mathfrak{g}^*$ , so  $df_x$  is a linear form on  $T_x\mathfrak{g}^* = \mathfrak{g}^*$ , that is,  $df_x \in \mathfrak{g}$ .

Suppose that  $V$  admits a filtration  $V_0 \subset V_1 \subset V_2 \subset \dots$ . Then, the associated graded space  $Gr(V) = V_0 \oplus \bigoplus_{n \geq 1} (V_n/V_{n+1})$  is also filtered. (Indeed, every graded space  $W = \bigoplus_{n \geq 0} W_n$  admits the filtration  $W_0 \subset W_0 \oplus W_1 \subset W_0 \oplus W_1 \oplus W_2 \subset \dots$ ) A natural question is: When do we have  $V \simeq Gr(V)$  as filtered spaces?

For the filtered space  $U\mathfrak{g}$ , the answer is a consequence of the following theorem.

**Theorem 8.1** (Poincaré-Birkhoff-Witt). *Let  $e_1, \dots, e_n$  be any linear basis for  $\mathfrak{g}$ . Let us also denote by  $e_1, \dots, e_n$  the image of this basis in the universal enveloping algebra  $U\mathfrak{g}$ . Then the monomials  $e_1^{m_1} \dots e_n^{m_n}$  form a basis for  $U\mathfrak{g}$ .*

**Corollary 8.2.** *There is an isomorphism of filtered spaces  $U\mathfrak{g} \simeq Gr(U\mathfrak{g})$ .*

*Proof of the corollary.* In  $S(\mathfrak{g})$ ,  $e_1^{m_1} \dots e_n^{m_n}$  also forms a basis, so we get an isomorphism  $U\mathfrak{g} \simeq S(\mathfrak{g})$  of filtered vector spaces by simple identification of the bases. Since  $Gr(U\mathfrak{g}) \simeq S(\mathfrak{g})$  as graded algebras, the corollary is proved.  $\square$

*Remark 8.3.* The point is that these spaces are isomorphic as *filtered* vector spaces. Saying that two infinite-dimensional vector spaces are isomorphic is totally useless.

*Proof of the theorem.* By definition, the unordered monomials  $e_{i_1} \dots e_{i_k}$  for  $k \leq p$  span the subspace  $T_0 \oplus \dots \oplus T_p$  of  $T(\mathfrak{g})$ , where  $T_i = \mathfrak{g}^{\otimes i}$ . Hence, they also span the quotient  $(U\mathfrak{g})_p := T_0 \oplus \dots \oplus T_p / \langle x \otimes y - y \otimes x - [x, y] \rangle$ . The goal is now to show that the *ordered* monomials  $e_1^{m_1} \dots e_n^{m_n}$  for  $m_1 + \dots + m_n \leq p$  still span  $(U\mathfrak{g})_p$ . Let's prove this by induction on  $p \geq 0$ . The case  $p = 0$  being trivial, consider  $e_{i_1} \dots e_{i_a} \dots e_{i_k}$ , with  $k \leq p$ , and assume that  $i_a$  has

the smallest value among the indices  $i_1, \dots, i_k$ . We can move  $e_{i_a}$  to the left as follows

$$e_{i_1} \cdots e_{i_a} \cdots e_{i_k} = e_{i_a} e_{i_1} \cdots \hat{e}_{i_a} \cdots e_{i_k} + \sum_{b=1}^{a-1} e_{i_1} \cdots e_{i_{b-1}} [e_{i_b}, e_{i_a}] \cdots \hat{e}_{i_a} \cdots e_{i_k}.$$

Using the commutation relations  $[e_{i_b}, e_{i_a}] = \sum_{\ell} c_{i_b i_a}^{\ell} e_{\ell}$ , we see that the term to the right belongs to  $(U\mathfrak{g})_{k-1}$ . Iterating this procedure leads to an equation of the form

$$e_{i_1} \cdots e_{i_a} \cdots e_{i_k} = e_1^{m_1} \cdots e_n^{m_n} + \text{terms in } (U\mathfrak{g})_{k-1},$$

with  $m_1 + \cdots + m_n = k \leq p$ . We are done by induction. The proof of the theorem is completed by the following homework. [This should really be done here]  $\square$

► **Exercise 8.1.** Prove that these ordered monomials are linearly independent.

Let's "generalize" the situation. We have  $U\mathfrak{g}$  and  $S(\mathfrak{g})$ , both of which are quotients of  $T(\mathfrak{g})$ , with kernels  $\langle x \otimes y - y \otimes x - [x, y] \rangle$  and  $\langle x \otimes y - y \otimes x \rangle$ . For any  $\varepsilon \in \mathbb{C}$ , consider the associative algebra  $S_{\varepsilon}(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - y \otimes x - \varepsilon[x, y] \rangle$ . By construction,  $S_0(\mathfrak{g}) = S(\mathfrak{g})$  and  $S_1(\mathfrak{g}) = U\mathfrak{g}$ . Recall that they are isomorphic as filtered vector spaces.

*Remark 8.4.* If  $\varepsilon \neq 0$ , the linear map  $\phi_{\varepsilon} : S_{\varepsilon}(\mathfrak{g}) \rightarrow U\mathfrak{g}$  given by  $\phi_{\varepsilon}(x) = \varepsilon x$  for all  $x \in \mathfrak{g}$  is an isomorphism of filtered algebras. So, we have nothing new here.

We can think of  $S_{\varepsilon}(\mathfrak{g})$  as a non-commutative deformation of the associative commutative algebra  $S(\mathfrak{g})$ . (Note that commutative deformations of the algebra of functions on a variety correspond to deformations of the variety.)

## Deformations of associative algebras

Let  $(A, m : A \otimes A \rightarrow A)$  be an associative algebra, that is, the linear map  $m$  satisfies the quadratic equation

$$m(m(a, b), c) = m(a, m(b, c)). \quad (\dagger)$$

Note that if  $\varphi : A \rightarrow A$  is a linear automorphism, the multiplication  $m_{\varphi}$  given by  $m_{\varphi}(a, b) = \varphi^{-1}(m(\varphi(a), \varphi(b)))$  is also associative. We like to think of  $m$  and  $m_{\varphi}$  as equivalent associative algebra structures on  $A$ . The "moduli space" of associative algebras on the vector space  $A$  is the set of solutions to  $(\dagger)$  modulo this equivalence relation.



One can come up with a notion of deformation for almost any kind of object. In these deformation theories, we are interested in some cohomology theories because they parameterize obstructions to deformations. The knowledge of the cohomology of a given Lie algebra  $\mathfrak{g}$ , enables us say a lot about the deformations of  $\mathfrak{g}$ . We'll come back to this question in the next lecture.

Let us turn to our original example: the family of associative algebras  $S_\varepsilon(\mathfrak{g})$ . Recall that by the PBW theorem, we have an isomorphism of filtered vector spaces  $S_\varepsilon(\mathfrak{g}) \xrightarrow{\psi} S(\mathfrak{g}) = \text{Pol}(\mathfrak{g}^*)$ , but this is not an isomorphism of associative algebras. Therefore, the multiplication defined by  $f * g := \psi(\psi^{-1}(f) \cdot \psi^{-1}(g))$  is not the normal multiplication on  $S(\mathfrak{g})$ . We claim that the result is of the form

$$f * g = fg + \sum_{n \geq 1} \varepsilon^n m_n(f, g),$$

where  $m_n$  is a bidifferential operator of order  $n$ , that is, it is of the form

$$m_n(f, g) = \sum_{I, J} p_n^{I, J} \partial^I f \partial^J g,$$

where  $I$  and  $J$  are multi-indices of length  $n$ , and  $p_n^{I, J} \in \text{Pol}(\mathfrak{g}^*)$ . The idea of the proof is to check this for  $f = \psi(e_1^{r_1} \cdots e_n^{r_n})$  and  $g = \psi(e_1^{l_1} \cdots e_n^{l_n})$  by writing

$$e_1^{r_1} \cdots e_n^{r_n} \cdot e_1^{l_1} \cdots e_n^{l_n} = e_1^{l_1+r_1} \cdots e_n^{l_n+r_n} + \sum_{k \geq 1} \varepsilon^k m_k(e_1^{r_1} \cdots e_n^{r_n}, e_1^{l_1} \cdots e_n^{l_n})$$

in  $S_\varepsilon(\mathfrak{g})$  using the commuting relations.

► **Exercise 8.2.** Compute the  $p_n^{I, J}$  for the Lie algebra  $\mathfrak{g}$  generated by  $X$ ,  $Y$ , and  $H$  with bracket  $[X, Y] = H$ ,  $[H, X] = [H, Y] = 0$ . This is called the Heisenberg Lie algebra.

So we have a family of products on  $\text{Pol}(\mathfrak{g}^*)$  which depend on  $\varepsilon$  in the following way:

$$f * g = fg + \sum_{n \geq 1} \varepsilon^n m_n(f, g)$$

Since  $f, g$  are polynomials and  $m_n$  is a bidifferential operator of order  $n$ , this series terminates, so it is a polynomial in  $\varepsilon$ . If we try to extend this product to  $C^\infty(\mathfrak{g}^*)$ , then there are questions about the convergence of the product  $*$ . There are two ways to deal with this problem. The first one is to take these matters of convergence seriously, consider some topology on  $C^\infty(\mathfrak{g}^*)$  and demand that the series converges. The other solution is to forget about convergence and just think in terms of formal power series in  $\varepsilon$ . This is the so-called “formal deformation” approach. As we shall see, there are interesting things to say with this seemingly rudimentary point of view.

### Formal deformations of associative algebras

Let  $(A, m_0)$  be an associative algebra over  $\mathbb{C}$ . Then, a formal deformation of  $(A, m_0)$  is a  $\mathbb{C}[[\hbar]]$ -linear map  $m : A[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} A[[\hbar]] \rightarrow A[[\hbar]]$  such that

$$m(a, b) = m_0(a, b) + \sum_{n \geq 1} \hbar^n m_n(a, b)$$

for all  $a, b \in A$ , and such that  $(A[[\hbar]], m)$  is an associative algebra. We say that two formal deformations  $m$  and  $\tilde{m}$  are equivalent if there is a  $\mathbb{C}[[\hbar]]$ -automorphism  $A[[\hbar]] \xrightarrow{\varphi} A[[\hbar]]$  such that  $\tilde{m} = m_\varphi$ , with  $\varphi(x) = x + \sum_{n \geq 1} \hbar^n \varphi_n(x)$  for all  $x \in A$ , where  $\varphi_n$  is an endomorphism of  $A$ .

Question: Describe the equivalence classes of formal deformations of a given associative algebra.

When  $(A, m_0)$  is a commutative algebra, the answer is known. Philosophically and historically, this case is relevant to quantum mechanics. In classical mechanics, observables are smooth functions on a phase space  $M$ , i.e they form a commutative associative algebra  $C^\infty(M)$ . But when you quantize this system (which is needed to describe something on the order of the Planck scale), you cannot think of observables as functions on phase space anymore. You need to deform the commutative algebra  $C^\infty(M)$  to a noncommutative algebra. And it works...

From now on, let  $(A, m_0)$  be a commutative associative algebra. Let's write  $m_0(a, b) = ab$ , and  $m(a, b) = a * b$ . (This is called a star product, and the terminology goes back to the sixties and the work of J. Vey). Then we have

$$a * b = ab + \sum_{n \geq 1} \hbar^n m_n(a, b).$$

Demanding the associativity of  $*$  imposes an infinite number of equations for the  $m_n$ 's, one for each order:

$$\hbar^0: a(bc) = (ab)c$$

$$\hbar^1: am_1(b, c) + m_1(a, bc) = m_1(a, b)c + m_1(ab, c)$$

$$\hbar^2: \dots$$

$\vdots$

► **Exercise 8.3.** Show that the bracket  $\{a, b\} = m_1(a, b) - m_1(b, a)$  defines a Poisson structure on  $A$ . This means that we can think of a Poisson structure on an algebra as the remnants of a deformed product where  $a * b - b * a = \hbar\{a, b\} + O(\hbar^2)$ .

One easily checks that if two formal deformations  $m$  and  $\tilde{m}$  are equivalent via  $\varphi$  (i.e:  $\tilde{m} = m_\varphi$ ), then the associated  $m_1, \tilde{m}_1$  are related by  $m_1(a, b) = \tilde{m}_1(a, b) + \varphi_1(ab) - \varphi_1(a)b - a\varphi_1(b)$ . In particular, two equivalent formal deformations induce the same Poisson structure. Also, it is possible to choose a representative in an equivalence class such that  $m_1$  is skew-symmetric (and then,  $m_1(a, b) = \frac{1}{2}\{a, b\}$ ). This leads to the following program for the classification problem:

1. Classify all Poisson structures on  $A$ .
2. Given a Poisson algebra  $(A, \{ , \})$ , classify all equivalence classes of star products on  $A$  such that  $m_1(a, b) = \frac{1}{2}\{a, b\}$ .

Under some mild assumption, it can be assumed that a star product is symmetric, i.e. that it satisfies the equation  $m_n(a, b) = (-1)^n m_n(b, a)$  for all  $n$ . The program given above was completed by Maxim Kontsevitch for the algebra of smooth functions on a manifold  $M$ . Recall that Poisson structures on  $C^\infty(M)$  are given by bivector fields on  $M$  that satisfy the Jacobi identity.

**Theorem 8.5** (Kontsevich, 1994). *Let  $A$  be the commutative associative algebra  $C^\infty(M)$ , and let us fix a Poisson bracket  $\{ , \}$  on  $A$ . Equivalence classes of symmetric star products on  $A$  with  $m_1(a, b) = \frac{1}{2}\{a, b\}$  are in bijection with formal deformations of  $\{ , \}$  modulo formal diffeomorphisms of  $M$ .*

A formal deformation of  $\{ , \}$  is a Poisson bracket  $\{ , \}_h$  on  $A[[h]]$  such that

$$\{a, b\}_h = \{a, b\} + \sum_{n \geq 1} h^n \mu_n(a, b)$$

for all  $a, b$  in  $A$ . A formal diffeomorphism of  $M$  is an automorphism  $\varphi$  of  $A[[h]]$  such that  $\varphi(f) = f + \sum_{n \geq 1} h^n \varphi_n(f)$  and  $\varphi(fg) = \varphi(f)\varphi(g)$  for all  $f, g$  in  $A$ .

We won't prove the theorem (it would take about a month). As Poisson algebras are Lie algebras, it relates deformations of associative algebras to deformations of Lie algebras.

## Formal deformations of Lie algebras

Given a Lie algebra  $(\mathfrak{g}, [ , ])$ , you want to know how many formal deformations of  $\mathfrak{g}$  there are. Sometimes, there are none (like in the case of  $\mathfrak{sl}_n$ , as we will see later). Sometimes, there are plenty (as for triangular matrices). The goal is now to construct some invariants of Lie algebras which will tell you whether there are deformations, and how many of them there are. In order to do this, we should consider cohomology theories for Lie algebras. We will focus first on the standard complex  $C^*(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n \geq 0} C^n(\mathfrak{g}, \mathfrak{g})$ , where  $C^n(\mathfrak{g}, \mathfrak{g}) = \text{Hom}(\Lambda^n \mathfrak{g}, \mathfrak{g})$ .

## Lecture 9

Let's summarize what has happened in the last couple of lectures.

1. We talked about  $T(\mathfrak{g})$ , and then constructed three algebras:
  - $U\mathfrak{g} = T(\mathfrak{g})/\langle x \otimes y - y \otimes x - [x, y] \rangle$ , with  $U\mathfrak{g} = S_1(\mathfrak{g}) \simeq S_\varepsilon(\mathfrak{g})$  as filtered associative algebras, for all non-zero  $\varepsilon \in \mathbb{C}$ .
  - $S_\varepsilon(\mathfrak{g}) = T(\mathfrak{g})/\langle x \otimes y - y \otimes x - \varepsilon[x, y] \rangle$  is a family of associative algebras, with  $S_\varepsilon(\mathfrak{g}) \simeq S_0(\mathfrak{g})$  as filtered vector spaces.
  - $S_0(\mathfrak{g}) \cong \text{Pol}(\mathfrak{g}^*) = T(\mathfrak{g})/\langle x \otimes y - y \otimes x \rangle = S_0(\mathfrak{g})$  is an associative, commutative algebra with a Poisson structure defined by the Lie bracket.
2. We have two “pictures” of deformations of an algebra
  - (a) There is a simple “big” algebra  $B$  (such as  $B = T(\mathfrak{g})$ ) and a family of ideals  $I_\varepsilon$ . Then we get a family  $B/I_\varepsilon = A_\varepsilon$ . This becomes a deformation family of the associative algebra  $A_0$  if we identify  $A_\varepsilon \simeq A_0$  as vector spaces (these are called *torsion free* deformations). Fixing this isomorphism gives a family of associative products on  $A_0$ .  
We can think of this geometrically as a family of (embedded) varieties.
  - (b) Alternatively, we can talk about deformations intrinsically (i.e., without referring to some bigger  $B$ ). Suppose we have  $A_0$  and a family of associative products  $a *_\varepsilon b$  on  $A_0$ .

**Example 9.1.** Let  $\text{Pol}(\mathfrak{g}^*) \xrightarrow{\phi} S_\varepsilon(\mathfrak{g})$  be the isomorphism of the PBW theorem. Then define  $f * g = \phi^{-1}(\phi(f) \cdot \phi(g)) = fg + \sum_{n \geq 1} \varepsilon^n m_n(f, g)$ .

Understanding deformations makes a connection between representation theory and Poisson geometry. A second course on Lie theory should discuss symplectic leaves of  $\text{Pol}(\mathfrak{g}^*)$ , which happen to be coadjoint orbits and correspond to representations. This is why deformations are relevant to representation theory.

Let  $A$  be a Poisson algebra with bracket  $\{ , \}$ , so it is a commutative algebra, and a Lie algebra, with the bracket acting by derivations. Typically,  $A = C^\infty(M)$ . Equivalence classes of formal (i.e., formal power series) symmetric (i.e.,  $m_n(f, g) = (-1)^n m_n(g, f)$ ) star products on  $C^\infty(M)$  are in bijection with equivalence classes of formal deformations of  $\{ , \}$  on  $C^\infty(M)[[h]]$ .

Apply this to the case  $A = C^\infty(\mathfrak{g}^*)$ . The associative product on  $S_\varepsilon(\mathfrak{g})$  comes from the product on  $T(\mathfrak{g})$ . The question is, “how many equivalence classes of star products are there on  $A$ ?” Any formal deformation of the Poisson structure on  $(A, \{ , \}_\mathfrak{g})$  is a PBW deformation of some formal deformation of the Lie algebra  $C^\infty(\mathfrak{g}^*)$  (with Lie bracket  $\{f, g\}(x) = x(df \wedge dg)$ ). Such a deformation is equivalent to a formal deformation of the Lie algebra structure on  $\mathfrak{g}$ . This is one of the reasons that deformations of Lie algebras are important — they describe deformations of certain associative algebras. When one asks such questions, some cohomology theory always shows up.

### Lie algebra cohomology

Recall that  $(M, \phi)$  is a  $\mathfrak{g}$ -module if  $\phi : \mathfrak{g} \rightarrow \text{End}(M)$  is a Lie algebra homomorphism. We will write  $xm$  for  $\phi(x)m$ . Define  $C^\bullet(\mathfrak{g}, M) = \bigoplus_{q \geq 0} C^q(\mathfrak{g}, M)$  where  $C^q(\mathfrak{g}, M) = \text{Hom}(\Lambda^q \mathfrak{g}, M)$  (linear maps). We define  $d : C^q \rightarrow C^{q+1}$  by

$$\begin{aligned} dc(x_1 \wedge \cdots \wedge x_{q+1}) &= \\ &= \sum_{1 \leq s < t \leq q+1} (-1)^{s+t-1} c([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_{q+1}) \\ &\quad + \sum_{s=1}^{q+1} (-1)^s x_s c(x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_{q+1}) \end{aligned}$$

► **Exercise 9.1.** Show that  $d^2 = 0$ .

Motivation: If  $\mathfrak{g} = \text{Vect}(\mathcal{M})$ ,  $M = C^\infty(\mathcal{M})$ , then  $C^q(\mathfrak{g}, M) = \Omega^q(\mathcal{M})$ , with the Cartan formula

$$\begin{aligned} (d\omega)(\xi_1 \wedge \cdots \wedge \xi_{q+1}) &= \\ &= \sum_{1 \leq s < t \leq q+1} (-1)^{s+t-1} \omega([\xi_s, \xi_t] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_s \wedge \cdots \wedge \hat{\xi}_t \wedge \cdots \wedge \xi_{q+1}) \\ &\quad + \sum_{s=1}^{q+1} (-1)^s \xi_s \omega(\xi_1 \wedge \cdots \wedge \hat{\xi}_s \wedge \cdots \wedge \xi_{q+1}) \end{aligned}$$

for vector fields  $\xi_i$ .

Another motivation comes from the following proposition.

**Proposition 9.2.**  $C^\bullet(\mathfrak{g}, \mathbb{C}) \simeq \Omega_R^\bullet(G) \subseteq \Omega^\bullet(G)$  where  $\mathbb{C}$  is the 1-dimensional trivial module over  $\mathfrak{g}$  (so  $xm = 0$ ).

► **Exercise 9.2.** Prove it.

*Remark 9.3.* This was Cartan's original motivation for Lie algebra cohomology. It turns out that the inclusion  $\Omega_R^\bullet(G) \hookrightarrow \Omega^\bullet(G)$  is a homotopy equivalence of complexes (i.e. the two complexes have the same homology), and the proposition above tells us that  $C^\bullet(\mathfrak{g}, \mathbb{C})$  is homotopy equivalent to  $\Omega_R^\bullet(G)$ . Thus, by computing the Lie algebra cohomology of  $\mathfrak{g}$  (the homology of the complex  $C^\bullet(\mathfrak{g}, \mathbb{C})$ ), one obtains the De Rham cohomology of  $G$  (the homology of the complex  $\Omega^\bullet(G)$ ).

Define  $H^q(\mathfrak{g}, M) = \ker(d : C^q \rightarrow C^{q+1}) / \text{im}(d : C^{q-1} \rightarrow C^q)$  as always. Let's focus on the case  $M = \mathfrak{g}$ , the adjoint representation:  $x \cdot m = [x, m]$ .

$H^0(\mathfrak{g}, \mathfrak{g})$  We have that  $C^0 = \text{Hom}(\mathbb{C}, \mathfrak{g}) \cong \mathfrak{g}$ , and

$$dc(y) = y \cdot c = [y, c].$$

so  $\ker(d : C^0 \rightarrow C^1)$  is the set of  $c \in \mathfrak{g}$  such that  $[y, c] = 0$  for all  $y \in \mathfrak{g}$ . That is, the kernel is the center of  $\mathfrak{g}$ ,  $Z(\mathfrak{g})$ . So  $H^0(\mathfrak{g}, \mathfrak{g}) = Z(\mathfrak{g})$ .

$H^1(\mathfrak{g}, \mathfrak{g})$  The kernel of  $d : C^1(\mathfrak{g}, \mathfrak{g}) \rightarrow C^2(\mathfrak{g}, \mathfrak{g})$  is

$$\{\mu : \mathfrak{g} \rightarrow \mathfrak{g} \mid d\mu(x, y) = \mu([x, y]) - [x, \mu(y)] - [\mu(x), y] = 0 \text{ for all } x, y \in \mathfrak{g}\},$$

which is exactly the set of derivations of  $\mathfrak{g}$ . The image of  $d : C^0(\mathfrak{g}, \mathfrak{g}) \rightarrow C^1(\mathfrak{g}, \mathfrak{g})$  is the set of *inner derivations*,  $\{dc : \mathfrak{g} \rightarrow \mathfrak{g} \mid dc(y) = [y, c]\}$ . The Liebniz rule is satisfied because of the Jacobi identity. So

$$H^1(\mathfrak{g}, \mathfrak{g}) = \{\text{derivations}\} / \{\text{inner derivations}\} =: \text{outer derivations}.$$

$H^2(\mathfrak{g}, \mathfrak{g})$  Let's compute  $H^2(\mathfrak{g}, \mathfrak{g})$ . Suppose  $\mu \in C^2$ , so  $\mu : \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{g}$  is a linear map. What does  $d\mu = 0$  mean?

$$\begin{aligned} d\mu(x_1, x_2, x_3) &= \mu([x_1, x_2], x_3) - \mu([x_1, x_3], x_2) + \mu([x_2, x_3], x_1) \\ &\quad - [x_1, \mu(x_2, x_3)] + [x_2, \mu(x_1, x_3)] - [x_3, \mu(x_1, x_2)] \\ &= -\mu(x_1, [x_2, x_3]) - [x_1, \mu(x_2, x_3)] + \text{cyclic permutations} \end{aligned}$$

Where does this kind of thing show up naturally?

Consider deformations of Lie algebras:

$$[x, y]_h = [x, y] + \sum_{n \geq 1} h^n m_n(x, y)$$

where the  $m_n : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  are bilinear. The deformed bracket  $[\cdot, \cdot]_h$  must satisfy the Jacobi identity,

$$[a, [b, c]_h]_h + [b, [c, a]_h]_h + [c, [a, b]_h]_h = 0$$

which gives us relations on the  $m_n$ . In degree  $h^N$ , we get

$$\begin{aligned} [a, m_N(b, c)] + m_N(a, [b, c]) + \sum_{k=1}^{N-1} m_k(a, m_{N-k}(b, c)) + \\ [b, m_N(c, a)] + m_N(b, [c, a]) + \sum_{k=1}^{N-1} m_k(b, m_{N-k}(c, a)) + \\ [c, m_N(a, b)] + m_N(c, [a, b]) + \sum_{k=1}^{N-1} m_k(c, m_{N-k}(a, b)) = 0 \quad (\dagger) \end{aligned}$$

► **Exercise 9.3.** Derive  $(\dagger)$ .

Define  $[m_K, m_{N-K}](a, b, c)$  as

$$m_K(a, m_{N-K}(b, c)) + m_K(b, m_{N-K}(c, a)) + m_K(c, m_{N-K}(a, b)).$$

Then  $(\dagger)$  can be written as

$$dm_N = \sum_{k=1}^{N-1} [m_k, m_{N-k}] \quad (\ddagger)$$

**Theorem 9.4.** Assume that for all  $n \leq N - 1$ , we have the relation  $dm_n = \sum_{k=1}^{n-1} [m_k, m_{n-k}]$ . Then  $d(\sum_{k=1}^{N-1} [m_k, m_{N-k}]) = 0$ .

► **Exercise 9.4.** Prove it.

The theorem tells us that if we have a “partial deformation” (i.e. we have found  $m_1, \dots, m_{N-1}$ ), then the expression  $\sum_{k=1}^{N-1} [m_k, m_{N-k}]$  is a 3-cocycle. Furthermore,  $(\ddagger)$  tells us that if we are to extend our deformation to one higher order,  $\sum_{k=1}^{N-1} [m_k, m_{N-k}]$  must represent zero in  $H^3(\mathfrak{g}, \mathfrak{g})$ .

Taking  $N = 1$ , we get  $dm_1 = 0$ , so  $\ker(d : C^2 \rightarrow C^3) =$  space of first coefficients of formal deformations of  $[ , ]$ . It will turn out that  $H^2$  is the space of equivalence classes of  $m_1$ .

It is worth noting that the following “pictorial calculus” may make some of the above computations easier. In the following pictures, arrows are considered to be oriented downwards, and trivalent vertices with two lines coming in and one going out represent the Lie bracket. So, for example, the antisymmetry of the Lie bracket is expressed as

$$\begin{array}{c} \diagup \quad \diagdown \\ \circlearrowleft \end{array} = - \begin{array}{c} \diagdown \quad \diagup \\ \circlearrowright \end{array}$$

and the Jacobi identity is

$$\begin{array}{c}
 \begin{array}{ccc}
 \text{Y-shape} & + & \text{Loop-shape} & + & \text{Cross-shape} & = \\
 \text{Y-shape} & - & \text{Y-shape} & + & \text{Cross-shape} & = 0
 \end{array}
 \end{array}$$

We can also use pictures to represent cocycles. Take  $\mu \in H^n(\mathfrak{g}, \mathfrak{g})$ . Then we draw  $\mu$  as



with  $n$  lines going in. Then, the Cartan formula for the differential says that

$$d \left( \begin{array}{c} \dots \\ \mu \end{array} \right) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} \begin{array}{c} i \quad j \\ \mu \end{array} + \sum_{1 \leq i \leq n+1} \begin{array}{c} i \\ \mu \end{array}$$

and the bracket of two cocycles  $\mu \in H^m$  and  $\nu \in H^n$  is

$$[\mu, \nu] = \sum_{1 \leq i \leq n} \begin{array}{c} i \\ \mu \\ \nu \end{array} - \sum_{1 \leq i \leq m} \begin{array}{c} i \\ \nu \\ \mu \end{array}$$

► **Exercise 9.5.** Use pictures to show that  $d[\mu, \nu] = \pm[d\mu, \nu] \pm [\mu, d\nu]$ .

Also, these pictures can be used to do the calculations in Exercises 9.3 and 9.4.



## Lecture 10

Here is the take-home exam, it's due on Tuesday:

- (1)  $B \subset SL_2(\mathbb{C})$  are upper triangular matrices, then
  - Describe  $X = SL_2(\mathbb{C})/B$
  - $SL_2(\mathbb{C})$  acts on itself via left multiplication implies that it acts on  $X$ . Describe the action.

$$(2) \text{ Find } \exp \begin{pmatrix} 0 & x_1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & x_{n-1} \\ 0 & & & 0 \end{pmatrix}$$

- (3) Prove that if  $V, W$  are filtered vector spaces (with increasing filtration) and  $\phi : V \rightarrow W$  satisfies  $\phi(V_i) \subseteq W_i$ , and  $Gr(\phi) : Gr(V) \xrightarrow{\sim} Gr(W)$  an isomorphism, then  $\phi$  is a linear isomorphism of filtered spaces.

### Lie algebra cohomology

Recall  $C^*(\mathfrak{g}, M)$  from the previous lecture, for  $M$  a finite dimensional representation of  $\mathfrak{g}$  (and  $\mathfrak{g}$  finite dimensional). There is a book by D. Fuchs, *Cohomology of  $\infty$ -dimensional Lie algebras* [Fuc50].

We computed that  $H^0(\mathfrak{g}, \mathfrak{g}) = Z(\mathfrak{g}) \simeq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$  and that  $H^1(\mathfrak{g}, \mathfrak{g})$  is the space of exterior derivations of  $\mathfrak{g}$ . Say  $c \in Z^1(\mathfrak{g}, \mathfrak{g})$ ,<sup>1</sup> so  $[c] \in H^1(\mathfrak{g}, \mathfrak{g})$ . Define  $\tilde{\mathfrak{g}}_c = \mathfrak{g} \oplus \mathbb{C}\partial_c$  with the bracket  $[(x, t), (y, s)] = ([x, y] - tc(y) + sc(x), 0)$ . So if  $e_1, \dots, e_n$  is a basis in  $\mathfrak{g}$  with the usual relations  $[e_i, e_j] = c_{ij}^k e_k$ , then we get one more generator  $\partial_c$  such that  $[\partial_c, x] = c(x)$ . Then  $H^1(\mathfrak{g}, \mathfrak{g})$  is the space of equivalence classes of extensions

$$0 \rightarrow \mathfrak{g} \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathbb{C} \rightarrow 0$$

up to the equivalences  $f$  such that the diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{g} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathbb{C} \longrightarrow 0 \\ & & \downarrow \text{Id} & & \downarrow f & & \downarrow \text{Id} \\ 0 & \longrightarrow & \mathfrak{g} & \longrightarrow & \tilde{\mathfrak{g}}' & \longrightarrow & \mathbb{C} \longrightarrow 0 \end{array}$$

This is the same as the space of exterior derivations.

<sup>1</sup> $Z^n(\mathfrak{g}, M)$  is the space of  $n$ -cocycles, i.e. the kernel of  $d : C^n(\mathfrak{g}, M) \rightarrow C^{n+1}(\mathfrak{g}, M)$ .

## $H^2(\mathfrak{g}, \mathfrak{g})$ and Deformations of Lie algebras

A *deformation* of  $\mathfrak{g}$  is the vector space  $\mathfrak{g}[[h]]$  with a bracket  $[a, b]_h = [a, b] + \sum_{n \geq 1} h^n m_n(a, b)$  such that  $m_n(a, b) = -m_n(b, a)$  and

$$[a, [b, c]_h]_h + [b, [c, a]_h]_h + [c, [a, b]_h]_h = 0.$$

The  $h^N$  order term of the Jacobi identity is the condition

$$[a, m_N(b, c)] + m_N(a, [b, c]) + \sum_{k=1}^{N-1} m_k(a, m_{N-k}(b, c)) + \text{cycle} = 0 \quad (\dagger\dagger)$$

For  $\mu \in C^2(\mathfrak{g}, \mathfrak{g})$ , we compute

$$d\mu(a, b, c) = -[a, \mu(b, c)] - \mu(a, [b, c]) + \text{cycle}.$$

Define

$$\{m_k, m_{N-k}\}(a, b, c) \stackrel{\text{def}}{=} m_k(a, m_{N-k}(b, c)) + \text{cycle}$$

This is called the Gerstenhaber bracket ... do a Google search for it if you like ... it is a tiny definition from a great big theory.

Then we can rewrite  $(\dagger\dagger)$  as

$$dm_N = \sum_{k=1}^{N-1} \{m_k, m_{N-k}\}. \quad (\ddagger)$$

In particular,  $dm_1 = 0$ , so  $m_1$  is in  $Z^2(\mathfrak{g}, \mathfrak{g})$ .

Equivalences:  $[a, b]'_h \simeq [a, b]_h$  if  $[a, b]'_h = \phi^{-1}([\phi(a), \phi(b)]_h)$  for some  $\phi(a) = a + \sum_{n \geq 1} h^n \phi_n(a)$ . then

$$m'_1(a, b) = m_1(a, b) - \phi_1([a, b]) + [a, \phi_1(b)] + [\phi_1(a), b].$$

which we can write as  $m'_1 = m_1 + d\phi_1$ . From this we can conclude

**Claim.** *The space of equivalence classes of possible  $m_1$  is exactly  $H^2(\mathfrak{g}, \mathfrak{g})$ .*

**Claim** (was HW). *If  $m_1$  is a 2-cocycle, and  $m_{N-1}, \dots, m_2$  satisfy the equations we want, then*

$$d \left( \sum_{k=1}^{N-1} \{m_k, m_{N-k}\} \right) = 0.$$

This is not enough; we know that  $\sum_{k=1}^{N-1} \{m_k, m_{N-k}\}$  is in  $Z^3(\mathfrak{g}, \mathfrak{g})$ , but to find  $m_N$ , we need it to be trivial in  $H^3(\mathfrak{g}, \mathfrak{g})$  because of  $(\ddagger)$ . If the cohomology class of  $\sum_{k=1}^{N-1} \{m_k, m_{N-k}\}$  is non-zero, it's class in  $H^3(\mathfrak{g}, \mathfrak{g})$  is called an *obstruction* to  $n$ -th order deformation. If  $H^3(\mathfrak{g}, \mathfrak{g})$  is zero, then any

first order deformation (element of  $H^2(\mathfrak{g}, \mathfrak{g})$ ) extends to a deformation, but if  $H^3(\mathfrak{g}, \mathfrak{g})$  is non-zero, then we don't know that we can always extend. Thus,  $H^3(\mathfrak{g}, \mathfrak{g})$  is the space of all possible obstructions to extending a deformation.

Let's keep looking at cohomology spaces. Consider  $C^*(\mathfrak{g}, \mathbb{C})$ , where  $\mathbb{C}$  is a one dimensional trivial representation of  $\mathfrak{g}$  given by  $x \mapsto 0$  for any  $x \in \mathfrak{g}$ .

First question: take  $U\mathfrak{g}$ , with the corresponding 1-dimensional representation  $\varepsilon : U\mathfrak{g} \rightarrow \mathbb{C}$  given by  $\varepsilon(x) = 0$  for  $x \in \mathfrak{g}$ .

► **Exercise 10.1.** Show that  $(U\mathfrak{g}, \varepsilon, \Delta, S)$  is a Hopf algebra with the  $\varepsilon$  above,  $\Delta(x) = 1 \otimes x + x \otimes 1$ , and  $S(x) = -x$  for  $x \in \mathfrak{g}$ . Remember that  $\Delta$  and  $\varepsilon$  are algebra homomorphisms, and that  $S$  is an anti-homomorphism.

Let's compute  $H^1(\mathfrak{g}, \mathbb{C})$  ( $H^0$  is boring, just a point). This is  $\ker(C^1 \xrightarrow{d} C^2)$ . Well,  $C^1(\mathfrak{g}, \mathbb{C}) = \text{Hom}(\mathfrak{g}, \mathbb{C})$ ,  $C^2(\mathfrak{g}, \mathbb{C}) = \text{Hom}(\Lambda^2 \mathfrak{g}, \mathbb{C})$ , and

$$dc(x, y) = c([x, y]).$$

So the kernel is the set of  $c$  such that  $c([x, y]) = 0$  for all  $x, y \in \mathfrak{g}$ . Thus,  $\ker(d) \subseteq C^1(\mathfrak{g}, \mathbb{C})$  is the space of  $\mathfrak{g}$ -invariant linear functionals. Recall that  $\mathfrak{g}$  acts on  $\mathfrak{g}$  by the adjoint action, and on  $\mathfrak{g}^* = C^1(\mathfrak{g}, \mathfrak{g})$  by the coadjoint action ( $x : l \mapsto l_x$  where  $l_x(y) = l([x, y])$ ). Under the coadjoint action,  $l \in \mathfrak{g}^*$  is  $\mathfrak{g}$ -invariant if  $l_x = 0$ . Note that  $C^0$  is just one point, so its image doesn't have anything in it.

Now let's compute  $H^2(\mathfrak{g}, \mathbb{C}) = \ker(d : C^2 \rightarrow C^3) / \text{im}(d : C^1 \rightarrow C^2)$ . Let  $c \in Z^2$ , then

$$dc(x, y, z) = c([x, y], z) - c([x, z], y) + c([y, z], x) = 0$$

for all  $x, y, z \in \mathfrak{g}$ . Now let's find the image of  $d : C^1 \rightarrow C^2$ : it is the set of functions of the form  $dl(x, y) = l([x, y])$  where  $l \in \mathfrak{g}^*$ . It is clear that  $l([x, y])$  are (trivial) 2-cocycles because of the Jacobi identity. Let's see what can we cook with this  $H^2$ .

**Definition 10.1.** A *central extension* of  $\mathfrak{g}$  is a short exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0.$$

Two such extensions are equivalent if there is a Lie algebra isomorphism  $f : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}'$  such that the diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \longrightarrow 0 \\ & & \downarrow \text{Id} & & \downarrow f & & \downarrow \text{Id} \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \tilde{\mathfrak{g}}' & \longrightarrow & \mathfrak{g} \longrightarrow 0 \end{array}$$

**Theorem 10.2.**  $H^2(\mathfrak{g}, \mathbb{C})$  is isomorphic to the space of equivalence classes of central extensions of  $\mathfrak{g}$ .

*Proof.* Let's describe the map in one direction. If  $c \in Z^2$ , then consider  $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$  with the bracket  $[(x, t), (y, s)] = ([x, y], c(x, y))$ . Equivalences of extensions boil down to  $c(x, y) \mapsto c(x, y) + l([x, y])$ .

► **Exercise 10.2.** Finish this proof. □

Let's do some (infinite dimensional) examples of central extensions.

**Example 10.3** (Affine Kac-Moody algebras). If  $\mathfrak{g} \subseteq \mathfrak{gl}_n$ , then we define the *loop space* or *loop algebra*  $\mathcal{L}\mathfrak{g}$  to be the set of maps  $S^1 \rightarrow \mathfrak{g}$ . To make the space more manageable, we only consider Laurent polynomials,  $z \mapsto \sum_{m \in \mathbb{Z}} a_m z^m$  for  $a_m \in \mathfrak{g}$  with all but finitely many of the  $a_m$  equal to zero. The bracket is given by  $[f, g]_{\mathcal{L}\mathfrak{g}}(z) = [f(z), g(z)]_{\mathfrak{g}}$ .

Since  $\mathfrak{g} \subseteq \mathfrak{gl}_n$ , there is an induced trace  $tr : \mathfrak{g} \rightarrow \mathbb{C}$ . This gives a non-degenerate inner product on  $\mathcal{L}\mathfrak{g}$ :

$$(f, g) := \oint_{|z|=1} tr(f(z^{-1})g(z)) \frac{dz}{z}.$$

There is a natural 2-cocycle on  $\mathcal{L}\mathfrak{g}$ , given by

$$c(f, g) = \frac{1}{2\pi i} \oint_{|z|=1} tr(f(z)g'(z)) \frac{dz}{z} = \operatorname{Res}_{z=0} \left( tr(f(z)g'(z)) \right),$$

and a natural outer derivation  $\partial : \mathcal{L}\mathfrak{g} \rightarrow \mathcal{L}\mathfrak{g}$  given by  $\partial x(z) = \frac{\partial x(z)}{\partial z}$ .

The Kac-Moody algebra is  $\mathcal{L}\mathfrak{g} \oplus \mathbb{C}\partial \oplus \mathbb{C}c$ . A second course on Lie theory should have some discussion of the representation theory of this algebra.

**Example 10.4.** Let  $\mathfrak{gl}_\infty$  be the algebra of matrices with finitely many non-zero entries. It is not very interesting. Let  $\mathfrak{gl}_\infty^1$  be the algebra of matrices with finitely many non-zero diagonals.  $\mathfrak{gl}_\infty^1$  is "more infinite dimensional" than  $\mathfrak{gl}_\infty$ , and it is more interesting.

► **Exercise 10.3.** Define

$$J = \left( \begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array} \right).$$

For  $x, y \in \mathfrak{gl}_\infty$ , show that

$$c(x, y) = tr(x[J, y])$$

is well defined (i.e. is a finite sum).

This  $c$  is a non-trivial 1-cocycle, i.e.  $[c] \in H^2(\mathfrak{gl}_\infty^1, \mathbb{C})$  is non-zero. By the way, instead of just using linear maps, we require that the maps  $\Lambda^2 \mathfrak{gl}_\infty^1 \rightarrow \mathbb{C}$  are graded linear maps. This is  $H_{\text{graded}}^2$ .

Notice that in  $\mathfrak{gl}_n$ ,  $\text{tr}(x[J, y]) = \text{tr}(J[x, y])$  is a trivial cocycle (it is  $d$  of  $l(x) = \text{tr}(Jx)$ ). So we have that  $H^2(\mathfrak{gl}_n, \mathbb{C}) = \{0\}$ .

We can define  $a_\infty = \mathfrak{gl}_\infty \oplus \mathbb{C}c$ . This is some non-trivial central extension.

To summarize the last lectures:

1. We related Lie algebras and Lie Groups. If you're interested in representations of Lie Groups, looking at Lie algebras is easier.
2. From a Lie algebra  $\mathfrak{g}$ , we constructed  $U\mathfrak{g}$ , the universal enveloping algebra. This got us thinking about associative algebras and Hopf algebras.
3. We learned about dual pairings of Hopf algebras. For example,  $\mathbb{C}[\Gamma]$  and  $C(\Gamma)$  are dual, and  $U\mathfrak{g}$  and  $C(G)$  are dual (if  $G$  is affine algebraic and we are looking at polynomial functions). This pairing is a starting point for many geometric realizations of representations of  $G$ . Conceptually, the notion of the universal enveloping algebra is closely related to the notion of the group algebra  $\mathbb{C}[\Gamma]$ .
4. Finally, we talked about deformations.

## Lecture 11 - Engel's Theorem and Lie's Theorem

The new lecturer is Vera Serganova.

We will cover

1. Classification of semisimple Lie algebras. This will include root systems and Dynkin diagrams.
2. (1) will lead to a classification of compact connected Lie Groups.
3. Finally, we will talk about representation theory of semisimple Lie algebras and the Weyl character formula.

A reference for this material is Fulton and Harris [FH91].

The first part is purely algebraic: we will study Lie algebras.  $\mathfrak{g}$  will be a Lie algebra, usually finite dimensional, over a field  $k$  (usually of characteristic 0).

Any Lie algebra  $\mathfrak{g}$  contains the ideal  $\mathcal{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ , the vector subspace generated by elements of the form  $[X, Y]$  for  $X, Y \in \mathfrak{g}$ .

► **Exercise 11.1.** Show that  $\mathcal{D}\mathfrak{g}$  is an ideal in  $\mathfrak{g}$ .

► **Exercise 11.2.** Let  $G$  be a simply connected Lie group with Lie algebra  $\mathfrak{g}$ . Then  $[G, G]$  is the subgroup of  $G$  generated by elements of the form  $ghg^{-1}h^{-1}$  for  $g, h \in G$ . Show that  $[G, G]$  is a connected closed normal Lie subgroup of  $G$ , with Lie algebra  $\mathcal{D}\mathfrak{g}$ .

⚠ *Warning 11.1.* Exercise 11.2 is a tricky problem. Here are some potential pitfalls:

1. For  $G$  connected, we do not necessarily know that the exponential map is surjective, because  $G$  may not be complete. For example,  $\exp : \mathfrak{sl}_2(\mathbb{C}) \rightarrow SL_2(\mathbb{C})$  is not surjective.<sup>1</sup>
2. If  $H \subseteq G$  is a subgroup with Lie algebra  $\mathfrak{h}$ , then  $\mathfrak{h} \subseteq \mathfrak{g}$  closed is not enough to know that  $H$  is closed in  $G$ . For example, take  $G$  to be a torus, and  $H$  to be a line with irrational slope.
3. This exercise *requires*  $G$  to be simply connected. Let

$$H := \left\{ \left( \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right) \right\} \times S^1$$

$$K := \left\{ \left( \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, c^n \right) \mid n \in \mathbb{Z} \right\} \subseteq H$$

<sup>1</sup>Assume  $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$  is in the image, then its pre-image must have eigenvalues  $(2n+1)i\pi$  and  $-(2n+1)i\pi$  for some integer  $n$ . So the pre-image has distinct eigenvalues, so it is diagonalizable. But that implies that  $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$  is diagonalizable, contradiction.

where  $c$  is an element of  $S^1$  of infinite order. Then  $K$  is normal in  $H$  and  $G = H/K$  is a counterexample.

**Definition 11.2.** Define  $\mathcal{D}^0\mathfrak{g} = \mathfrak{g}$ , and  $\mathcal{D}^n\mathfrak{g} = [\mathcal{D}^{n-1}\mathfrak{g}, \mathcal{D}^{n-1}\mathfrak{g}]$ . This is called the *derived series* of  $\mathfrak{g}$ . We say  $\mathfrak{g}$  is *solvable* if  $\mathcal{D}^n\mathfrak{g} = 0$  for some  $n$  sufficiently large.

**Definition 11.3.** We can also define  $\mathcal{D}_0\mathfrak{g} = \mathfrak{g}$ , and  $\mathcal{D}_n\mathfrak{g} = [\mathfrak{g}, \mathcal{D}_{n-1}\mathfrak{g}]$ . This is called the *lower central series* of  $\mathfrak{g}$ . We say that  $\mathfrak{g}$  is *nilpotent* if  $\mathcal{D}_n\mathfrak{g} = 0$  for some  $n$  sufficiently large.

We will denote  $\mathcal{D}_1\mathfrak{g} = \mathcal{D}^1\mathfrak{g}$  by  $\mathcal{D}\mathfrak{g}$ . Solvable and nilpotent lie algebras are hard to classify. Instead, we will do the classification of semisimple Lie algebras (see Definition 11.15).

The following example is in some sense universal (see corollaries 11.7 and 11.12):

**Example 11.4.** Let  $\mathfrak{gl}(n)$  be the Lie algebra of all  $n \times n$  matrices, and let  $\mathfrak{b}$  be the subalgebra of upper triangular matrices. I claim that  $\mathfrak{b}$  is solvable. To see this, note that  $\mathcal{D}\mathfrak{b}$  is the algebra of *strictly* upper triangular matrices, and in general,  $\mathcal{D}^k\mathfrak{b}$  has zeros on the main diagonal and the  $2^{k-2}$  diagonals above the main diagonal (for  $k \geq 2$ ). Let  $\mathfrak{n} = \mathcal{D}\mathfrak{b}$ . You can check that  $\mathfrak{n}$  is in fact nilpotent.

Useful facts about solvable/nilpotent Lie algebras:

1. If you have an exact sequence of Lie algebras

$$0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a} \rightarrow 0$$

then  $\mathfrak{g}$  is solvable if and only if  $\mathfrak{a}$  and  $\mathfrak{g}/\mathfrak{a}$  are solvable.

2. If you have an exact sequence of Lie algebras

$$0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a} \rightarrow 0$$

then if  $\mathfrak{g}$  is nilpotent, so are  $\mathfrak{a}$  and  $\mathfrak{g}/\mathfrak{a}$ .



*Warning 11.5.* The converse is not true. Diagonal matrices  $\mathfrak{d}$  is nilpotent, and we have

$$0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{b} \rightarrow \mathfrak{d} \rightarrow 0.$$

Note that  $\mathfrak{b}$  is not nilpotent, because  $\mathcal{D}\mathfrak{b} = \mathcal{D}_n\mathfrak{b} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ .

3. If  $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$  are solvable ideals, then the sum  $\mathfrak{a} + \mathfrak{b}$  is solvable. To see this, note that we have

$$0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{a} + \mathfrak{b} \rightarrow \underbrace{(\mathfrak{a} + \mathfrak{b})/\mathfrak{a}}_{\simeq \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})} \rightarrow 0$$

$\mathfrak{a}$  is solvable by assumption, and  $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$  is a quotient of a solvable algebra, so it is solvable by (1). Applying (1) again,  $\mathfrak{a} + \mathfrak{b}$  is solvable.

4. If  $k \subseteq F$  is a field extension, with a Lie algebra  $\mathfrak{g}$  over  $k$ , we can make a lie algebra  $\mathfrak{g} \otimes_k F$  over  $F$ . Note that  $\mathfrak{g} \otimes_k F$  is solvable (nilpotent) if and only if  $\mathfrak{g}$  is.


We will now prove Engel's theorem and Lie's theorem.

For any Lie algebra  $\mathfrak{g}$ , we have the adjoint representation:  $X \mapsto ad_X \in \mathfrak{gl}(\mathfrak{g})$  given by  $ad_X(Y) = [X, Y]$ . If  $\mathfrak{g}$  is nilpotent, then  $ad_X$  is a nilpotent operator for any  $X \in \mathfrak{g}$ . The converse is also true as we will see shortly (Cor. 11.9).

**Theorem 11.6** (Engel's Theorem). *Let  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ , and assume that  $X$  is nilpotent for any  $X \in \mathfrak{g}$ . Then there is a vector  $v \in V$  such that  $\mathfrak{g} \cdot v = 0$ .*

Note that the theorem holds for any representation  $\rho$  of  $\mathfrak{g}$  in which every element acts nilpotently; just replace  $\mathfrak{g}$  in the statement of the theorem by  $\rho(\mathfrak{g})$ .

**Corollary 11.7.** *If  $V$  is a representation of  $\mathfrak{g}$  in which every element acts nilpotently, then one can find  $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$  a complete flag such that  $\mathfrak{g}(V_i) \subseteq V_{i-1}$ . That is, there is a basis in which all of the elements of  $\mathfrak{g}$  are strictly upper triangular.*

 **Warning 11.8.** Note that the theorem isn't true if you say "suppose  $\mathfrak{g}$  is nilpotent" instead of the right thing. For example, the set of diagonal matrices  $\mathfrak{d} \subset \mathfrak{gl}(V)$  is nilpotent.

*Proof.* Let's prove the theorem by induction on  $\dim \mathfrak{g}$ . We first show that  $\mathfrak{g}$  has an ideal  $\mathfrak{a}$  of codimension 1. To see this, take a maximal proper subalgebra  $\mathfrak{a} \subset \mathfrak{g}$ . Look at the representation of  $\mathfrak{a}$  on the quotient space  $\mathfrak{g}/\mathfrak{a}$ . This representation,  $\mathfrak{a} \rightarrow \mathfrak{gl}(\mathfrak{g}/\mathfrak{a})$ , satisfies the condition of the theorem,<sup>2</sup> so by induction, there is some  $X \in \mathfrak{g}$  such that  $ad_{\mathfrak{a}}(X) = 0$  modulo  $\mathfrak{a}$ . So  $[\mathfrak{a}, X] \subseteq \mathfrak{a}$ , so  $\mathfrak{b} = kX \oplus \mathfrak{a}$  is a new subalgebra of  $\mathfrak{g}$  which is larger, so it must be all of  $\mathfrak{g}$ . Thus,  $\mathfrak{a}$  must have had codimension 1. Therefore,  $\mathfrak{a} \subseteq \mathfrak{g}$  is actually an ideal (because  $[X, \mathfrak{a}] = 0$ ).

<sup>2</sup>For any  $X \in \mathfrak{a}$ , since  $X$  is nilpotent,  $ad_X$  is also nilpotent.




Next, we prove the theorem. Let  $V_0 = \{v \in V \mid \mathfrak{a}v = 0\}$ , which is non-zero by the inductive hypothesis. We claim that  $\mathfrak{g}V_0 \subseteq V_0$ . To see this, take  $x \in \mathfrak{g}$ ,  $v \in V_0$ , and  $y \in \mathfrak{a}$ . We have to check that  $y(xv) = 0$ . But

$$yxv = x \underbrace{yv}_0 + \underbrace{[y, x]v}_{\in \mathfrak{a}} = 0.$$

Now, we have that  $\mathfrak{g} = kX \oplus \mathfrak{a}$ , and  $\mathfrak{a}$  kills  $V_0$ , and that  $X : V_0 \rightarrow V_0$  is nilpotent, so it has a kernel. Thus, there is some  $v \in V_0$  which is killed by  $X$ , and so  $v$  is killed by all of  $\mathfrak{g}$ .  $\square$

**Corollary 11.9.** *If  $\text{ad}_X$  is nilpotent for every  $X \in \mathfrak{g}$ , then  $\mathfrak{g}$  is nilpotent as a Lie algebra.*

*Proof.* Let  $V = \mathfrak{g}$ , so we have  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ , which has kernel  $Z(\mathfrak{g})$ . By Engel's theorem, we know that there is an  $x \in \mathfrak{g}$  such that  $(\text{ad } \mathfrak{g})(x) = 0$ . This implies that  $Z(\mathfrak{g}) \neq 0$ . By induction we can assume  $\mathfrak{g}/Z(\mathfrak{g})$  is nilpotent. But then  $\mathfrak{g}$  itself must be nilpotent as well because  $\mathcal{D}_n(\mathfrak{g}/Z(\mathfrak{g})) = 0$  implies  $\mathcal{D}_{n+1}(\mathfrak{g}) = 0$ .  $\square$

 **Warning 11.10.** If  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  is a nilpotent subalgebra, it does not imply that every  $X \in \mathfrak{g}$  is nilpotent (take diagonal matrices for example).

**Theorem 11.11** (Lie's Theorem). *Let  $k$  be algebraically closed and of characteristic 0. If  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  is a solvable subalgebra, then all elements of  $\mathfrak{g}$  have a common eigenvector in  $V$ .*

This is a generalization of the statement that two commuting operators have a common eigenvector.

**Corollary 11.12.** *If  $\mathfrak{g}$  is solvable, then there is a complete flag  $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$  such that  $\mathfrak{g}(V_i) \subseteq V_i$ . That is, there is a basis in which all elements of  $\mathfrak{g}$  are upper triangular.*

*Proof.* If  $\mathfrak{g}$  is solvable, take any subspace  $\mathfrak{a} \subset \mathfrak{g}$  of codimension 1 containing  $\mathcal{D}\mathfrak{g}$ , then  $\mathfrak{a}$  is an ideal. We're going to try to do the same kind of induction as in Engel's theorem.

For a linear functional  $\lambda : \mathfrak{a} \rightarrow k$ , let

$$V_\lambda = \{v \in V \mid Xv = \lambda(X)v \text{ for all } X \in \mathfrak{a}\}.$$

$V_\lambda \neq 0$  for some  $\lambda$  by induction hypothesis.

**Claim.**  $\mathfrak{g}(V_\lambda) \subseteq V_\lambda$ .

*Proof of Claim.* Choose  $v \in V_\lambda$  and  $X \in \mathfrak{a}$ ,  $Y \in \mathfrak{g}$ . Then

$$X(Yv) = Y(\underbrace{Xv}_{\lambda(X)v}) + \underbrace{[X, Y]v}_{\lambda([X, Y])v}$$

We want to show that  $\lambda[X, Y] = 0$ . There is a trick. Let  $r$  be the largest integer such that  $v, Yv, Y^2v, \dots, Y^rv$  is a linearly independent set. We know that  $Xv = \lambda(X)v$  for any  $X \in \mathfrak{a}$ . We claim that  $XY^jv \equiv \lambda(X)Y^jv \pmod{\text{span}\{v, Yv, \dots, Y^{j-1}v\}}$ . This is clear for  $j = 0$ , and by induction, we have

$$\begin{aligned} XY^jv &= Y\underbrace{XY^{j-1}v}_{\equiv \lambda(X)Y^{j-1}v \pmod{\text{span}\{v, \dots, Y^{j-2}v\}}} + \underbrace{[X, Y]Y^{j-1}v}_{\equiv \lambda([X, Y])Y^{j-1}v \pmod{\text{span}\{v, \dots, Y^{j-2}v\}}} \\ &\equiv \lambda(X)Y^jv \pmod{\text{span}\{v, \dots, Y^{j-1}v\}} \end{aligned}$$

So the matrix for  $X$  can be written as  $\lambda(X)$  on the diagonal and stuff above the diagonal (in this basis). So the trace of  $X$  is  $(r+1)\lambda(X)$ . Then we have that  $\text{tr}([X, Y]) = (r+1)\lambda([X, Y])$ , since the above statement was proved for any  $X \in \mathfrak{a}$  and  $[X, Y] \in \mathfrak{a}$ . But the trace of a commutator is always 0. Since the characteristic of  $k$  is 0, we can conclude that  $\lambda[X, Y] = 0$ .

□Claim

To finish the proof, write  $\mathfrak{g} = kT \oplus \mathfrak{a}$ , with  $T : V_\lambda \rightarrow V_\lambda$  (we can do this because of the claim). Since  $k$  is algebraically closed,  $T$  has a non-zero eigenvector  $w$  in  $V_\lambda$ . This  $w$  is the desired common eigenvector. □

*Remark 11.13.* If  $k$  is not algebraically closed, the theorem doesn't hold. For example, consider the (one-dimensional) Lie algebra generated by a rotation of  $\mathbb{R}^2$ .

The theorem also fails if  $k$  is not characteristic 0. Say  $k$  is characteristic  $p$ , then let  $x$  be the permutation matrix of the  $p$ -cycle  $(p \ p-1 \ \dots \ 2 \ 1)$  (i.e. the matrix  $\begin{pmatrix} 0 & I_{p-1} \\ 1 & 0 \end{pmatrix}$ ), and let  $y$  be the diagonal matrix  $\text{diag}(0, 1, 2, \dots, p-1)$ . Then  $[x, y] = x$ , so the Lie algebra generated by  $x$  and  $y$  is solvable. However,  $y$  is diagonal, so we know all of its eigenvectors, and none of them is an eigenvector of  $x$ .

**Corollary 11.14.** *Let  $k$  be of characteristic 0. Then  $\mathfrak{g}$  is solvable if and only if  $\mathcal{D}\mathfrak{g}$  is nilpotent.*

If  $\mathcal{D}\mathfrak{g}$  is nilpotent, then  $\mathfrak{g}$  is solvable from the definitions. If  $\mathfrak{g}$  is solvable, then look at everything over the algebraic closure of  $k$ , where  $\mathfrak{g}$  looks like upper triangular matrices, so  $\mathcal{D}\mathfrak{g}$  is nilpotent. All this is independent of coefficients (by useful fact (4)).

## The radical

There is a unique maximal solvable ideal in  $\mathfrak{g}$  (by useful fact (3): sum of solvable ideals is solvable), which is called the radical of  $\mathfrak{g}$ .

**Definition 11.15.** We call  $\mathfrak{g}$  *semisimple* if  $\text{rad } \mathfrak{g} = 0$ .

► **Exercise 11.3.** Show that  $\mathfrak{g}/\text{rad } \mathfrak{g}$  is always semisimple.

If  $\mathfrak{g}$  is one dimensional, generated by  $X$ , then we have that  $[\mathfrak{g}, \mathfrak{g}] = 0$ , so  $\mathfrak{g}$  cannot be semisimple.

If  $\mathfrak{g}$  is two dimensional, generated by  $X$  and  $Y$ , then we have that  $[\mathfrak{g}, \mathfrak{g}]$  is one dimensional, spanned by  $[X, Y]$ . Thus,  $\mathfrak{g}$  cannot be semisimple because  $\mathcal{D}\mathfrak{g}$  is a solvable ideal.

There is a semisimple Lie algebra of dimension 3, namely  $\mathfrak{sl}_2$ .

Semisimple algebras have really nice properties. Cartan's criterion (Theorem 12.7) says that  $\mathfrak{g}$  is semisimple if and only if the Killing form (see next lecture) is non-degenerate. Whitehead's theorem (Theorem 12.10) says that if  $V$  is a non-trivial irreducible representation of a semisimple Lie algebra  $\mathfrak{g}$ , then  $H^i(\mathfrak{g}, V) = 0$  for all  $i$ . Weyl's theorem (Theorem 12.14) says that every finite-dimensional representation of a semisimple Lie algebra is the direct sum of irreducible representations. If  $G$  is simply connected and compact, the  $\mathfrak{g}$  is semisimple .

## Lecture 12 - Cartan Criterion, Whitehead and Weyl Theorems

### Invariant forms and the Killing form

Let  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a representation. To make the notation cleaner, we will write  $\bar{X}$  for  $\rho(X)$ . We can define a bilinear form on  $\mathfrak{g}$  by  $B_V(X, Y) := \text{tr}(\bar{X}\bar{Y})$ . This form is symmetric because  $\text{tr}(AB) = \text{tr}(BA)$  for any linear operators  $A$  and  $B$ .

We also have that

$$\begin{aligned} B_V([X, Y], Z) &= \text{tr}(\bar{X}\bar{Y}\bar{Z} - \bar{Y}\bar{X}\bar{Z}) = \text{tr}(\bar{X}\bar{Y}\bar{Z}) - \text{tr}(\bar{X}\bar{Z}\bar{Y}) \\ &= \text{tr}(\bar{X}\bar{Y}\bar{Z} - \bar{X}\bar{Z}\bar{Y}) = B_V(X, [Y, Z]), \end{aligned}$$

so  $B$  satisfies

$$B([X, Y], Z) = B(X, [Y, Z]).$$

Such a form is called an *invariant form*. It is called invariant because it is implied by  $B$  being *Ad*-invariant. Assume that for any  $g \in G$  and  $X, Z \in \mathfrak{g}$ , we  $B(\text{Ad}_g X, \text{Ad}_g Z) = B(X, Z)$ . Let  $\gamma$  be a path in  $G$  with  $\gamma'(0) = Y$ . We get that

$$B([Y, X], Z) + B(X, [Y, Z]) = \left. \frac{d}{dt} \right|_{t=0} B(\text{Ad}_{\gamma(t)}(X), \text{Ad}_{\gamma(t)}(Z)) = 0.$$

**Definition 12.1.** The *Killing form*, denoted by  $B$ , is the special case where  $\rho$  is the adjoint representation. That is,  $B(X, Y) := \text{tr}(\text{ad}_X \circ \text{ad}_Y)$ .

► **Exercise 12.1.** Let  $\mathfrak{g}$  be a simple Lie algebra over an algebraically closed field. Check that two invariant forms on  $\mathfrak{g}$  are proportional.

► **Exercise 12.2** (In class). If  $\mathfrak{g}$  is solvable, then  $B(\mathfrak{g}, \mathcal{D}\mathfrak{g}) = 0$ .

*Solution.* First note that if  $Z = [X, Y] \in \mathcal{D}\mathfrak{g}$ , then  $\text{ad}_Z = [\text{ad}_X, \text{ad}_Y] \in \mathcal{D}(\text{ad } \mathfrak{g})$  since the adjoint representation is a Lie algebra homomorphism. Moreover,  $\mathfrak{g}$  solvable implies the image of the adjoint representation,  $\text{ad}(\mathfrak{g}) \simeq \mathfrak{g}/Z(\mathfrak{g})$ , is solvable. Therefore, in some basis of  $V$  of a representation of  $\text{ad}(\mathfrak{g})$ , all matrices of  $\text{ad}(\mathfrak{g})$  are upper triangular (by Lie's Theorem), and those of  $\mathcal{D}(\text{ad } \mathfrak{g})$  are all strictly upper triangular. The product of an upper triangular matrix and a strictly upper triangular matrix will be strictly upper triangular and therefore have trace 0. ■

The converse of this exercise is also true. It will follow as a corollary of our next theorem (Corollary 12.6 below).

**Theorem 12.2.** Suppose  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ ,  $\text{char } k = 0$ , and  $B_V(\mathfrak{g}, \mathfrak{g}) = 0$ . Then  $\mathfrak{g}$  is solvable.

For the proof, we will need the following facts from linear algebra.

**Lemma 12.3.**<sup>1</sup> *Let  $X$  be a diagonalizable linear operator in  $V$ , with  $k$  algebraically closed. If  $X = A \cdot \text{diag}(\lambda_1, \dots, \lambda_n) \cdot A^{-1}$  and  $f : k \rightarrow k$  is a function, we define  $f(X)$  as  $A \cdot \text{diag}(f(\lambda_1), \dots, f(\lambda_n)) \cdot A^{-1}$ . Suppose  $\text{tr}(X \cdot f(X)) = 0$  for any  $\mathbb{Q}$ -linear map  $f : k \rightarrow k$  such that  $f$  is the identity on  $\mathbb{Q}$ , then  $X = 0$ .*

*Proof.* Consider only  $f$  such that the image of  $f$  is  $\mathbb{Q}$ . Let  $\lambda_1, \dots, \lambda_m$  be the eigenvalues of  $X$  with multiplicities  $n_1, \dots, n_m$ . We obtain  $\text{tr}(X \cdot f(X)) = n_1 \lambda_1 f(\lambda_1) + \dots + n_m \lambda_m f(\lambda_m) = 0$ . Apply  $f$  to this identity to obtain  $n_1 f(\lambda_1)^2 + \dots + n_m f(\lambda_m)^2 = 0$  which implies  $f(\lambda_i) = 0$  for all  $i$ . If some  $\lambda_i$  is not zero, we can choose  $f$  so that  $f(\lambda_i) \neq 0$ , so  $\lambda_i = 0$  for all  $i$ . Since  $X$  is diagonalizable,  $X = 0$ .  $\square$

**Lemma 12.4** (Jordan Decomposition). *Given  $X \in \mathfrak{gl}(V)$ , there are unique  $X_s, X_n \in \mathfrak{gl}(V)$  such that  $X_s$  is diagonalizable,  $X_n$  is nilpotent,  $[X_s, X_n] = 0$ , and  $X = X_n + X_s$ . Furthermore,  $X_s$  and  $X_n$  are polynomials in  $X$ .*

*Proof.* All but the last statement is standard; see, for example, Corollary 2.5 of Chapter XIV of [Lan02]. To see the last statement, let the characteristic polynomial of  $X$  be  $\prod_i (x - \lambda_i)^{n_i}$ . By the Chinese remainder theorem, we can find a polynomial  $f$  such that  $f(x) \equiv \lambda_i \pmod{(x - \lambda_i)^{n_i}}$ . Choose a basis so that  $X$  is in Jordan form and compute  $f(X)$  block by block. On a block with  $\lambda_i$  along the diagonal  $(X - \lambda_i I)^{n_i}$  is 0, so  $f(X)$  is  $\lambda_i I$  on this block. Then  $f(X) = X_s$  is diagonalizable and  $X_n = X - f(X)$  is nilpotent.  $\square$

**Lemma 12.5.** *Let  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ . The adjoint representation  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  preserves Jordan decomposition:  $\text{ad}_{X_s} = (\text{ad}_X)_s$  and  $\text{ad}_{X_n} = (\text{ad}_X)_n$ . In particular,  $\text{ad}_{X_s}$  is a polynomial in  $\text{ad}_X$ .*

*Proof.* Suppose the eigenvalues of  $X_s$  are  $\lambda_1, \dots, \lambda_m$ , and we are in a basis where  $X_s$  is diagonal. Check that  $\text{ad}_{X_s}(E_{ij}) = [X_s, E_{ij}] = (\lambda_i - \lambda_j)E_{ij}$ . So  $X_s$  diagonalizable implies  $\text{ad}_{X_s}$  is diagonalizable (because it has a basis of eigenvectors). We have that  $\text{ad}_{X_n}$  is nilpotent because the monomials in the expansion of  $(\text{ad}_{X_n})^k(Y)$  have  $X_n$  to at least the  $k/2$  power on one side of  $Y$ . So we have that  $\text{ad}_X = \text{ad}_{X_s} + \text{ad}_{X_n}$ , with  $\text{ad}_{X_s}$  diagonalizable,  $\text{ad}_{X_n}$  nilpotent, and the two commute, so by uniqueness of the Jordan decomposition,  $\text{ad}_{X_s} = (\text{ad}_X)_s$  and  $\text{ad}_{X_n} = (\text{ad}_X)_n$ .  $\square$

*Proof of Theorem 12.2.* It is enough to show that  $\mathcal{D}\mathfrak{g}$  is nilpotent. Let  $X \in \mathcal{D}\mathfrak{g}$ , so  $X = \sum [Y_i, Z_i]$ . It suffices to show that  $X_s = 0$ . To do this, let

<sup>1</sup>This is different from what we did in class. There is an easier way to do this if you are willing to assume  $k = \mathbb{C}$  and use complex conjugation. See Fulton and Harris for this method.

$f : k \rightarrow k$  be any  $\mathbb{Q}$ -linear map fixing  $\mathbb{Q}$ .

$$\begin{aligned}
 B_V(f(X_s), X_s) &= B_V(f(X_s), X) && (X_n \text{ doesn't contribute}) \\
 &= B_V\left(f(X_s), \sum_i [Y_i, Z_i]\right) \\
 &= \sum_i B_V(\underbrace{[f(X_s), Y_i]}_{\in \mathfrak{g}}, Z_i) && (B_V \text{ invariant}) \\
 &= 0 && (\text{assuming } [f(X_s), Y_i] \in \mathfrak{g})
 \end{aligned}$$

Then by Lemma 12.3,  $X_s = 0$ .

To see that  $[f(X_s), Y_i] \in \mathfrak{g}$ , suppose the eigenvalues of  $X_s$  are  $\lambda_1, \dots, \lambda_m$ . Then the eigenvalues of  $f(X_s)$  are  $f(\lambda_i)$ , the eigenvalues of  $ad_{X_s}$  are of the form  $\mu_{ij} := \lambda_i - \lambda_j$ , and eigenvalues of  $ad_{f(X_s)}$  are  $\nu_{ij} := f(\lambda_i) - f(\lambda_j) = f(\mu_{ij})$ . If we define  $g$  to be a polynomial such that  $g(\mu_{ij}) = \nu_{ij}$ , then  $ad_{f(X_s)}$  and  $g(ad_{X_s})$  are diagonal (in some basis) with the same eigenvalues in the same places, so they are equal. So we have

$$\begin{aligned}
 [f(X_s), Y_i] &= g(ad_{X_s})(Y_i) \\
 &= h(ad_X)(Y_i) \in \mathfrak{g} && (\text{using Lemma 12.5})
 \end{aligned}$$

for some polynomial  $h$ .

The above arguments assume  $k$  is algebraically closed, so if it's not apply the above to  $\mathfrak{g} \otimes_k \bar{k}$ . Then  $\mathfrak{g} \otimes_k \bar{k}$  solvable implies  $\mathfrak{g}$  solvable as mentioned in the previous lecture.  $\square$

**Corollary 12.6.**  $\mathfrak{g}$  is solvable if and only if  $B(\mathcal{D}\mathfrak{g}, \mathfrak{g}) = 0$ .

*Proof.* ( $\Leftarrow$ ) We have that  $B(\mathcal{D}\mathfrak{g}, \mathfrak{g}) = 0$  implies  $B(\mathcal{D}\mathfrak{g}, \mathcal{D}\mathfrak{g}) = 0$  which implies that  $ad(\mathcal{D}\mathfrak{g})$  is solvable. The adjoint representation of  $\mathcal{D}\mathfrak{g}$  gives the exact sequence

$$0 \rightarrow Z(\mathcal{D}\mathfrak{g}) \rightarrow \mathcal{D}\mathfrak{g} \rightarrow ad(\mathcal{D}\mathfrak{g}) \rightarrow 0.$$

Since  $Z(\mathcal{D}\mathfrak{g})$  and  $ad(\mathcal{D}\mathfrak{g})$  are solvable,  $\mathcal{D}\mathfrak{g}$  is solvable by useful fact (1) of Lecture 11, so  $\mathfrak{g}$  is solvable.

( $\Rightarrow$ ) This is exercise 12.2.  $\square$

**Theorem 12.7** (Cartan's Criterion). *The Killing form is non-degenerate if and only if  $\mathfrak{g}$  is semi-simple.*

*Proof.* Say  $\mathfrak{g}$  is semisimple. Let  $\mathfrak{a} = \ker B$ . Because  $B$  is invariant, we get that  $\mathfrak{a}$  is an ideal, and  $B|_{\mathfrak{a}} = 0$ . By the previous theorem (12.2), we have that  $\mathfrak{a}$  is solvable, so  $\mathfrak{a} = 0$  (by definition of semisimple).

Suppose that  $\mathfrak{g}$  is not semisimple, so  $\mathfrak{g}$  has a non-trivial solvable ideal. Then the last non-zero term in its upper central series is some abelian ideal

$\mathfrak{a} \subseteq \mathfrak{g}$ <sup>2</sup>. For any  $X \in \mathfrak{a}$ , the matrix of  $ad_X$  is of the form  $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$  with respect to the decomposition  $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}/\mathfrak{a}$ , and for  $Y \in \mathfrak{g}$ ,  $ad_Y$  is of the form  $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ . Thus, we have that  $tr(ad_X \circ ad_Y) = 0$  so  $X \in \ker B$ , so  $B$  is degenerate.  $\square$

**Theorem 12.8.** *Any semisimple Lie algebra is a direct sum of simple algebras.*

*Proof.* If  $\mathfrak{g}$  is simple, then we are done. Otherwise, let  $\mathfrak{a} \subseteq \mathfrak{g}$  be an ideal, then the restriction  $B|_{\mathfrak{a}}$  is nondegenerate. Note that  $\mathfrak{a}^\perp$  is also an ideal by invariance of  $B$ . Look at  $\mathfrak{a} \cap \mathfrak{a}^\perp$ . On their intersection,  $B$  is zero, so the intersection is a solvable ideal, so it is zero. Thus, we have that  $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ . By induction on dimension, we're done.  $\square$

*Remark 12.9.* In particular, if  $\mathfrak{g} = \bigoplus \mathfrak{g}_i$  is semisimple, with each  $\mathfrak{g}_i$  simple, we have that  $\mathcal{D}\mathfrak{g} = \bigoplus \mathcal{D}\mathfrak{g}_i$ . But  $\mathcal{D}\mathfrak{g}_i$  is either 0 or  $\mathfrak{g}_i$ , and it cannot be 0 (lest  $\mathfrak{g}_i$  be a solvable ideal). Thus  $\mathcal{D}\mathfrak{g} = \mathfrak{g}$ .

**Theorem 12.10** (Whitehead). *If  $\mathfrak{g}$  is semisimple and  $V$  is an irreducible non-trivial representation of  $\mathfrak{g}$ , then  $H^i(\mathfrak{g}, V) = 0$  for all  $i \geq 0$ .*

*Proof.* The proof uses the Casimir operator,  $C_V \in \mathfrak{gl}(V)$ . Assume for the moment that  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ . Choose a basis  $e_1, \dots, e_n$  in  $\mathfrak{g}$ , with dual basis  $f_1, \dots, f_n$  in  $\mathfrak{g}$  (dual with respect to  $B_V$ , so  $B_V(e_i, f_j) = \delta_{ij}$ ). It is necessary that  $B_V$  be non-degenerate for such a dual basis to exist, and this is where we use that  $\mathfrak{g}$  is semisimple. The<sup>3</sup> Casimir operator is defined to be  $C_V = \sum e_i \circ f_i \in \mathfrak{gl}(V)$  (where  $\circ$  is composition of linear operators on  $V$ ). The main claim is that  $[C_V, X] = 0$  for any  $X \in \mathfrak{g}$ . This can be checked directly: put  $[X, f_i] = \sum a_{ij} f_j$ ,  $[X, e_i] = \sum b_{ij} e_j$ , then apply  $B_V$  to obtain  $a_{ji} = B_V(e_i, [X, f_j]) = B_V([e_i, X], f_j) = -b_{ij}$ , where the middle equality is by invariance of  $B_V$ .

$$\begin{aligned} [X, C_V] &= \sum_i X e_i f_i - e_i X f_i + e_i X f_i - e_i f_i X \\ &= \sum_i [X, e_i] f_i + e_i [X, f_i] \\ &= \sum_i \sum_j b_{ij} e_j f_i + a_{ij} e_i f_j \\ &= \sum_i \sum_j (a_{ij} + b_{ji}) e_i f_j = 0. \end{aligned}$$

<sup>2</sup>Quick exercise: why is  $\mathfrak{a}$  an ideal?

<sup>3</sup>We will soon see that  $C_V$  is independent of the basis  $e_1, \dots, e_n$ , so the article "the" is appropriate.

Suppose  $V$  is irreducible, and  $k$  is algebraically closed. Then the condition  $[C_V, X] = 0$  means precisely that  $C_V$  is an intertwiner so by Schur's lemma,  $C_V = \lambda \text{Id}$ . We can compute

$$\begin{aligned} \text{tr}_V C_V &= \sum_{i=1}^{\dim \mathfrak{g}} \text{tr}(e_i f_i) \\ &= \sum B_V(e_i, f_i) = \dim \mathfrak{g}. \end{aligned}$$

Thus, we have that  $\lambda = \frac{\dim \mathfrak{g}}{\dim V}$ , in particular, it is non-zero.

For any representation  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ , we can still talk about  $C_V$ , but we define it for the image  $\rho(\mathfrak{g})$ , so  $C_V = \frac{\dim \rho(\mathfrak{g})}{\dim V} \text{Id}$ . We get that  $[C_V, \rho(X)] = 0$ . The point is that if  $V$  is non-trivial irreducible, we have that  $C_V$  is non-zero.

Now consider the complex calculating the cohomology:

$$\text{Hom}(\Lambda^k \mathfrak{g}, V) \xrightarrow{d} \text{Hom}(\Lambda^{k+1} \mathfrak{g}, V)$$

We will construct a chain homotopy<sup>4</sup>  $\gamma : C^{k+1} \rightarrow C^k$  between the zero map on the complex and the map  $C_V = \frac{\dim \rho(\mathfrak{g})}{\dim V} \text{Id}$ :

$$\gamma c(x_1, \dots, x_k) = \sum_i e_i c(f_i, x_1, \dots, x_k)$$

► **Exercise 12.3.** Check directly that  $(\gamma d + d\gamma)c = C_V c$ .

Thus  $\gamma d + d\gamma = C_V = \lambda \text{Id}$  (where  $\lambda = \frac{\dim \rho(\mathfrak{g})}{\dim V}$ ). Now suppose  $dc = 0$ . Then we have that  $d\gamma(c) = \lambda c$ , so  $c = \frac{d(\gamma(c))}{\lambda}$ . Thus,  $\ker d / \text{im } d = 0$ , as desired.  $\square$

*Remark 12.11.* What is  $H^1(\mathfrak{g}, k)$ , where  $k$  is the trivial representation of  $\mathfrak{g}$ ? Recall that the cochain complex is

$$k \rightarrow \text{Hom}(\mathfrak{g}, k) \xrightarrow{d} \text{Hom}(\Lambda^2 \mathfrak{g}, k) \rightarrow \dots$$

If  $c \in \text{Hom}(\mathfrak{g}, k)$  and  $c \in \ker d$ , then  $dc(x, y) = c([x, y]) = 0$ , so  $c$  is 0 on  $\mathcal{D}\mathfrak{g} = \mathfrak{g}$ . So we get that  $H^1(\mathfrak{g}, k) = (\mathfrak{g}/\mathcal{D}\mathfrak{g})^* = 0$ .

However, it is not true that  $H^i(\mathfrak{g}, k) = 0$  for  $i \geq 2$ . Recall from Lecture 10 that  $H^2(\mathfrak{g}, k)$  parameterizes central extensions of  $\mathfrak{g}$  (Theorem 10.2).

► **Exercise 12.4.** Compute  $H^j(\mathfrak{sl}_2, k)$  for all  $j$ .

<sup>4</sup>Don't worry about the term "chain homotopy" for now. It just means that  $\gamma$  satisfies the equation in Exercise 12.3. See Proposition 2.12 of [Hat02].



*Remark 12.12.* Note that for  $\mathfrak{g}$  semisimple, we have  $H^1(\mathfrak{g}, M) = 0$  for any finite dimensional representation  $M$  (not just irreducibles). We have already seen that this holds when  $M$  is trivial and Whitehead's Theorem shows this when  $M$  is non-trivial irreducible. If  $M$  is not irreducible, use short exact sequences to long exact sequences in cohomology: if

$$0 \rightarrow W \rightarrow M \rightarrow V \rightarrow 0$$

is an exact sequence of representations of  $\mathfrak{g}$ , then

$$\rightarrow H^1(\mathfrak{g}, V) \rightarrow H^1(\mathfrak{g}, M) \rightarrow H^1(\mathfrak{g}, W) \rightarrow$$

is exact. The outer guys are 0 by induction on dimension, so the middle guy is zero.

We need a lemma before we do Weyl's Theorem.

**Lemma 12.13.** *Say we have a short exact sequence*

$$0 \rightarrow W \rightarrow M \rightarrow V \rightarrow 0.$$

*If  $H^1(\mathfrak{g}, \underbrace{\text{Hom}_k(V, W)}_{V^* \otimes W}) = 0$ , then the sequence splits.*

*Proof.* Let  $X \in \mathfrak{g}$ . Let  $X_W$  represent the induced linear operator on  $W$ . Then we can write  $X_M = \begin{pmatrix} X_W & c(X) \\ 0 & X_V \end{pmatrix}$ . What is  $c(X)$ ? It is an element of  $\text{Hom}_k(V, W)$ . So  $c$  is a linear function from  $\mathfrak{g}$  to  $\text{Hom}_k(V, W)$ . It will be a 1-cocycle: we have  $[X_M, Y_M] = [X, Y]_M$  because these are representations, which gives us

$$X_W c(Y) - c(Y) X_V - (Y_W c(X) - c(X) Y_V) = c([X, Y]).$$

In general,  $dc(X, Y) = c([X, Y]) - Xc(Y) + Yc(X)$ , where  $Xc(Y)$  is given by the action of  $X \in \mathfrak{g}$  on  $V^* \otimes W$ , which is not necessarily composition. In our case this action is by commutation, where  $c(Y)$  is extended to an endomorphism of  $V \oplus W$  by writing it as  $\begin{pmatrix} 0 & c(Y) \\ 0 & 0 \end{pmatrix}$ . The line above says exactly that  $dc = 0$ .

Put  $\Gamma = \begin{pmatrix} 1_W & K \\ 0 & 1_V \end{pmatrix}$ . Conjugating by  $\Gamma$  gives an equivalent representation. We have

$$\Gamma X_M \Gamma^{-1} = \begin{pmatrix} X_W & c(X) + KX_V - X_W K \\ 0 & X_V \end{pmatrix}$$

We'd like to kill the upper right part (to show that  $X$  acts on  $V$  and  $W$  separately). We have  $c \in \text{Hom}(\mathfrak{g}, V^* \otimes W)$ ,  $K \in V^* \otimes W$ . Since the first cohomology is zero,  $dc = 0$ , so we can find a  $K$  such that  $c = dK$ . Since  $c(X) = dK(X) = X(K) = X_W K - KX_V$ , the upper right part is indeed 0.  $\square$

**Theorem 12.14** (Weyl). *If  $\mathfrak{g}$  is semi-simple and  $V$  is a finite-dimensional representation of  $\mathfrak{g}$ , then  $V$  is semi-simple<sup>5</sup> (i.e. completely reducible).*

*Proof.* The theorem follows immediately from Lemma 12.13 and Remark 12.12.  $\square$

Weyl proved this using the unitary trick, which involves knowing about compact real forms.

*Remark 12.15.* We know from Lecture 10 that deformations of  $\mathfrak{g}$  are enumerated by  $H^2(\mathfrak{g}, \mathfrak{g})$ . This means that semisimple Lie algebras do not have any deformations! This suggests that the variety of semisimple Lie algebras is discrete. Perhaps we can classify them.

Whenever you have a Lie algebra  $\mathfrak{g}$ , you can ask about  $\text{Aut } \mathfrak{g}$ .  $\text{Aut } \mathfrak{g}$  will be a Lie group. What is the connected component of the identity? We can also ask about the Lie algebra of the group of automorphisms. Let  $X(t)$  be a path in  $\text{Aut } \mathfrak{g}$  such that  $X(0) = 1$ . Then we compute  $\left. \frac{d}{dt} X(t) \right|_{t=0} = \phi$ . We have that

$$[X(t)Y, X(t)Z] = X(t)([Y, Z])$$

We can differentiate this and get  $\phi \in \text{End } \mathfrak{g}$  with  $\phi[Y, Z] = [\phi Y, Z] + [Y, \phi Z]$  – derivations.

As an immediate corollary of the Jacobi identity, we get that  $ad_X$  is a derivation on  $\mathfrak{g}$ . So  $ad(\mathfrak{g}) \subseteq \text{Der}(\mathfrak{g})$ .

► **Exercise 12.5.** Check that  $ad(\mathfrak{g})$  is an ideal.

We have seen in lecture 9 (page 38) that  $\text{Der}(\mathfrak{g})/ad(\mathfrak{g}) \simeq H^1(\mathfrak{g}, \mathfrak{g})$ . The conclusion is that  $\text{Der}(\mathfrak{g}) = ad(\mathfrak{g})$ — that is, all derivations on a semisimple Lie algebra are inner.

What does this say about the connected component of the identity of  $\text{Aut } \mathfrak{g}$ ? It has the same Lie algebra as  $\mathfrak{g}$ , so it is just  $AdG$ .

Next time we'll talk about root decomposition of semisimple Lie algebras.

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<sup>5</sup>For any invariant subspace  $W \subseteq V$ , there is an invariant  $W' \subseteq V$  so that  $V = W \oplus W'$ .

## Lecture 13 - The root system of a semi-simple Lie algebra

The goal for today is to start with a semisimple Lie algebra over a field  $k$  (assumed algebraically closed and characteristic zero), and get a root system.

Recall Jordan decomposition. For  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ , any  $x \in \mathfrak{g}$  can be written (uniquely) as  $x = x_s + x_n$ , where  $x_s$  is semisimple and  $x_n$  is nilpotent, both of which are polynomials in  $x$ . In general,  $x_s$  and  $x_n$  are in  $\mathfrak{gl}(V)$ , but not necessarily in  $\mathfrak{g}$ .

**Proposition 13.1.** *If  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  is semisimple, then  $x_s, x_n \in \mathfrak{g}$ .*

*Proof.* Notice that  $\mathfrak{g}$  acts on  $\mathfrak{gl}(V)$  via commutator, and  $\mathfrak{g}$  is an invariant subspace. By complete reducibility (Theorem 12.14), we can write  $\mathfrak{gl}(V) = \mathfrak{g} \oplus \mathfrak{m}$  where  $\mathfrak{m}$  is  $\mathfrak{g}$ -invariant, so

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g} \quad \text{and} \quad [\mathfrak{g}, \mathfrak{m}] \subseteq \mathfrak{m}.$$

We have that  $ad_{x_s}$  and  $ad_{x_n}$  are polynomials in  $ad_x$  (by Lemma 12.5), so

$$[x_n, \mathfrak{g}] \subseteq \mathfrak{g}, \quad [x_s, \mathfrak{g}] \subseteq \mathfrak{g} \quad \text{and} \quad [x_n, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [x_s, \mathfrak{m}] \subseteq \mathfrak{m}. \quad (\dagger)$$

Take  $x_n = a + b \in \mathfrak{g} \oplus \mathfrak{m}$ , where  $a \in \mathfrak{g}$  and  $b \in \mathfrak{m}$ . We would like to show that  $b = 0$ , for then we would have that  $x_n \in \mathfrak{g}$ , from which it would follow that  $x_s \in \mathfrak{g}$ .

Decompose  $V = V_1 \oplus \cdots \oplus V_n$  with the  $V_i$  irreducible. Since  $x_n$  is a polynomial in  $x$ , we have that  $x_n(V_i) \subseteq V_i$ , and  $a(V_i) \subseteq V_i$  since  $a \in \mathfrak{g}$ , so  $b(V_i) \subseteq V_i$ . Moreover, we have that

$$[x_n, \mathfrak{g}] = \underbrace{[a, \mathfrak{g}]}_{\in \mathfrak{g}} + \underbrace{[b, \mathfrak{g}]}_{\in \mathfrak{m}} \subseteq \mathfrak{g},$$

so  $[b, \mathfrak{g}] = 0$  (i.e.  $b$  is an intertwiner). By Schur's lemma,  $b$  must be a scalar operator on  $V_i$  (i.e.  $b|_{V_i} = \lambda_i \text{Id}$ ). We have  $\text{tr}_{V_i}(x_n) = 0$  because  $x_n$  is nilpotent. Also  $\text{tr}_{V_i}(a) = 0$  because  $\mathfrak{g}$  is semisimple implies  $\mathcal{D}\mathfrak{g} = \mathfrak{g}$ , so  $a = \sum [x_k, y_k]$ , and the traces of commutators are 0. Thus,  $\text{tr}_{V_i}(b) = 0$ , so  $\lambda_i = 0$  and  $b = 0$ . Now  $x_n = a \in \mathfrak{g}$ , and so  $x_s \in \mathfrak{g}$ .  $\square$

Since the image of a semisimple Lie algebra is semisimple, the proposition tells us that for any representation  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ , the semisimple and nilpotent parts of  $\rho(x)$  are in the image of  $\mathfrak{g}$ . In fact, the following corollary shows that there is an *absolute* Jordan decomposition  $x = x_s + x_n$  within  $\mathfrak{g}$ .

**Corollary 13.2.** *If  $\mathfrak{g}$  is semisimple, and  $x \in \mathfrak{g}$ , then there are  $x_s, x_n \in \mathfrak{g}$  such that for any representation  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ , we have  $\rho(x_s) = \rho(x)_s$  and  $\rho(x_n) = \rho(x)_n$ .*

*Proof.* Consider the (faithful) representation  $ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ . By the proposition, there are some  $x_s, x_n \in \mathfrak{g}$  such that  $(ad_x)_s = ad_{x_s}$  and  $(ad_x)_n = ad_{x_n}$ . Since  $ad$  is faithful,  $ad_x = ad_{x_n} + ad_{x_s}$  and  $ad_{[x_n, x_s]} = [ad_{x_n}, ad_{x_s}] = 0$  tell us that  $x = x_n + x_s$  and  $[x_s, x_n] = 0$ . These are our candidates for the absolute Jordan decomposition.

Given any surjective Lie algebra homomorphism  $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}'$ , we have that  $ad_{\sigma(y)}(\sigma(z)) = \sigma(ad_y(z))$ , from which it follows that  $ad_{\sigma(x_s)}$  is diagonalizable and  $ad_{\sigma(x_n)}$  is nilpotent (note that we've used surjectivity of  $\sigma$ ). Thus,  $\sigma(x)_n = \sigma(x_n)$  and  $\sigma(x)_s = \sigma(x_s)$ . That is, our candidates are preserved by surjective homomorphisms.

Now given any representation  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ , the previous paragraph allows us to replace  $\mathfrak{g}$  by its image, so we may assume  $\rho$  is faithful. By the proposition, there are some  $y, z \in \mathfrak{g}$  such that  $\rho(x)_s = \rho(y)$ ,  $\rho(x)_n = \rho(z)$ . Then  $[\rho(y), -]_{\mathfrak{gl}(\rho(\mathfrak{g}))}$  is a diagonalizable operator on  $\mathfrak{gl}(\rho(\mathfrak{g})) \cong \mathfrak{gl}(\mathfrak{g})$ , and  $[\rho(z), -]_{\mathfrak{gl}(\rho(\mathfrak{g}))}$  is nilpotent. Uniqueness of the Jordan decomposition implies that  $\rho(y) = \rho(x_s)$  and  $\rho(z) = \rho(x_n)$ . Since  $\rho$  is faithful, it follows that  $y = x_s$  and  $z = x_n$ .  $\square$

**Definition 13.3.** We say  $x \in \mathfrak{g}$  is *semisimple* if  $ad_x$  is diagonalizable. We say  $x$  is *nilpotent* if  $ad_x$  is nilpotent.

Given any representation  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  with  $\mathfrak{g}$  semisimple, the corollary tells us that if  $x$  is semisimple, then  $\rho(x)$  is diagonalizable, and if  $x$  is nilpotent, then  $\rho(x)$  is nilpotent. If  $\rho$  is faithful, then  $x$  is semisimple (resp. nilpotent) if and only if  $\rho(x)$  is semisimple (resp. nilpotent).

**Definition 13.4.** We denote the set of all semisimple elements in  $\mathfrak{g}$  by  $\mathfrak{g}_{ss}$ . We call an  $x \in \mathfrak{g}_{ss}$  *regular* if  $\dim(\ker ad_x)$  is minimal (i.e. the dimension of the centralizer is minimal).

**Example 13.5.** Let  $\mathfrak{g} = \mathfrak{sl}_n$ . Semisimple elements of  $\mathfrak{sl}_n$  are exactly the diagonalizable matrices, and nilpotent elements are exactly the nilpotent matrices. If  $x \in \mathfrak{g}$  is diagonalizable, then the centralizer is minimal exactly when all the eigenvalues are distinct. So the regular elements are the diagonalizable matrices with distinct eigenvalues.

Let  $h \in \mathfrak{g}_{ss}$  be regular. We have that  $ad_h$  is diagonalizable, so we can write  $\mathfrak{g} = \bigoplus_{\mu \in k} \mathfrak{g}_\mu$ , where  $\mathfrak{g}_\mu = \{x \in \mathfrak{g} \mid [h, x] = \mu x\}$  are eigenspaces of  $ad_h$ . We know that  $\mathfrak{g}_0 \neq 0$  because it contains  $h$ . There are some other properties:

1.  $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subseteq \mathfrak{g}_{\mu+\nu}$ .
2.  $\mathfrak{g}_0 \subseteq \mathfrak{g}$  is a subalgebra.
3.  $B(\mathfrak{g}_\mu, \mathfrak{g}_\nu) = 0$  if  $\mu \neq -\nu$  (here,  $B$  is the Killing form, as usual).

4.  $B|_{\mathfrak{g}_\mu \oplus \mathfrak{g}_{-\mu}}$  is non-degenerate, and  $B|_{\mathfrak{g}_0}$  is non-degenerate.

*Proof.* Property 1 follows from the Jacobi identity: if  $x \in \mathfrak{g}_\mu$  and  $y \in \mathfrak{g}_\nu$ , then

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \mu[x, y] + \nu[x, y],$$

so  $[x, y] \in \mathfrak{g}_{\mu+\nu}$ . Property 2 follows immediately from 1. Property 3 follows from 1 because  $ad_x \circ ad_y : \mathfrak{g}_\gamma \rightarrow \mathfrak{g}_{\gamma+\mu+\nu}$ , so  $B(x, y) = tr(ad_x \circ ad_y) = 0$  whenever  $\mu + \nu \neq 0$ . Finally, Cartan's criterion says that  $B$  must be non-degenerate, so property 4 follows from 3.  $\square$

**Proposition 13.6.** *In the situation above ( $\mathfrak{g}$  is semisimple and  $h \in \mathfrak{g}_{ss}$  is regular),  $\mathfrak{g}_0$  is abelian.*

*Proof.* Take  $x \in \mathfrak{g}_0$ , and write  $x = x_s + x_n$ . Since  $ad_{x_n}$  is a polynomial of  $ad_x$ , we have  $[x_n, h] = 0$ , so  $x_n \in \mathfrak{g}_0$ , from which we get  $x_s \in \mathfrak{g}_0$ . Since  $[x_s, h] = 0$ , we know that  $ad_{x_s}$  and  $ad_h$  are simultaneously diagonalizable (recall that  $ad_{x_s}$  is diagonalizable). Thus, for generic  $t \in k$ , we have that  $\ker ad_{h+tx_s} \subseteq \ker ad_h$ . Since  $h$  is regular,  $\ker ad_{x_s} = \ker ad_h = \mathfrak{g}_0$ . So  $[x_s, \mathfrak{g}_0] = 0$ , which implies that  $\mathfrak{g}_0$  is nilpotent by Corollary 11.9 to Engel's Theorem. Now we have that  $ad_x : \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$  is nilpotent, and we want  $ad_x$  to be the zero map. Notice that  $B(\mathfrak{g}_0, \mathcal{D}\mathfrak{g}_0) = 0$  since  $\mathfrak{g}_0$  is nilpotent, but  $B|_{\mathfrak{g}_0}$  is non-degenerate by property 4 above, so  $\mathcal{D}\mathfrak{g}_0 = 0$ , so  $\mathfrak{g}_0$  is abelian.  $\square$

**Definition 13.7.** We call  $\mathfrak{h} := \mathfrak{g}_0$  the *Cartan subalgebra* of  $\mathfrak{g}$  (associated to  $h$ ).

In Theorem 14.1, we will show that any two Cartan subalgebras of a semisimple Lie algebra  $\mathfrak{g}$  are related by an automorphism of  $\mathfrak{g}$ , but for now we just fix one. See [Hum78, §15] for a more general definition of Cartan subalgebras.

► **Exercise 13.1.** Show that if  $\mathfrak{g}$  is semisimple,  $\mathfrak{h}$  consists of semisimple elements.

All elements of  $\mathfrak{h}$  are simultaneously diagonalizable because they are all diagonalizable (by the above exercise) and they all commute (by the above proposition). For  $\alpha \in \mathfrak{h}^* \setminus \{0\}$  consider

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

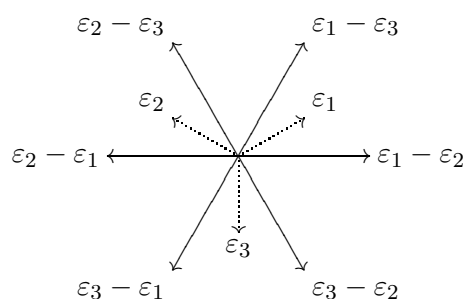
If this  $\mathfrak{g}_\alpha$  is non-trivial, it is called a *root space* and the  $\alpha$  is called a *root*. The *root decomposition* (or *Cartan decomposition*) of  $\mathfrak{g}$  is  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha$ .

**Example 13.8.**  $\mathfrak{g} = \mathfrak{sl}(2)$ . Take  $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , a regular element. The Cartan subalgebra is  $\mathfrak{h} = k \cdot H$ , a one-dimensional subspace. We have  $\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \right\}$  and  $\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \right\}$ , and  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$ .

**Example 13.9.**  $\mathfrak{g} = \mathfrak{sl}(3)$ . Take

$$\mathfrak{h} = \left\{ \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & x_3 \end{pmatrix} \mid x_1 + x_2 + x_3 = 0 \right\}.$$

Let  $E_{ij}$  be the elementary matrices. We have that  $[\text{diag}(x_1, x_2, x_3), E_{ij}] = (x_i - x_j)E_{ij}$ . If we take the basis  $\varepsilon_i(x_1, x_2, x_3) = x_i$  for  $\mathfrak{h}^*$ , then we have roots  $\varepsilon_i - \varepsilon_j$ . They can be arranged in a diagram:



We will see that this generalizes to  $\mathfrak{sl}(n)$

The *rank* of  $\mathfrak{g}$  is defined to be  $\dim \mathfrak{h}$ . In particular, the rank of  $\mathfrak{sl}(n)$  is going to be  $n - 1$ .

Basic properties of the root decomposition are:

1.  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ .
2.  $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$  if  $\alpha + \beta \neq 0$ .
3.  $B|_{\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}}$  is non-degenerate.
4.  $B|_{\mathfrak{h}}$  is non-degenerate

Note that 3 implies that  $\alpha$  is a root if and only if  $-\alpha$  is a root.

► **Exercise 13.2.** Check these properties.

Now let's try to say as much as we can about this root decomposition. Define  $\mathfrak{h}_\alpha \subseteq \mathfrak{h}$  as  $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ . Take  $x \in \mathfrak{g}_\alpha$  and  $y \in \mathfrak{g}_{-\alpha}$  and  $h \in \mathfrak{h}$ . Then compute

$$\begin{aligned} B(\overbrace{[x, y]}^{\in \mathfrak{h}_\alpha}, h) &= B(x, \overbrace{[y, h]}^{\in \mathfrak{g}_\alpha}) && (B \text{ is invariant}) \\ &= \alpha(h)B(x, y) && (\text{since } y \in \mathfrak{g}_\alpha) \end{aligned}$$

It follows that  $\mathfrak{h}_\alpha^\perp = \ker(\alpha)$ , which is of codimension one. Thus,  $\mathfrak{h}_\alpha$  is one-dimensional.

**Proposition 13.10.** *If  $\mathfrak{g}$  is semisimple and  $\alpha$  is a root, then  $\alpha(\mathfrak{h}_\alpha) \neq 0$ .*

*Proof.* Assume that  $\alpha(\mathfrak{h}_\alpha) = 0$ . Then pick  $x \in \mathfrak{g}_\alpha$ ,  $y \in \mathfrak{g}_{-\alpha}$  such that  $[x, y] = h \neq 0$ . If  $\alpha(h) = 0$ , then we have that  $[h, x] = \alpha(h)x = 0$ ,  $[h, y] = 0$ . Thus  $\langle x, y, h \rangle$  is a copy of the Heisenberg algebra, which is solvable (in fact, nilpotent). By Lie's Theorem,  $ad_{\mathfrak{g}}(x)$  and  $ad_{\mathfrak{g}}(y)$  are simultaneously upper triangularizable, so  $ad_{\mathfrak{g}}(h) = [ad_{\mathfrak{g}}(x), ad_{\mathfrak{g}}(y)]$  is nilpotent. This is a contradiction because  $h$  is an element of the Cartan subalgebra, so it is semisimple.  $\square$

For each root  $\alpha$ , we will take  $H_\alpha \in \mathfrak{h}_\alpha$  such that  $\alpha(H_\alpha) = 2$  (we can always scale  $H_\alpha$  to get this). We can choose  $X_\alpha \in \mathfrak{g}_\alpha$  and  $Y_\alpha \in \mathfrak{g}_{-\alpha}$  such that  $[X_\alpha, Y_\alpha] = H_\alpha$ . We have that  $[H_\alpha, X_\alpha] = \alpha(H_\alpha)X_\alpha = 2X_\alpha$  and  $[H_\alpha, Y_\alpha] = -2Y_\alpha$ . That means we have a little copy of  $\mathfrak{sl}(2) \cong \langle H_\alpha, X_\alpha, Y_\alpha \rangle$ . Note that this makes  $\mathfrak{g}$  a representation of  $\mathfrak{sl}_2$  via  $\mathfrak{sl}_2 \xrightarrow{ad} \mathfrak{gl}(\mathfrak{g})$ .

We normalize  $\alpha(h_\alpha)$  to 2 so that we get the standard basis of  $\mathfrak{sl}_2$ . This way, the representations behave well (namely, that various coefficients are integers). Next we study these representations.

### Irreducible finite dimensional representations of $\mathfrak{sl}(2)$

Let  $H, X, Y$  be the standard basis of  $\mathfrak{sl}(2)$ , and let  $V$  be an irreducible representation. By Corollary 13.2, the action of  $H$  on  $V$  is diagonalizable and the actions of  $X$  and  $Y$  on  $V$  are nilpotent. By Lie's Theorem (applied to the solvable subalgebra generated by  $H$  and  $X$ ),  $X$  and  $H$  have a common eigenvector  $v$ :  $Hv = \lambda v$  and  $Xv = 0$  (since  $X$  is nilpotent, its only eigenvalues are zero). Verify by induction that

$$\begin{aligned} HY^r v &= YHY^{r-1}v + [H, Y]Y^{r-1}v = (\lambda - 2(r-1))Y^r v + 2Y^r v \\ &= (\lambda - 2r)Y^r v \end{aligned} \quad (*)$$

$$\begin{aligned} XY^r v &= YXY^{r-1}v + [X, Y]Y^{r-1}v \\ &= (r-1)(\lambda - (r-1) + 1)Y^{r-1}v + (\lambda - 2(r-1))Y^{r-1}v \\ &= r(\lambda - r + 1)Y^{r-1}v \end{aligned} \quad (\dagger)$$

Thus, the span of  $v, Yv, Y^2v, \dots$  is a subrepresentation, so it must be all of  $V$  (since  $V$  is irreducible). Since  $Y$  is nilpotent, there is a minimal  $n$  such that  $Y^n v = 0$ . From  $(\dagger)$ , we get that  $\lambda = n - 1$  is a non-negative integer. Since  $v, Yv, \dots, Y^{n-1}v$  have distinct eigenvalues (under  $H$ ), they are linearly independent.

Conclusion: For every non-negative integer  $n$ , there is exactly one irreducible representation of  $\mathfrak{sl}_2$  of dimension  $n + 1$ , and the  $H$ -eigenvalues on that representation are  $n, n - 2, n - 4, \dots, 2 - n, -n$ .

*Remark 13.11.* As a consequence, we have that in a general root decomposition,  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ , each root space is one dimensional. Assume that  $\dim \mathfrak{g}_{-\alpha} > 1$ . Consider an  $\mathfrak{sl}(2)$  in  $\mathfrak{g}$ , generated by  $\langle X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha] \rangle$  where  $Y_\alpha \in \mathfrak{g}_{-\alpha}$  and  $X_\alpha \in \mathfrak{g}_\alpha$ . Then there is some  $Z \in \mathfrak{g}_{-\alpha}$  such that  $[X_\alpha, Z] = 0$  (since  $\mathfrak{h}_\alpha$  is one-dimensional). Hence,  $Z$  is a highest vector with respect to the adjoint action of this  $\mathfrak{sl}(2)$ . But we have that  $ad_{H_\alpha}(Z) = -2Z$ , and the eigenvalue of a highest vector must be positive! This shows that the choice of  $X_\alpha$  and  $Y_\alpha$  is really unique.

**Definition 13.12.** Thinking of  $\mathfrak{g}$  as a representation of  $\mathfrak{sl}_2 = \langle X_\alpha, Y_\alpha, H_\alpha \rangle$ , the irreducible subrepresentation containing  $\mathfrak{g}_\beta$  is called the  $\alpha$ -string through  $\beta$ .

Let  $\Delta$  denote the set of roots. Then  $\Delta$  is a finite subset of  $\mathfrak{h}^*$  with the following properties:

1.  $\Delta$  spans  $\mathfrak{h}^*$ .
2. If  $\alpha, \beta \in \Delta$ , then  $\beta(H_\alpha) \in \mathbb{Z}$ , and  $\beta - (\beta(H_\alpha))\alpha \in \Delta$ .
3. If  $\alpha, c\alpha \in \Delta$ , then  $c = \pm 1$ .

► **Exercise 13.3.** Prove these properties.



## Lecture 14 - More on Root Systems

Assume  $\mathfrak{g}$  is semisimple. Last time, we started with a regular element  $h \in \mathfrak{g}_{ss}$  and constructed the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ , where  $\Delta \subseteq \mathfrak{h}^*$  is the set of roots. We proved that each  $\mathfrak{g}_\alpha$  is one dimensional (we do not call 0 a root). For each root, we associated an  $\mathfrak{sl}(2)$  subalgebra. Given  $X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}$ , we set  $H_\alpha = [X_\alpha, Y_\alpha]$ , and normalized so that  $\alpha(H_\alpha) = 2$ .

Furthermore, we showed that

1.  $\Delta$  spans  $\mathfrak{h}^*$ ,
2.  $\alpha(H_\beta) \in \mathbb{Z}$ , with  $\alpha - \alpha(H_\beta)\beta \in \Delta$  for all  $\alpha, \beta \in \Delta$ , and
3. if  $\alpha, k\alpha \in \Delta$ , then  $k = \pm 1$ .

How unique is this decomposition? We started with some choice of a regular semisimple element. Maybe a different one would have produced a different Cartan subalgebra.

**Theorem 14.1.** *Let  $\mathfrak{h}$  and  $\mathfrak{h}'$  be two Cartan subalgebras of a semisimple Lie algebra  $\mathfrak{g}$  (over an algebraically closed field of characteristic zero). Then there is some  $\phi \in AdG$  such that  $\phi(\mathfrak{h}) = \mathfrak{h}'$ . Here,  $G$  is the Lie group associated to  $\mathfrak{g}$ .*

*Proof.* Consider the map

$$\begin{aligned} \Phi : \mathfrak{h}^{\text{reg}} \times \mathfrak{g}_{\alpha_1} \times \cdots \times \mathfrak{g}_{\alpha_N} &\rightarrow \mathfrak{g} \\ (h, x_1, \dots, x_N) &\mapsto \exp(ad_{x_1}) \cdots \exp(ad_{x_N})h. \end{aligned}$$

Note that  $ad_{x_i}h$  is linear in both  $x_i$  and  $h$ , and each  $ad_{\mathfrak{g}_{\alpha_i}}$  is nilpotent, so the power series for  $\exp$  is finite. It follows that  $\Phi$  is a polynomial function. Since  $\frac{d}{dt} \exp(ad_{tx_i})h \Big|_{t=0} = \alpha_i(h)x_i \in \mathfrak{g}_{\alpha_i}$ , the differential of  $\Phi$  at  $(h, 0, \dots, 0)$  is

$$D\Phi|_{(h,0,\dots,0)} = \left( \begin{array}{c|ccc} \text{Id}_{\mathfrak{h}} & & 0 & \\ \hline & \alpha_1(h) & & 0 \\ & & \ddots & \\ 0 & & 0 & \alpha_N(h) \end{array} \right)$$

with respect to the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\alpha_N}$ .  $D\Phi|_{(h,0,\dots,0)}$  is non-degenerate because  $h \in \mathfrak{h}^{\text{reg}}$  implies that  $\alpha_i(h) \neq 0$ . So  $\text{im } \Phi$  contains a Zariski open set. Let  $\Phi'$  be the analogous map for  $\mathfrak{h}'$ . Since Zariski open sets are dense, we have that  $\text{im } \Phi \cap \text{im } \Phi' \neq \{0\}$ . So there is some  $\psi \in AdG, \psi' \in AdG$ , and  $h \in \mathfrak{h}, h' \in \mathfrak{h}'$  such that  $\psi(h) = \psi'(h')$ . Thus, we have that  $\mathfrak{h} = \psi^{-1}\psi'(\mathfrak{h}')$ .  $\square$

### Abstract Root systems

We'd like to forget that any of this came from a Lie algebra. Let's just study an abstract set of vectors in  $\mathfrak{h}^*$  satisfying some properties. We know that  $B$  is non-degenerate on  $\mathfrak{h}$ , so there is an induced isomorphism  $s : \mathfrak{h} \rightarrow \mathfrak{h}^*$ . By definition,  $\langle s(h), h' \rangle = B(h, h')$ .

Let's calculate

$$\begin{aligned}
 \langle sH_\beta, H_\alpha \rangle &= B(H_\beta, H_\alpha) = B(H_\alpha, H_\beta) && (B \text{ symmetric}) \\
 &= B(H_\alpha, [X_\beta, Y_\beta]) = B([H_\alpha, X_\beta], Y_\beta) && (B \text{ invariant}) \\
 &= B(X_\beta, Y_\beta)\beta(H_\alpha) \\
 &= \frac{1}{2}B([H_\beta, X_\beta], Y_\beta)\beta(H_\alpha) && (2X_\beta = [H_\beta, X_\beta]) \\
 &= \frac{1}{2}B(H_\beta, H_\beta)\beta(H_\alpha) && (B \text{ invariant})
 \end{aligned}$$

Thus, we have that  $s(H_\beta) = \frac{B(H_\beta, H_\beta)}{2}\beta$ . Also, compute

$$\begin{aligned}
 (\beta, \beta) &:= \langle \beta, s^{-1}\beta \rangle \\
 &= \beta \left( \frac{2H_\beta}{B(H_\beta, H_\beta)} \right) \\
 &= \frac{4}{B(H_\beta, H_\beta)} && (\beta(H_\beta) = 2)
 \end{aligned}$$

So  $s(H_\beta) = \frac{2\beta}{(\beta, \beta)}$ . This is sometimes called the *coroot* of  $\beta$ , and denoted  $\check{\beta}$ . Thus, we may rewrite fact 2 from last time as:

$$\text{For } \alpha, \beta \in \Delta, \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}, \text{ and } \alpha - \frac{2(\alpha, \beta)}{(\beta, \beta)}\beta \in \Delta. \quad (2')$$

Now you can define  $r_\beta : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$  given by  $r_\beta(x) = x - \frac{2(x, \beta)}{(\beta, \beta)}\beta$ . This is the reflection through the hyperplane orthogonal to  $\beta$  in  $\mathfrak{h}^*$ . The group generated by the  $r_\beta$  for  $\beta \in \Delta$  is a Coxeter group. If we want to study these things, we'd better classify root systems.

We want to be working in Euclidean space, but we are now in  $\mathfrak{h}^*$ . Let  $\mathfrak{h}_r$  be the real span of the  $H_\alpha$ 's. We claim that  $B$  is positive definite on  $\mathfrak{h}_r$ . To see this, note that  $X_\alpha, Y_\alpha, H_\alpha$  make a little  $\mathfrak{sl}(2)$  in  $\mathfrak{g}$ , and that  $\mathfrak{g}$  is therefore a representation of  $\mathfrak{sl}(2)$  via the adjoint actions  $ad_{X_\alpha}, ad_{Y_\alpha}, ad_{H_\alpha}$ . But we know that in any representation of  $\mathfrak{sl}(2)$ , the eigenvalues of  $H_\alpha$  must be integers. Thus,  $ad_{H_\alpha} \circ ad_{H_\alpha}$  has only positive eigenvalues, so  $B(H_\alpha, H_\alpha) = tr(ad_{H_\alpha} \circ ad_{H_\alpha}) > 0$ .

Thus, we may think of our root systems in Euclidean space, where the inner product on  $\mathfrak{h}^*$  is given by  $(\mu, \nu) \stackrel{def}{=} B(s^{-1}(\mu), s^{-1}(\nu)) = \langle \mu, s^{-1}\nu \rangle$ .

**Definition 14.2.** An *abstract reduced root system* is a finite set  $\Delta \subseteq \mathbb{R}^n \setminus \{0\}$  which satisfies

(RS1)  $\Delta$  spans  $\mathbb{R}^n$ ,

(RS2) for  $\alpha, \beta \in \Delta$ ,  $\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$ , with  $r_\beta(\Delta) = \Delta$

(i.e.  $\alpha, \beta \in \Delta \Rightarrow r_\beta(\alpha) \in \Delta$ , with  $\alpha - r_\beta(\alpha) \in \mathbb{Z}\beta$ ), and

(RS3)  $\alpha, k\alpha \in \Delta \Rightarrow k = \pm 1$  (this is the “reduced” part).

The number  $n$  is called the *rank* of  $\Delta$ .

Notice that given root systems  $\Delta_1 \subset \mathbb{R}^n, \Delta_2 \subset \mathbb{R}^m$ , we have that  $\Delta_1 \amalg \Delta_2 \subset \mathbb{R}^n \oplus \mathbb{R}^m$  is a root system.

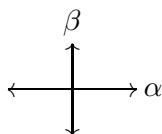
**Definition 14.3.** A root system is *irreducible* if it cannot be decomposed into the union of two systems of smaller rank.

► **Exercise 14.1.** Let  $\mathfrak{g}$  be a semisimple Lie algebra and let  $\Delta$  be its root system. Show that  $\Delta$  is irreducible if and only if  $\mathfrak{g}$  is simple.

Now we will classify all systems of rank 2. Observe that  $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \frac{2(\alpha, \beta)}{(\beta, \beta)} = 4 \cos^2 \theta$ , where  $\theta$  is the angle between  $\alpha$  and  $\beta$ . This thing must be an integer. Thus, there are not many choices for  $\theta$ :

$\cos \theta$	0	$\pm \frac{1}{2}$	$\pm \frac{1}{\sqrt{2}}$	$\pm \frac{\sqrt{3}}{2}$
$\theta$	$\frac{\pi}{2}$	$\frac{\pi}{3}, \frac{2\pi}{3}$	$\frac{\pi}{4}, \frac{3\pi}{4}$	$\frac{\pi}{6}, \frac{5\pi}{6}$

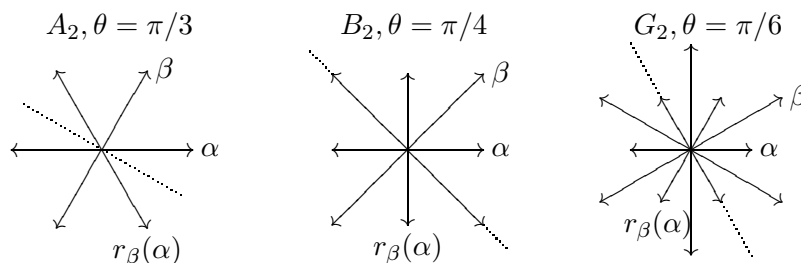
Choose two vectors with minimal angle between them. If the minimal angle is  $\pi/2$ , then the system is reducible.



Notice that  $\alpha$  and  $\beta$  can be scaled independently.

If the minimal angle is smaller than  $\pi/2$ , then  $r_\beta(\alpha) \neq \alpha$ , so the difference  $\alpha - r_\beta(\alpha)$  is a non-zero integer multiple of  $\beta$  (in fact, a positive multiple of  $\beta$  since  $\theta < \pi/2$ ). If we assume  $\|\alpha\| \leq \|\beta\|$  (we can always switch them), we get that  $\|\alpha - r_\beta(\alpha)\| < 2\|\alpha\| \leq 2\|\beta\|$ , so  $\alpha - r_\beta(\alpha) = \beta$ . Thus, once we set the direction of  $\beta$ , its length is determined (relative to  $\alpha$ ). Then we can obtain the remaining elements of the root system from the condition that  $\Delta$  is invariant under  $r_\alpha$  and  $r_\beta$ , observing that no additional vectors can be

added without violating **RS2**, **RS3**, or the prescribed minimal angle. Thus, all the irreducible rank two root systems are



### The Weyl group

Given a root system  $\Delta = \{\alpha_1, \dots, \alpha_N\}$ , we call the group generated by the  $r_{\alpha_i}$ s the *Weyl group*, denoted  $W$ .

*Remark 14.4.* Each  $r_\alpha \in W$  can be constructed in terms of the group  $G$  of  $\mathfrak{g}$ . If you have  $X_\alpha, Y_\alpha, H_\alpha$  generating an  $\mathfrak{sl}(2)$ , then you know that there is a subgroup  $SL(2)$  of  $G$ . There is a special element corresponding to  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  in there. It is given by  $s_\alpha = \exp(X_\alpha)\exp(-Y_\alpha)\exp(X_\alpha)$ . We have that  $\langle \beta, Ad_{s_\alpha}(h) \rangle = \langle r_\alpha\beta, h \rangle$ . So this reflection is coming from the embedding of  $SL(2)$  in  $G$ .

► **Exercise 14.2.** Let  $\mathfrak{g}$  be a semisimple Lie algebra, and let  $\mathfrak{h}$  be a Cartan subalgebra. For each root  $\alpha$  define

$$s_\alpha = \exp(ad_{X_\alpha}) \exp(-ad_{Y_\alpha}) \exp(ad_{X_\alpha}).$$

Prove that  $s_\alpha(\mathfrak{h}) = \mathfrak{h}$  and

$$\langle \lambda, s_\alpha(h) \rangle = \langle r_\alpha(\lambda), h \rangle$$

for any  $h \in \mathfrak{h}$  and  $\lambda \in \mathfrak{h}^*$ , where  $r_\alpha$  is the reflection in  $\alpha^\perp$ .

**Example 14.5.** The root system  $A_n$  is the root system of  $\mathfrak{sl}_{n+1}$ . We pick an orthonormal basis  $\varepsilon_1, \dots, \varepsilon_{n+1}$  of  $\mathbb{R}^{n+1}$ , the the root system is the set of all the differences:  $\Delta = \{\varepsilon_i - \varepsilon_j | i \neq j\}$ . We have that

$$r_{\varepsilon_i - \varepsilon_j}(\varepsilon_k) = \begin{cases} \varepsilon_k & k \neq i, j \\ \varepsilon_j & k = i \\ \varepsilon_i & k = j \end{cases}$$

is a transposition, so we have that  $W \simeq S_{n+1}$

Now back to classification of abstract root systems.

Draw a hyperplane in general position (so that it doesn't contain any roots). This divides  $\Delta$  into two parts,  $\Delta = \Delta^+ \amalg \Delta^-$ . The roots in  $\Delta^+$

are called *positive roots*, and the elements of  $\Delta^-$  are called negative roots. We say that  $\alpha \in \Delta^+$  is *simple* if it cannot be written as the sum of other positive roots. Let  $\Pi$  be the set of simple roots, sometimes called a base. It has the properties

1. Any  $\alpha \in \Delta^+$  is a sum of simple roots (perhaps with repetition):  $\alpha = \sum_{\beta \in \Pi} m_\beta \beta$  where  $m_\beta \in \mathbb{Z}_{\geq 0}$ .

2. If  $\alpha, \beta \in \Pi$ , then  $(\alpha, \beta) \leq 0$ .

This follows from the fact that if  $(\alpha, \beta) > 0$ , then  $\alpha - \beta$  and  $\beta - \alpha$  are again roots (as we showed when we classified rank 2 root systems), and one of them is positive, say  $\alpha - \beta$ . Then  $\alpha = \beta + (\alpha - \beta)$ , contradicting simplicity of  $\alpha$ .

3.  $\Pi$  is a linearly independent set.

If they were linearly dependent, the relation  $\sum_{\alpha_i \in \Pi} a_i \alpha_i = 0$  must have some negative coefficients (because all of  $\Pi$  is in one half space), so we can always write

$$0 \neq a_1 \alpha_1 + \cdots + a_r \alpha_r = a_{r+1} \alpha_{r+1} + \cdots + a_n \alpha_n$$

with all the  $a_i \geq 0$ . Taking the inner product with the left hand side, we get

$$\begin{aligned} & \|a_1 \alpha_1 + \cdots + a_r \alpha_r\|^2 \\ &= (a_1 \alpha_1 + \cdots + a_r \alpha_r, a_{r+1} \alpha_{r+1} + \cdots + a_n \alpha_n) \leq 0 \end{aligned}$$

by **2**, which is absurd.

*Remark 14.6.* Notice that the hyperplane is  $t^\perp$  for some  $t$ , and the positive roots are the  $\alpha \in \Delta$  for which  $(t, \alpha) > 0$ . This gives an order on the roots. You can inductively prove **1** using this order.

*Remark 14.7.* Notice that when you talk about two roots, they always generate one of the rank 2 systems, and we know what all the rank 2 systems are.

**Claim.** Suppose we have chosen  $\Delta^+ \supset \Pi$ . Then for  $\alpha \in \Pi$ , we have that  $r_\alpha(\Delta^+) = \Delta^+ \cup \{-\alpha\} \setminus \{\alpha\}$ .

*Proof.* For  $\beta \neq \alpha$  simple, we have  $r_\alpha(\beta) = \beta + k\alpha$  for some non-negative  $k$ ; this must be on the positive side of the hyperplane, so it is a positive root. Now assume you have a positive root of the form  $\gamma = m\alpha + \sum_{\alpha_i \neq \alpha} m_i \alpha_i$ , with  $m, m_i \geq 0$ . Then we have that  $r_\alpha(\gamma) = -m\alpha + \sum m'_i \alpha_i \in \Delta$ . If any of the  $m_i$  were non-zero, we know that some  $m'_i$  is positive (by the

base case above), but every root can be (uniquely) written as either a non-negative or a non-positive combination of the simple roots. Since  $r_\alpha(\gamma)$  has some positive coefficient, it must be a non-negative combination, so it is a positive root.  $\square$

Suppose we have two different sets of positive roots  $\Delta^+$  and  $\Delta_1^+$ , then one can get  $\Delta_1^+$  from  $\Delta^+$  by a chain of simple reflections. You have to be a little careful because the set of simple roots changes, so you're picking your reflections from different sets. Fix a base  $\Pi = \{\alpha_1, \dots, \alpha_n\}$ , and call  $r_i := r_{\alpha_i}$  simple reflections.

**Proposition 14.8.** *Simple reflections act transitively on the set of sets of positive roots.*

*Proof.* Start with  $\Delta^+$ , and apply  $r_i$  to get a new  $\Delta_1^+$ , then from the new base apply some  $r_\alpha$  to get  $\Delta_2^+$ , so you have  $\Delta_2^+ = r_\alpha r_i \Delta^+$ . The important thing is that for any element of the Weyl group, we have  $w r_\alpha w^{-1} = r_{w\alpha}$ . So if  $\alpha = r_i(\alpha_j)$ , we have that  $r_\alpha r_i = r_i r_j$ . Induct.  $\square$

**Corollary 14.9.**  *$W$  is generated by simple reflections.*

*Proof.* Any root is a simple root for some choice of positive roots. To show this, draw your hyperplane really close to your vector. Then we know that  $\alpha$  is obtained from our initial set  $\Pi$  by simple reflections. We get that if  $\alpha = r_{i_1} \cdots r_{i_k}(\alpha_j)$ , then  $r_\alpha = (r_{i_1} \cdots r_{i_k}) r_j (r_{i_1} \cdots r_{i_k})^{-1}$ .  $\square$

Let's see how it works for  $A_n$ . Choose a vector  $t = t_1 \varepsilon_1 + \cdots + t_{n+1} \varepsilon_{n+1}$  in such a way that  $t_1 > \cdots > t_{n+1}$ . Then we have that  $(\varepsilon_i - \varepsilon_j, t) = t_i - t_j$ , so the positive roots are the  $\varepsilon_i - \varepsilon_j$  for which  $i < j$ . Then the simple roots are  $\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_n - \varepsilon_{n+1}\}$

## Lecture 15 - Dynkin diagrams, Classification of root systems

Last time, we talked about root systems  $\Delta \subset \mathbb{R}^n$ . We constructed the Weyl group  $W$ , the finite group generated by reflections. We considered  $\Pi \subset \Delta$ , the simple roots.  $\Pi$  forms a basis for  $\mathbb{R}^n$ , and showed that every root is a non-negative (or non-positive) linear combination of simple roots.

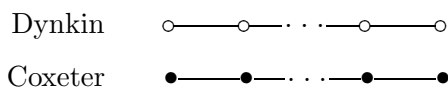
We showed that if you have  $\alpha, \beta \in \Pi$ , then  $n_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}_{\leq 0}$ . There are not many options for this value. It can only be 0, -1, -2, or -3. Also, we have that  $n_{\alpha\beta}n_{\beta\alpha} = 4 \cos^2 \theta_{\alpha\beta}$ . If  $n_{\alpha\beta} = 0$ , the two are orthogonal. If  $n_{\alpha\beta} = -1$ , then the angle must be  $2\pi/3$  and the two are the same length. If  $n_{\alpha\beta} = -2$ , the angle must be  $3\pi/4$  and  $\|\beta\| = \sqrt{2} \|\alpha\|$ . If  $n_{\alpha\beta} = -3$ , the angle is  $5\pi/6$  and  $\|\beta\| = \sqrt{3} \|\alpha\|$ . Thus we get:

$n_{\beta\alpha}$	$n_{\alpha\beta}$	relationship	Dynkin picture
0	0		$\alpha \circ \quad \circ \beta$
-1	-1		$\alpha \circ \text{---} \circ \beta$
-2	-1		$\alpha \circ \text{---} \leftarrow \circ \beta$
-3	-1		$\alpha \circ \text{---} \leftarrow \leftarrow \circ \beta$

We will encode this information in a *Dynkin diagram*. The nodes of the diagram are simple roots. We join two nodes by  $n_{\alpha\beta}n_{\beta\alpha}$  lines. The arrows go from the longer root to the shorter root in the case of two or three lines. (As always, the alligator eats the big one.)

There is also something called a *Coxeter diagram*, which is not directed. You look at  $e_i = \frac{\alpha}{(\alpha, \alpha)^{1/2}}$ ,  $i = 1, \dots, n$ . Then the number of lines between two vertices is  $n_{\alpha\beta}n_{\beta\alpha} = 4 \cdot \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = 4(e_i, e_j)^2$ .

**Example 15.1.**  $A_n$  has  $n$  simple roots, given by  $\varepsilon_i - \varepsilon_{i+1}$ . So the graphs are



Properties of the Coxeter graph:

(CX1) A subgraph of a Coxeter graph is a Coxeter graph. This is trivial.

(CX2) A Coxeter graph is acyclic.

*Proof.* Let  $e_1, \dots, e_k$  be a cycle in the Coxeter graph. Then look at

$$\left( \sum e_i, \sum e_i \right) = k + \sum_{\substack{i < j \\ i, j \text{ adj}}} \underbrace{2(e_i, e_j)}_{\leq -1} \leq 0$$

which contradicts that the inner product is positive definite.  $\square$

(CX3) The degree of each vertex is less than or equal to 3, where we count double and triple edges as adding two and three to the degree, respectively.

*Proof.* Let  $e_0$  have  $e_1, \dots, e_k$  adjacent to it. Since there are no cycles,  $e_1, \dots, e_k$  are orthogonal to each other. So we can compute

$$\begin{aligned} \left( e_0 - \sum (e_0, e_i)e_i, e_0 - \sum (e_0, e_i)e_i \right) &> 0 \\ 1 - \sum (e_0, e_i)^2 &> 0 \end{aligned}$$

but  $(e_0, e_i)^2$  is one fourth of the number of edges connecting  $e_0$  and  $e_i$ . So  $k$  cannot be bigger than 3.  $\square$

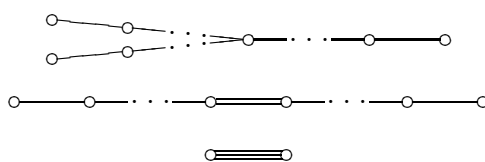
(CX4) Suppose a Coxeter graph has a subgraph of type  $A_n$ , and only the endpoints of this subgraph have additional edges (say  $\Gamma_1$  at one end and  $\Gamma_2$  at the other end). Then we can forget about the stuff in the middle and just fuse  $\Gamma_1$  with  $\Gamma_2$ , and we will still get a Coxeter graph.



To see this, let  $e_1, \dots, e_k$  be the vertices in the  $A_k$ . Let  $e_0 = e_1 + \dots + e_k$ . Then we can compute that  $(e_0, e_0) = 1$ , with  $(e_0, e_s) = (e_1, e_s)$  and  $(e_0, e_t) = (e_k, e_t)$ .

Thus, a Coxeter graph must be one of the following forms. There can only be one fork (two could be glued to give valence 4), and only one double edge, and if there is a triple edge, nothing else can be connected to it.

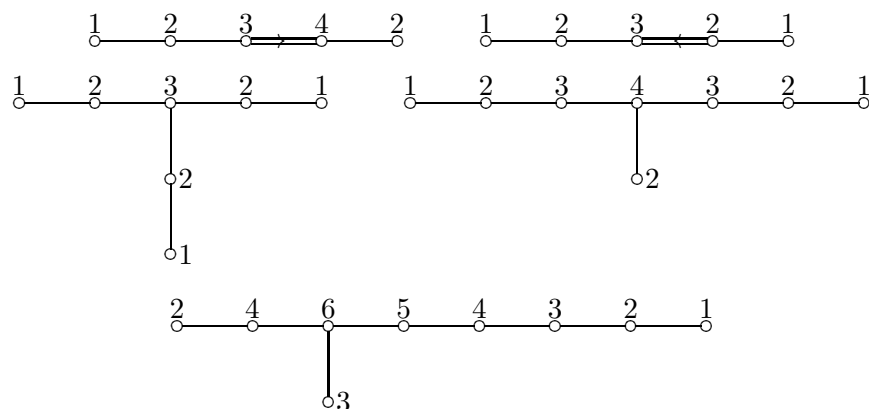
So the only possible Dynkin diagrams so far are of the form





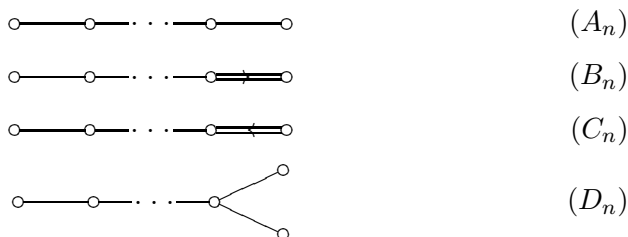
Now we use the property that a subgraph of a Dynkin graph is a Dynkin graph. Now we will calculate that some things are forbidden. We label the vertex corresponding to  $\alpha_i$  with a number  $m_i$ , and check that

$$\left(\sum m_i \alpha_i, \sum m_i \alpha_i\right) = 0.$$

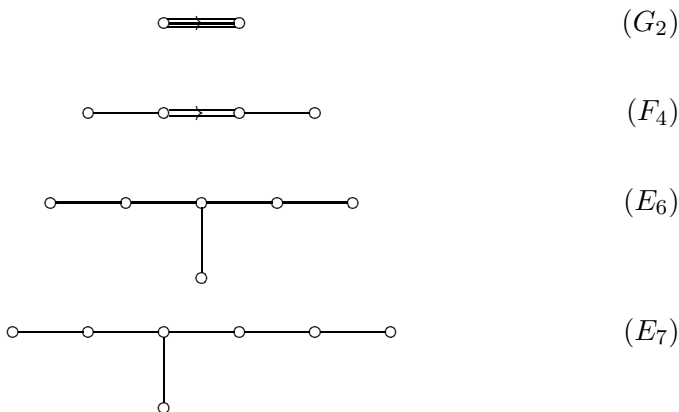


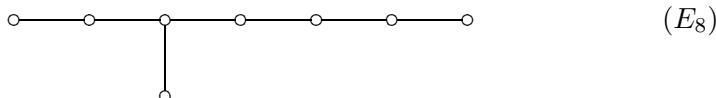
Thus we have narrowed our list of possible Dynkin diagrams to a short list.

Connected Dynkin diagrams:  $n$  is the total number of vertices.



Those are the “classical diagrams”, and the others are





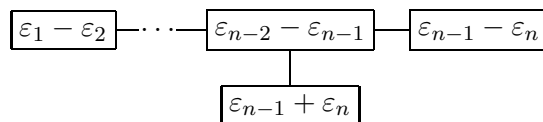
Each of these is indeed the diagram of some root system, which is what we will show now.

We've already constructed  $A_n$ . So let's jump to the next ones.

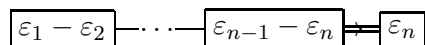
Root systems: we start with  $D_n$ . Let  $\varepsilon_1, \dots, \varepsilon_n$  be an orthonormal basis for  $\mathbb{R}^n$ . Then let the roots be

$$\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j) | i < j \leq n\}.$$

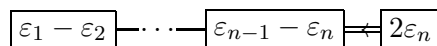
We choose the simple roots to be



To get the root system for  $B_n$ , take  $D_n$  and add  $\{\pm\varepsilon_i | i \leq n\}$ , in which case the simple roots are



To get  $C_n$ , take  $D_n$  and add  $\{\pm 2\varepsilon_i | i \leq n\}$ , then the simple roots are



Recall that we can define coroots  $\check{\alpha} = \frac{2\alpha}{(\alpha, \alpha)}$ . This will reverse the arrows in the Dynkin diagram. The dual root system is usually the same as the original, but is sometimes different. For example,  $C_n$  and  $B_n$  are dual.

Those are all the classical diagrams. Now let's do the others.

We've already done  $G_2$ .

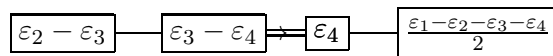
Now we will do  $F_4$  and  $E_8$ , then we know that the rest are subsets.

$F_4$  comes from some special properties of a cube in 4-space. Let  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$  be an orthonormal basis for  $\mathbb{R}^4$ . Then let the roots be

$$\left\{ \pm(\varepsilon_i \pm \varepsilon_j), \pm\varepsilon_i, \frac{\pm(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)}{2} \right\}$$

These are orthogonal to some faces of the 4-cube.

The simple roots are

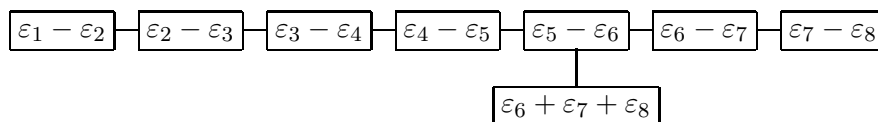


There are 48 roots total. Remember that the dimension of the lie algebra is the number of roots plus the dimension of the cartan (the rank of  $\mathfrak{g}$ , which is 4 here), so the dimension is 52 in this case.

Construction of  $E_8$ . Look at  $\mathbb{R}^9$  with our usual orthonormal basis. The trick is that we are going to project on to the plane orthogonal to  $\varepsilon_1 + \dots + \varepsilon_9$  (so just throw in the relation that that sum is 0, so that we work in the hyperplane). The roots are

$$\{\varepsilon_i - \varepsilon_j | i \neq j\} \cup \{\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k) | i \neq j \neq k\}$$

The total number of roots is  $|\Delta| = 9 \cdot 8 + 2 \binom{9}{3} = 240$ . So the dimension of the algebra is 248! The simple roots are



These are all the indecomposable root systems, the rest are just direct sums.

**Theorem 15.2** (Serre). *Say  $k$  algebraically closed, characteristic 0. For each indecomposable root system, there exists (unique up to isomorphism) a simple Lie algebra with the given root system.*

There are several ways to do this. You could construct all of them explicitly. The other way is to use Serre's relations, which is what we will do later.

Let's construct the Lie algebras for  $A_n, B_n, C_n, D_n$ . We know that  $A_n$  is  $\mathfrak{sl}(n+1)$ . Let's consider  $\mathfrak{so}(2n)$ , the linear maps of  $k^{2n}$  preserving the form. We can choose a basis for  $k^{2n}$  so that the matrix of the form is  $I = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ . Let  $X \in \mathfrak{so}(2n)$ , then we have that  $X^t I + IX = 0$ . So we can write

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

Where  $B^t = -B, C^t = -C$ . What should the Cartan subalgebra be? It should be a maximal abelian subalgebra. Notice that if we take  $A = \text{diag}(t_1, \dots, t_n)$  to be diagonal and  $B = C = 0$ , then we get an abelian subalgebra  $\mathfrak{h}$ . What are the root spaces? We have that  $E_{ij} - E_{j+n, i+n}$  has eigenvalue  $t_i - t_j$ . We have that  $E_{i, j+n} - E_{j, i+n}$  has eigenvalue  $t_i + t_j$ , and the symmetric term with eigenvalue  $-t_i - t_j$ . So we get the root system  $D_n$ .

Now let's look at  $\mathfrak{sp}(2n)$ , the linear operators which preserve a non-degenerate skew-symmetric form. In some basis, the form is  $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ , with  $X^t J + JX = 0$ . We get the same conditions, except that  $B = B^t, C =$

$C^t$ . The root system is the same as for  $\mathfrak{so}(2n)$  plus roots corresponding to  $E_{i,i+n}$ , with eigenvalue  $2t_i$ .

We still have to look at  $\mathfrak{so}(2n+1)$ . We'll choose a basis so that our form is a little different. We have a bilinear symmetric form, and we will write it as

$$B = \left( \begin{array}{c|c} 1 & 0 \\ \hline 0 & I \end{array} \right),$$

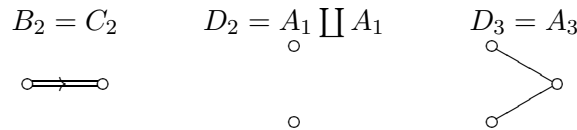
where  $I = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ .

So we have that  $\mathfrak{so}(2n) \subseteq \mathfrak{so}(2n+1)$  and

$$X = \left( \begin{array}{c|cccccc} 0 & x_1 & \cdots & x_n & y_1 & \cdots & y_n \\ \hline -y_1 & & & A & & & B \\ \vdots & & & & & & \\ -y_n & & & & & & \\ -x_1 & & & C & & & -A^t \\ \vdots & & & & & & \\ -y_n & & & & & & \end{array} \right)$$

**Isomorphisms of small dimension**

Let's say that we believe the theorem. Then you can see that  $B_2$  and  $C_2$  are the same. Also,  $D_2$  is just  $A_1 \amalg A_1$ , and  $D_3$  is  $A_3$ :



Thus, we have that  $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$ ,  $\mathfrak{so}(4) \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ ,  $\mathfrak{so}(6) \simeq \mathfrak{sl}(4)$  as Lie algebras.

You can see some of these directly: let's construct a map of groups  $SL(2) \times SL(2) \rightarrow SO(4)$ , whose kernel is discrete. Let  $W$  be the space of  $2 \times 2$  matrices, then the left group acts on  $W$ :  $(X, Y)w = XwY^{-1}$ , preserving the determinant of  $w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . The quadratic form  $ad - bc$  is preserved. The Lie group preserving such a form is  $SO(4)$ , so we have a map. You can check that the kernel is the matrices of the form  $(I, I)$  and  $(-I, -I)$ , and it is surjective (count dimensions). Thus, you get an isomorphism on the level of Lie algebras.

Let's also do  $\mathfrak{so}(6) \simeq \mathfrak{sl}(4)$ . The approach is the same. Let  $V$  be the standard 4-dimensional representation of  $\mathfrak{sl}(4)$ . Let  $W = \Lambda^2 V$ , which is a 6-dimensional representation of  $\mathfrak{sl}(4)$ . Note that you have a pairing

$$\Lambda^2 V \times \Lambda^2 V \rightarrow \Lambda^4 V \simeq k$$

where the last isomorphism is as representations of  $SL(4)$  (because determinant is 1 in  $SL(4)$ ). Thus,  $W = \Lambda^2 V$  has some  $SL(4)$ -invariant symmetric bilinear form which is preserved, so you get a map  $SL(4) \rightarrow SO(6)$ . It is not hard to check that the kernel is  $\pm I$ , and the dimensions match, so it is surjective. Thus, you get an isomorphism of Lie algebras.

Next time, we'll do this theorem and we'll talk about the Exceptional algebras.

## Lecture 16 - Serre's Theorem

Today we're going to prove Serre's theorem, and then work with some constructions of the exceptional algebras.

Start with a semi-simple Lie algebra  $\mathfrak{g}$  (over  $k = \bar{k}$ ), with Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ . Then we have the root system  $\Delta$ , with a fixed set of simple roots  $\Pi = \{\alpha_1, \dots, \alpha_n\}$ . We have  $X_i := X_{\alpha_i}, Y_i, H_i$  associated to each simple root.

The *Cartan matrix* is  $a_{ij} = \alpha_j(H_i) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ . We know that  $a_{ij} \in \mathbb{Z}_{\leq 0}$  for  $i \neq j$  and  $a_{ii} = 2$ .

Then we claim that the following relations are satisfied:

$$[H_i, H_j] = 0 \quad [H_i, X_j] = a_{ij}X_j \quad [H_i, Y_j] = -a_{ij}Y_j \quad [X_i, Y_j] = \delta_{ij}H_i \quad (1)$$

$$(\theta_{ij}^+) \quad (ad_{X_i})^{1-a_{ij}}X_j = 0 \quad (ad_{Y_i})^{1-a_{ij}}Y_j = 0 \quad (\theta_{ij}^-) \quad (2)$$

the second group are called Serre's relations (though Serre called them Weyl relations). We verify that these relations hold in our Lie algebra.

The tricky one from (1) is that  $[X_i, Y_j] = 0$  when  $i \neq j$ . You know that  $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_j}] \subseteq \mathfrak{g}_{\alpha_i - \alpha_j}$ , but  $\alpha_i - \alpha_j \notin \Delta$ ! So  $\mathfrak{g}_{\alpha_i - \alpha_j} = 0$ .

**Example 16.1.**  $\mathfrak{g} = \mathfrak{sl}(n+1)$ . We have  $X_i = E_{i, i+1}, Y_i = E_{i+1, i}, H_i = [X_i, Y_i] = E_{ii} - E_{i+1, i+1}$ .

Let's check that the second group of relations hold. We want to show that  $(ad_{X_i})^{1-a_{ij}}X_j = 0$ . What is  $a_{ij}$ ? We know that  $(ad_{X_i})^{1-a_{i, i+1}}X_{i+1} = 0$  by a calculation ... just do it.

To see why these relations hold in general, consider  $\theta_{ij}^-$ . We have that  $ad_{X_i}(Y_j) = 0$ , so  $Y_j$  is a highest vector with respect to this  $\mathfrak{sl}(2)$  action. We also have that  $ad_{H_i}(Y_j) = -a_{ij}Y_j$ , so the highest weight is  $-a_{ij}$ . We know what representations of  $\mathfrak{sl}(2)$  look like, so we know that  $(ad_{Y_i})^{1-a_{ij}}Y_j = 0$ . Thus,  $\theta_{ij}^-$  holds. To show  $\theta_{ij}^+$ , you can switch  $X_i$  and  $Y_i$ , and negate  $H_i$ .

We don't know that these are all the relations yet. We also do not yet know that for every root system there is an algebra. But we know that if such an algebra exists, then it satisfies these relations.

**Theorem 16.2.** *Let  $\mathfrak{g}$  be the Lie algebra with generators  $H_i, X_i, Y_i$  for  $1 \leq i \leq n$ , with a Cartan matrix  $a_{ij} = (\alpha_i, \check{\alpha}_j)$  from some root system and the relations given, then  $\mathfrak{g}$  is a finite dimensional semi-simple Lie algebra with the Cartan subalgebra spanned by  $H_1, \dots, H_n$ , and with the given root system  $\Delta$ .*

Note that this shows existence. But since any such Lie algebra satisfies these relations, we have uniqueness (you cannot add more relations without making the root system smaller).

*Remark 16.3.* When we talk about a Lie algebra with some generators and relations, how do you think about it? What is a free Lie algebra? It is tricky to think about the Jacobi identity, not like in an associative algebra. But a Lie algebra always has an associative algebra  $U\mathfrak{g}$ . What is  $U\mathfrak{g}$  going to be? The free associative algebra with the same generators. If you have generators  $g_1, \dots, g_n$ , look at the tensor algebra  $T(g_1, \dots, g_n)$ , and realize  $\mathfrak{g}$  inside as a Lie subalgebra generated by  $g_1, \dots, g_n$ .

*Proof. Step 1:* Take the free Lie algebra and quotient by the relations in group (1). Call the result  $\tilde{\mathfrak{g}}$ . Then the claim is that

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+$$

as vector spaces, where  $\mathfrak{h}$  is the abelian Lie algebra generated by  $H_1, \dots, H_n$ , and  $\tilde{\mathfrak{n}}^\pm$  are free Lie algebras generated by the  $X_i$  (resp.  $Y_i$ ). They should be free because there are no relations among the  $X_i$ 's, but it is not so easy. There is a standard trick for doing such things.

$U\tilde{\mathfrak{g}} = U(\tilde{\mathfrak{n}}^-)U(\mathfrak{h})U(\tilde{\mathfrak{n}}^+)$  (easy to show from relations (1)). In fact, we want  $U(\tilde{\mathfrak{g}}) = \underbrace{U(\tilde{\mathfrak{n}}^-)}_{T(\tilde{\mathfrak{n}}^-)} \otimes \underbrace{U(\mathfrak{h})}_{S(\mathfrak{h})} \otimes \underbrace{U(\tilde{\mathfrak{n}}^+)}_{T(\tilde{\mathfrak{n}}^+)}$ . We construct a representation of  $\tilde{\mathfrak{g}}$  in  $T(\tilde{\mathfrak{n}}^-) \otimes S(\mathfrak{h})$ .

We say  $Y_i(1 \otimes 1) = Y_i \otimes 1$ ,  $h(1 \otimes 1) = 1 \otimes h$ , and  $X_i(1 \otimes 1) = 0$ . Then we define things inductively:

$$\begin{aligned} Y_i(Y_j a \otimes b) &=_{def} Y_i Y_j a \otimes b \\ H_i(Y_j a \otimes b) &= Y_j(H_i(a \otimes b)) - a_{ij} Y_j(a \otimes b) \\ X_i(Y_j a \otimes b) &= Y_j(X_i(a \otimes b)) + \delta_{ij} H_i(a \otimes b) \end{aligned}$$

Your job is to check that this gives a representation (note that  $1 \otimes 1$  generates the whole representation).

So we get an isomorphism of  $U(\tilde{\mathfrak{n}}^-)$  and  $T(\tilde{\mathfrak{n}}^-)$ . We define  $U(\tilde{\mathfrak{n}}^- + \mathfrak{h}) \leftarrow T(\tilde{\mathfrak{n}}^-) \otimes S(\mathfrak{h})$  given by  $x \mapsto x(1 \otimes 1)$ . We do the same thing for  $\tilde{\mathfrak{n}}^+$ .

$$0 \rightarrow U(\tilde{\mathfrak{n}}^+) \rightarrow U(\tilde{\mathfrak{g}}) \rightarrow T(\tilde{\mathfrak{n}}^-) \otimes S(\mathfrak{h}) \rightarrow 0$$

$$0 \rightarrow U(\tilde{\mathfrak{n}}^-) \rightarrow U(\tilde{\mathfrak{g}}) \rightarrow T(\tilde{\mathfrak{n}}^+) \otimes S(\mathfrak{h}) \rightarrow 0$$

Hence  $U(\tilde{\mathfrak{g}}) \simeq T(\tilde{\mathfrak{n}}^-) \otimes S(\mathfrak{h}) \otimes T(\tilde{\mathfrak{n}}^+)$ . So  $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+$ .

*Step 2:* We now throw in relations from group (2). We have that  $\theta_{ij}^\pm \in \tilde{\mathfrak{n}}^\pm$ . We have to verify that

$$[Y_k, \theta_{ij}^+] = 0$$

which sort of follows from relations from group (1) (check for yourself). We have that

$$[H_k, \theta_{ij}^+] = c_{kij} \theta_{ij}^+$$

So the ideal  $j^+ \subset \tilde{\mathfrak{n}}^+$  generated by  $\{\theta_{ij}^+\}$  is an ideal in the whole algebra  $\tilde{\mathfrak{g}}$ . Similarly with  $j^- \subset \tilde{\mathfrak{n}}^-$  generated by  $\{\theta_{ij}^-\}$ .

Now we want  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ , where  $\mathfrak{n}^\pm = \tilde{\mathfrak{n}}^\pm / j^\pm$ . The main point is that throwing in Serre's relations doesn't disturb the Cartan subalgebra.

From relations in (1),  $\mathfrak{h}$  acts semi-simply on  $\mathfrak{g}$ , so we can write  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ , where the  $\mathfrak{g}_\alpha$  are eigenspaces

$$g_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x\}. \quad (*)$$

We can look at

$$R = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\} \subseteq \mathfrak{h}^*$$

We know that  $\pm\alpha_1, \dots, \pm\alpha_n \in R$ . We have yet to show that we get the right root system and the  $\mathfrak{g}$  is semisimple and finite dimensional.

If  $\alpha \in R$ , then it must be either a non-negative or non-positive combination of the  $\alpha_i$  (because of the decomposition (\*). We still have that  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ . So we know that  $R$  is  $R^+ \amalg R^-$  (property (a)). We also have that  $R \cap \mathbb{Z}\alpha_i = \{\pm\alpha_i\}$ , because  $[X_i, X_i] = 0$  (property (b)).

Step 3: Now we'll construct a Weyl group. The main idea is that the relations (1) and (2) imply that  $ad_{X_i}, ad_{Y_i}$  are locally nilpotent operators on  $\mathfrak{g}$  ( $A$  is locally nilpotent on  $V$  is for any vector  $v \in V$ , there is some  $n(v)$  such that  $A^{n(v)}v = 0$ ). You can check it on generators, and then by the Jacobi identity, you get it for any vector.

Let  $s_i = \exp(ad_{X_i}) \exp(-ad_{Y_i}) \exp(ad_{X_i})$ . Then  $s_i \in \text{Aut } \mathfrak{g}$

► **Exercise 16.1.** Check that  $s_i(\mathfrak{h}) \subseteq \mathfrak{h}$

We have that

$$\langle \lambda, s_i(h) \rangle = \langle r_i(\lambda), h \rangle \quad (**)$$

for any  $h \in \mathfrak{h}$  and any  $\lambda \in \mathfrak{h}^*$ , where  $r_i(\lambda) = \lambda - \lambda(H_i)\alpha_i$ .

Let  $W$  be the group generated by  $r_1, \dots, r_n$ . We have that  $w(R) = R$  because of the identity (\*\*)

On the other hand, we know that  $W$  is the Weyl group of the root system  $\Delta$ . Hence we have an embedding  $\Delta \subseteq R$ . Moreover, we have that  $\dim \mathfrak{g}_\alpha = 1$  for any  $\alpha \in \Delta$  because if we write  $w = r_{i_1} \cdots r_{i_k}$  and  $s = s_{i_1} \cdots s_{i_k}$ , and if  $\alpha = w(\alpha_i)$ , then  $\mathfrak{g}_\alpha = s(\mathfrak{g}_{\alpha_i})$ .

Step 4: Prove that  $\Delta = R$ . Let  $\lambda \in R \setminus \Delta$ . Then  $\lambda$  is not proportional to any  $\alpha \in \Delta$ . One can find some  $h$  in the real span of the  $H_i$  such that  $\langle \lambda, h \rangle = 0$  and  $\langle \alpha, h \rangle \neq 0$  for all  $\alpha \in \Delta$ . This divides  $\Delta = \Delta^+ \amalg \Delta^-$ . Choose  $\Pi \subseteq \Delta^+$  a new basis of simple roots  $\{\beta_1, \dots, \beta_n\}$ . We showed that  $W$  acts transitively on the sets of simple roots, so we can find some  $w \in W$  such that  $w(\alpha_i) = \beta_i$ . Then look at  $w^{-1}(\lambda)$ .

Since  $\lambda = \sum m_i \beta_i$  for some  $m_i > 0$ ,  $m_j < 0$ , and  $w^{-1}(\lambda) = \sum m_i \alpha_i$  for  $m_i > 0, m_j < 0$ ,  $w^{-1}(\lambda)$  is not a linear combination of only positive roots



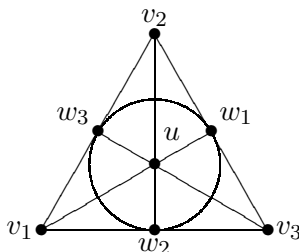
or of only negative roots. But  $R = R^+ \amalg R^-$ , so this is a contradiction and hence  $\Delta = R$ .

Step 5: Check that  $\mathfrak{g}$  is semi-simple. We have to show that there are no nontrivial abelian ideals in  $\mathfrak{g}$ . We already know that  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$  and that  $\mathfrak{g}_\alpha$  is 1-dimensional. We also know that the Serre relations hold. Notice that for any ideal  $\mathfrak{a}$ ,  $\mathfrak{a}$   $ad_{\mathfrak{h}}$ -invariant implies that  $\mathfrak{a} = \mathfrak{h}' \oplus \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$ . If  $\mathfrak{g}_\alpha \subseteq \mathfrak{a}$ , then  $X_\alpha \in \mathfrak{a}$ , so  $[Y_\alpha, X_\alpha] = H_\alpha \in \mathfrak{a}$ , so  $[H_\alpha, X_\alpha] = 2X_\alpha \in \mathfrak{a}$ . Thus, we have the whole  $\mathfrak{sl}(2)$  in  $\mathfrak{a}$ , so it cannot be abelian. Thus we have that  $\mathfrak{a} = \mathfrak{h}'$ . Take  $h \in \mathfrak{h}'$  non-zero. But  $\alpha_1, \dots, \alpha_n$  span  $\mathfrak{h}^*$ , so  $\alpha_i$  are such that  $\alpha_i(h) \neq 0$ , then  $[h, X_i] = \alpha_i(h)X_i \in \mathfrak{a}$ . Contradiction.  $\square$

So we know in each case what the Lie algebra is and how to work with them. In the non-exceptional cases, we have nice geometric descriptions of them. Next time, we will give some nice explanations of the exceptional Lie algebras. Next time will be the last lecture on the structure theory, and then we will do representation theory.

## Lecture 17 - Constructions of Exceptional simple Lie Algebras

We'll begin with the construction of  $G_2$ . Consider the picture



This is the projective plane over  $\mathbb{F}_2$ .

Consider the standard 7-dimensional representation of  $\mathfrak{gl}(7)$ , call it  $E$ . Take a basis given by points in the projective plane above.

► **Exercise 17.1.** Prove that  $\Lambda^3 E$  has an open  $\mathfrak{gl}(7)$ -orbit, by looking at the particular vector

$$\omega = v_1 \wedge v_2 \wedge v_3 + w_1 \wedge w_2 \wedge w_3 + u \wedge v_1 \wedge w_1 + u \wedge v_2 \wedge w_2 + u \wedge v_3 \wedge w_3$$

and checking that  $\mathfrak{gl}(7)\omega = \Lambda^3 E$ . (Remember  $x(v_1 \wedge v_2) = x(v_1) \wedge v_2 + v_1 \wedge x(v_2)$ .)

**Claim.**  $\mathfrak{g} = \{x \in \mathfrak{gl}(7) \mid x\omega = 0\}$  is a simple Lie algebra with root system  $G_2$ .

*Proof.* We have to choose a Cartan subalgebra of this  $\mathfrak{g}$ . Consider linear operators which are diagonal in the given basis  $u, v_1, v_2, v_3, w_1, w_2, w_3$ , take  $h = \text{diag}(c, a_1, a_2, a_3, b_1, b_2, b_3)$ . We want that  $h\omega = 0$ , which gives some linear equations. It is easy to check that  $h\omega = 0$  implies that  $c = 0$ ,  $a_1 + a_2 + a_3 = 0$ , and  $b_i = -a_i$ . This is a 2-dimensional subalgebra, it is our Cartan subalgebra.

If you consider the root diagram for  $G_2$  and look at only the long roots, you get a copy of  $A_2$ . This should mean that you have an embedding  $A_2 \subseteq G_2$ , so we should look for a copy of  $\mathfrak{sl}(3)$  in our  $\mathfrak{g}$ . We can write  $E = ku \oplus V \oplus W$ , where  $V = \langle v_1, v_2, v_3 \rangle$  and  $W = \langle w_1, w_2, w_3 \rangle$ . Let's consider the subalgebra which not only kills  $\omega$ , but also kills  $u$ . Let  $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid xu = 0\}$ . For  $x \in \mathfrak{g}_0$  we have  $x(v_1 \wedge v_2 \wedge v_3) = 0$  and  $x(w_1 \wedge w_2 \wedge w_3) = 0$ , so  $x$  preserves  $V$  and  $W$ . It also must kill the 2-form  $\alpha = v_1 \wedge w_1 + v_2 \wedge w_2 + v_3 \wedge w_3$ , since  $0 = x(u \wedge \alpha) = xu \wedge \alpha + u \wedge x\alpha = u \wedge x\alpha$  forces  $x\alpha = 0$ . This 2-form gives a pairing, so that  $V \simeq W^*$ . We can compute exactly what the pairing is,  $\langle v_i, w_j \rangle = \delta_{ij}$ . Therefore the operator  $x$  must be of the form

$$x = \left( \begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & A & 0 \\ 0 & 0 & -A^t \end{array} \right)$$

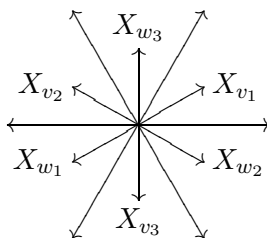
[[where  $A$  has trace zero?]]

The total dimension of  $G_2$  is 14, which we also know by the exercise is the dimension of  $\mathfrak{g}$ . We saw the Cartan subalgebra has dimension 2, and this  $\mathfrak{g}_0$  piece has dimension 6. So we still need another 6 dimensional piece.

For each  $v \in V$ , define a linear operator  $X_v$  which acts by  $X_v(u) = v$ .  $\Lambda^2 V \simeq_{\gamma} W$  is dual to  $V$  (since  $\Lambda^3 V = \mathbb{C}$ ). Then  $X_v$  acts by  $X_v(v') = \gamma(v \wedge v')$  and  $X_v(w) = 2\langle v, w \rangle u$ . Check that this kills  $\alpha$ , and hence is in  $\mathfrak{g}$ .

Similarly, you can define  $X_w$  for each  $w \in W$  by  $X_w(u) = w, X_w(w') = \gamma(w \wedge w'), X_w(v) = 2\langle w, v \rangle u$ .

If you think about a linear operator which takes  $u \mapsto v_i$ , it must be in some root space, this tells you about how it should act on  $V$  and  $W$ . This is how we constructed  $X_v$  and  $X_w$ , so we know that  $X_{v_i}$  and  $X_{w_i}$  are in some root spaces. We can check that their roots are the short roots in the diagram,



and so they span the remaining 6 dimensions of  $G_2$ . To properly complete this construction, we should check that this is semisimple, but we're not going to.  $\square$

We saw here that  $G_2$  is isomorphic to the Lie algebra of automorphisms of a generic 3-form in 7-dimensional space. The picture of the projective plane is related to Cayley numbers, an important nonassociative division algebra, of which  $G_2$  is the algebra of automorphisms.

Let's analyze what we did with  $G_2$ , so that we can do a similar thing to construct  $E_8$ . We discovered certain phenomena, we can write  $\mathfrak{g} = \underbrace{\mathfrak{g}_0}_{\mathfrak{sl}(3)} \oplus \underbrace{\mathfrak{g}_1}_V \oplus \underbrace{\mathfrak{g}_2}_W$ . This gives us a  $\mathbb{Z}/3$ -grading:  $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j \pmod{3}}$ . As an  $\mathfrak{sl}(3)$  representation, it has three components:  $ad$ , standard, and the dual to the standard. We get that  $W \cong V^* \simeq \Lambda^2 V$ . Similarly,  $V \simeq \Lambda^2 W$ . This is called *Triality*.

More generally, say we have  $\mathfrak{g}_0$  a semi-simple Lie algebra, and  $V, W$

representations of  $\mathfrak{g}_0$ , with intertwining maps

$$\begin{aligned}\alpha &: \Lambda^2 V \rightarrow W \\ \beta &: \Lambda^2 W \rightarrow V \\ V &\simeq W^*.\end{aligned}$$

We also have  $\gamma : V \otimes W \simeq V \otimes V^* \rightarrow \mathfrak{g}_0$  (representations are semi-simple, so the map  $\mathfrak{g}_0 \rightarrow \mathfrak{gl}(V) \simeq V \otimes V^*$  splits). We normalize  $\gamma$  in the following way. Let  $B$  be the Killing form, and normalize  $\gamma$  so that  $B(\gamma(v \otimes w), X) = \langle w, Xv \rangle$ . Make a Lie algebra  $\mathfrak{g} = \mathfrak{g}_0 \oplus V \oplus W$  by defining  $[X, v] = Xv$  and  $[X, w] = Xw$  for  $X \in \mathfrak{g}_0, v \in V, w \in W$ . We also need to define  $[\cdot, \cdot]$  on  $V, W$  and between  $V$  and  $W$ . These are actually forced, up to coefficients:

$$\begin{aligned}[v_1, v_2] &= a\alpha(v_1 \wedge v_2) \\ [w_1, w_2] &= b\beta(w_1 \wedge w_2) \\ [v, w] &= c\gamma(v \otimes w).\end{aligned}$$

There are some conditions on the coefficients  $a, b, c$  imposed by the Jacobi identity;  $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ . Suppose  $x \in \mathfrak{g}_0$ , with  $y, z \in \mathfrak{g}_i$  for  $i = 0, 1, 2$ , then there is nothing to check, these identities come for free because  $\alpha, \beta, \gamma$  are  $\mathfrak{g}_0$ -invariant maps. There are only a few more cases to check, and only one of them gives you a condition. Look at

$$[v_0, [v_1, v_2]] = ca\gamma(v_0 \otimes \alpha(v_1 \wedge v_2)) \quad (\text{RHS})$$

and it must be equal to

$$\begin{aligned}[[v_0, v_1], v_2] + [v_1, [v_0, v_2]] = \\ -ac\gamma(v_2 \otimes \alpha(v_0 \wedge v_1)) + ac\gamma(v_1 \otimes \alpha(v_0 \wedge v_1)) \quad (\text{LHS})\end{aligned}$$

This doesn't give a condition on  $ac$ , but we need to check that it is satisfied. It suffices to check that  $B(\text{RHS}, X) = B(\text{LHS}, X)$  for any  $X \in \mathfrak{g}_0$ . This gives us the following condition:

$$\langle \alpha(v_1 \wedge v_2), Xv_0 \rangle = \langle \alpha(v_0 \wedge v_2), Xv_1 \rangle - \langle \alpha(v_0 \wedge v_1), Xv_2 \rangle$$

The fact that  $\alpha$  is an intertwining map for  $\mathfrak{g}_0$  gives us the identity:

$$\langle \alpha(v_1 \wedge v_2), Xv_0 \rangle = \langle \alpha(Xv_1 \wedge v_2), v_0 \rangle - \langle \alpha(v_1 \wedge Xv_2), v_0 \rangle$$

and we also have that

$$\langle \alpha(v_1 \wedge v_2), v_0 \rangle = \langle \alpha(v_0 \wedge v_2), v_1 \rangle = \langle \alpha(v_0 \wedge v_1), v_2 \rangle$$

With these two identities it is easy to show that the equation (and hence this Jacobi identity) is satisfied.

We also get the Jacobi identity on  $[w, [v_1, v_2]]$ , which is equivalent to:

$$ab\beta(w \wedge \alpha(v_1 \wedge v_2)) = c(\gamma(v_1 \otimes w)v_2 - \gamma(v_2 \otimes w)v_1)$$

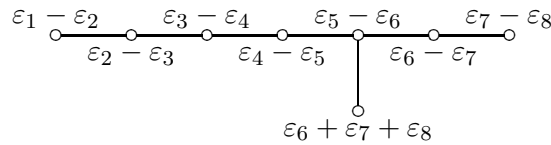
It suffices to show for any  $w' \in W$  that the pairings of each side with  $w'$  are equal,

$$ab\langle w', \beta(w \wedge \alpha(v_1 \wedge v_2)) \rangle = cB(\gamma(v_1 \otimes w), \gamma(v_2 \otimes w')) - cB(\gamma(v_2 \otimes w), \gamma(v_1 \otimes w'))$$

This time we will get a condition on  $a$ ,  $b$ , and  $c$ . You can check that any of the other cases of the Jacobi identity give you the same conditions.

Now we will use this to construct  $E_8$ . Write  $\mathfrak{g} = \mathfrak{g}_0 \oplus V \oplus W$ , where we take  $\mathfrak{g}_0 = \mathfrak{sl}(9)$ . Let  $E$  be the 9-dimensional representation of  $\mathfrak{g}_0$ . Then take  $V = \Lambda^3 E$  and  $W = \Lambda^3 E^* \simeq \Lambda^6 E$ . We have a pairing  $\Lambda^3 E \otimes \Lambda^6 E \rightarrow k$ , so we have  $V \simeq W^*$ . We would like to construct  $\alpha : \Lambda^2 \rightarrow W$ , but this is just given by  $v_1 \wedge v_2$  including into  $\Lambda^6 E \simeq W$ . Similarly, we get  $\beta : \Lambda^2 W \rightarrow V$ . You get that the rank of  $\mathfrak{g}$  is 8 ( $= \text{rank of } \mathfrak{g}_0$ ). Notice that  $\dim V = \binom{9}{3} = 84$ , which is the same as  $\dim W$ , and  $\dim \mathfrak{g}_0 = \dim \mathfrak{sl}(9) = 80$ . Thus, we have that  $\dim \mathfrak{g} = 84 + 84 + 80 = 248$ , which is the dimension of  $E_8$ , and this is indeed  $E_8$ .

Remember that we previously got  $E_7$  and  $E_6$  from  $E_8$ . Look at the diagram for  $E_8$ :



The extra guy,  $\epsilon_6 + \epsilon_7 + \epsilon_8$ , corresponds to the 3-form. When you cut out  $\epsilon_1 - \epsilon_2$ , you can figure out what is left and you get  $E_7$ . Then you can additionally cut out  $\epsilon_2 - \epsilon_3$  and get  $E_6$ .

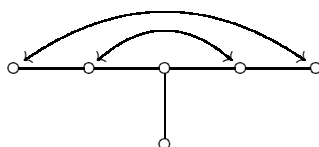
Fianally, we construct  $F_4$ :

We know that any simple algebra can be determined by generators and relations, with a  $X_i, Y_i, H_i$  for each node  $i$ . But sometimes our diagram has a symmetry, like switching the horns on a  $D_n$ , which induces an automorphism of the Lie algebra given by  $\gamma(X_i) = X_i$  for  $i < n - 1$  and switches  $X_{n-1}$  and  $X_n$ . Because the arrows are preserved, you can check that the Serre relations still hold. Thus, in general, an automorphism of the diagram induces an automorphism of the Lie algebra (in a very concrete way).

**Theorem 17.1.**  $(\text{Aut } \mathfrak{g})/(\text{Aut}_0 \mathfrak{g}) = \text{Aut } \Gamma$ . So the connected component of the identity gives some automorphisms, and the connected components are parameterized by automorphisms of the diagram.

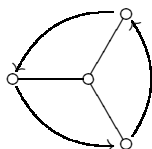
$D_n$  is the diagram for  $SO(2n)$ . We have that  $SO(2n) \subset O(2n)$ , and the group of automorphisms of  $SO(2n)$  is  $O(2n)$ . This isn't true of  $SO(2n+1)$ , because the automorphisms given by  $O(2n+1)$  are the same as those from  $SO(2n+1)$ . This corresponds to the fact that  $D_n$  has a nontrivial automorphism, but  $B_n$  doesn't.

Notice that  $E_6$  has a symmetry; the involution:

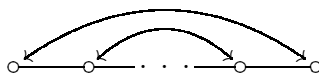


Define  $X'_1 = X_1 + X_5, X'_2 = X_2 + X_4, X'_3 = X_3, X'_6 = X_6$ , and the same with  $Y$ 's, the fixed elements of this automorphism. We have that  $H'_1 = H_1 + H_5$  (you have to check that this works), and similarly for the other  $H$ 's. As the set of fixed elements, you get an algebra of rank 4 (which must be our  $F_4$ ). You can check that  $\alpha'_1(H'_2) = -1, \alpha'_2(H'_1) = -1, \alpha'_3(H'_1) = 0, \alpha'_3(H'_2) = -2, \alpha'_2(H'_3) = -1$ , so this is indeed  $F_4$  as desired. In fact, any diagram with multiple edges can be obtained as the fixed algebra of some automorphism:

► **Exercise 17.2.** Check that  $G_2$  is the fixed algebra of the automorphism of  $D_4$ :



Check that  $B_n, C_n$  can be obtained from  $A_{2n}, A_{2n+1}$



next time, we'll do representation theory ... highest weight, etc.

## Lecture 18 - Representations

### Representations

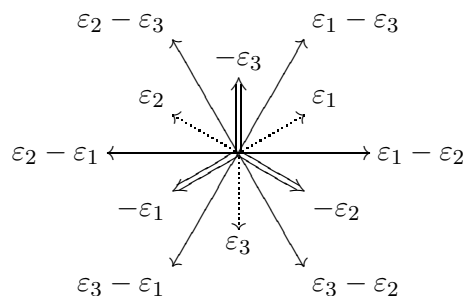
Let  $\mathfrak{g}$  be a semi-simple lie algebra over an algebraically closed field  $k$  of characteristic 0. Consider the root decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$ , and let  $V$  be a finite-dimensional representation of  $\mathfrak{g}$ . Because all elements of  $\mathfrak{h}$  are semisimple, and because Jordan decomposition is preserved, the elements of  $\mathfrak{h}$  can be simultaneously diagonalized. That is, we may write  $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_{\mu}$ , where  $V_{\mu} = \{v \in V | hv = \mu(h)v \ \forall h \in \mathfrak{h}\}$ . We call  $V_{\mu}$  a weight space, and  $\mu$  the weights. Define  $P(V) = \{\mu \in \mathfrak{h}^* | V_{\mu} \neq 0\}$ . The multiplicity of a weight  $\mu \in P(V)$  is  $\dim V_{\mu}$ , and is denoted  $m_{\mu}$ .

**Example 18.1.** You can take  $V = k$  (the trivial representation). Then  $P(V) = 0$  and  $m_0 = 1$ .

**Example 18.2.** If  $V = \mathfrak{g}$  and we take the adjoint representation, then we have that  $P(V) = \Delta \cup \{0\}$ , with  $m_{\alpha} = 1$  for  $\alpha \in \Delta$ , and  $m_0 =$  the rank of  $\mathfrak{g}$ .

We can plot the set of weights in  $\mathfrak{h}^*$ :

**Example 18.3.** Let  $\mathfrak{g} = \mathfrak{sl}(3)$ :



For the adjoint representation, the weights are shown in single lines.

If  $E$  is the standard 3-dimensional representation, then we get weights  $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ , shown in dotted lines.

We can also take the dual representation  $E^*$ . In general, the weights of the dual representation are the negatives of the original representation because  $\langle h\phi, v \rangle = -\langle \phi, hv \rangle$ , so we get the double lines.

Note that the sum of these is a  $G_2$ . (We noted earlier that  $G_2$  has some  $\mathfrak{sl}(3)$ s sitting around).

Properties: For  $V$  finite-dimensional

- (1)  $\mu(H_{\alpha}) \in \mathbb{Z}$  for any root  $\alpha$  and  $\mu \in P(V)$ .

This is because all weights of any representation of  $\mathfrak{sl}(2)$  are integers, so restricting to our little  $\mathfrak{sl}(2)$ , we get this result.

(2)  $\mathfrak{g}_\alpha V_\mu \subseteq V_{\mu+\alpha}$  for  $\alpha \in \Delta, \mu \in P(V)$ .

To check this:

$$\begin{aligned} h(x_\alpha v) &= x_\alpha h v + [h, x_\alpha] v \\ &= x_\alpha \mu(h) v + \alpha(h) x_\alpha v \\ &= (\mu + \alpha)(h) x_\alpha v \end{aligned}$$

(3) If  $\mu \in P(V)$  and  $w \in W$ , then  $w(\mu) \in P(V)$  and  $m_\mu = m_{w(\mu)}$ .

To see this, it is sufficient to check for simple reflections  $w = r_i$ . Recall that for each reflection, there is a special element of the group  $G$  itself (coming from the small  $SL(2) \subseteq G$ ). It is  $\exp X_i \exp(-Y_i) \exp X_i s_i \in G$ . Then  $s_i(\mathfrak{h}) = \mathfrak{h}$  and  $s_i(V_\mu) = V_{r_i \mu}$ .

**Example 18.4.** If  $\mathfrak{g} = \mathfrak{sl}(3)$ , then we get  $W = S_3 = D_3$ . The orbit of a point can have a couple of different forms. If the point is on a mirror, then you get a triangle. For a generic point, you get a hexagon (which is not regular, but still symmetric).

It is pretty clear that knowing the weights and multiplicities gives us a lot of information about the representation. However, it is hard to draw pictures in more than three dimensions, so instead you write the following expression:

$$ch V = \sum_{\mu \in P(V)} m_\mu e^\mu$$

You can formally add and multiply these expressions:  $ch$  is additive with respect to direct sum and multiplicative with respect to tensor products:

$$\begin{aligned} ch(V \oplus W) &= ch V + ch W \\ ch(V \otimes W) &= (ch V)(ch W) \end{aligned}$$

This is because  $V_\mu \otimes W_\nu \subseteq (V \otimes W)_{\mu+\nu}$ . You can also check that the  $ch$  of the dual is  $\sum m_\mu e^{-\mu}$ . If you don't care about multiplication, then you can do infinite-dimensional stuff.

Relations to group characters: if you have  $G \rightarrow GL(V)$ , then we have the characters  $\chi_V(g) = tr_V(g)$ . How are these related? Well, you can exponentiate  $\mathfrak{h}$  to get a subgroup  $H \subseteq G$ . To compute  $\chi_V(\exp h)$  for  $h \in \mathfrak{h}$ , you just need to add the  $\exp$ 's of the eigenvalues of  $h$ :

$$\chi_V(\exp h) = tr(\exp h) = \sum m_\mu e^{\mu(h)}.$$

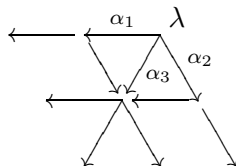
Recall that group characters are constant on conjugacy classes. If you look at different Cartan subalgebras, then something ... the character is determined by its values on the Cartan. . . This is a good way to keep track of weights.



## Highest weights

Fix a set of simple roots  $\Pi = \{\alpha_i\}$ . A highest weight is a  $\lambda \in P(V)$  such that  $\alpha_i + \lambda \notin P(V)$ ,  $\forall \alpha \in \Pi$ .

Say  $V$  is irreducible and  $\lambda$  is some highest weight. Let  $v \in V_\lambda$ . We have that  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ , and that  $\mathfrak{n}^+v = 0$ . Since  $v$  generates  $V$  (because  $V$  is irreducible), where  $V = (U\mathfrak{g})v$ . But by PBW,  $U(\mathfrak{g}) = U\mathfrak{n}^- \oplus U\mathfrak{h} \oplus U\mathfrak{n}^+$ , so  $V = U\mathfrak{n}^-v$ . Thus, all weights of the irreducible representation are given by applying negative roots. There are some relations among the different ways of applying roots:  $[Y_{\alpha_1}, Y_{\alpha_2}] = Y_{\alpha_1 + \alpha_2}$ . Note that in particular, we get that  $m_\lambda = 1$ .



The highest weight is unique in an irreducible representation: suppose  $\mu$  is another one. But all weights lie “below”  $\lambda$ . So we get  $\lambda \leq \mu$  and  $\mu \leq \lambda$ , where  $a \leq b$  means that  $b = a + \sum_{\alpha \in \Delta^+} n_\alpha \alpha$  with  $n_\alpha \geq 0$ . (Note that we are using the irreducible hypothesis to say that all weights are below the highest weight).

If  $\lambda$  is a highest weight of an irreducible representation with respect to some choice of simple roots  $\alpha_1, \dots, \alpha_n$ , then for any  $w \in W$ ,  $w(\lambda)$  will be the highest weight with respect to  $w(\alpha_1), \dots, w(\alpha_n)$ . Thus, we get that  $P(V)$  is a subset in the convex hull of the orbit of the highest weight under the Weyl group,  $W\lambda$ .

If  $\lambda$  is a highest weight of an irreducible  $V$ , then we know that  $\lambda(H_i) \in \mathbb{Z}_{\geq 0}$ . Sometimes this is written  $(\lambda, \check{\alpha}_i) = \lambda(H_i) \geq 0$ . To see this, note that we are restricting to our little  $\mathfrak{sl}(2)$  again.

There are a few lattices in  $\mathfrak{h}^*$ . If you look at the lattice generated by the roots, then we call that the *root lattice*, and it is denoted  $Q$ :

$$Q = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n.$$

There is another one, called the *weight lattice*, and is called  $P$ :

$$P = \{\mu \in \mathfrak{h}^* \mid (\mu, \check{\alpha}_i) \in \mathbb{Z}\}.$$

Let  $P^+ = \{\lambda \in P \mid (\lambda, \check{\alpha}_i) \geq 0\}$ . These are called the integral dominant elements. Both  $P$  and  $Q$  have the same rank, but  $P$  is usually bigger (and never smaller, obviously).

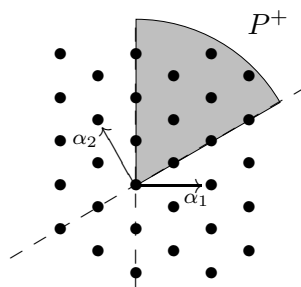
**Example 18.5.** For  $\mathfrak{sl}(2)$ , the root lattice is the even integers (because  $[H, X] = 2X$ ) While the weight lattice is the integers (because you can get any integer as a weight).

Thus, you can look at  $P/Q$ , which is the center of the simply connected group corresponding to  $\mathfrak{g}$ .

We have said that the highest weight is in  $P^+$ .

► **Exercise 18.1.** Show that  $P^+$  is the fundamental domain of the  $W$  action on  $P$ . (This follows from the fact that  $W$  acts simply and transitively on the set of bases).

In the case of  $\mathfrak{sl}(3)$ , we get:



For every  $\mu \in P$ , there is a unique element  $w \in W$  such that  $w\mu \in P^+$ .

**Theorem 18.6.** *There is a bijection between  $P^+$  and the set of (isomorphism classes of) finite-dimensional irreducible representations of  $\mathfrak{g}$ .*

If you have an irreducible representation  $V$ , then to it we associate its highest weight in  $P^+$ . We already know that the map is well-defined (because the highest weights are unique). For injectivity we must show that two different representations cannot have the same highest weight. For surjectivity, we need to show that every element of  $P^+$  is the highest weight of some irreducible representations.

We are going to use Verma modules because they are nice. A lot of people avoid using them, but I don't think that is a good approach.

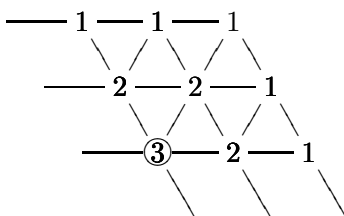
Let  $V$  be our representation with highest weight  $\lambda$ . That means that  $V_\lambda$  is a 1-dimensional representation of the subalgebra  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+ \subseteq \mathfrak{g}$ . There is an embedding  $V_\lambda \rightarrow V$ , which is a homomorphism of  $U\mathfrak{b}$ -modules (that is, of representations of  $\mathfrak{b}$ ). Thus, there is an induced object  $U(\mathfrak{g}) \otimes_{U\mathfrak{b}} V_\lambda$ , and there is a homomorphism of  $U\mathfrak{g}$ -modules from  $U(\mathfrak{g}) \otimes_{U\mathfrak{b}} V_\lambda$  to  $V$ . However, there is a problem: when we do the induction, we get something infinite dimensional.

**Definition 18.7.** A *Verma module* is  $M(\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}} V_\lambda$ .

Verma Modules are universal in the sense that all representations with highest weight  $\lambda$  are quotients of  $M(\lambda)$ . To understand what  $M(\lambda)$  is as a vector space, we use PBW to get that  $U\mathfrak{g} = U\mathfrak{n}^- \otimes U\mathfrak{h} \otimes U\mathfrak{n}^+ = U(\mathfrak{n}^-) \otimes U\mathfrak{b}$  (these tensor products are as vector spaces). Thus, we get  $M(\lambda) =$

$U\mathfrak{n}^- \otimes_k V_\lambda$ . The Verma module is something universal ... . We can choose a basis  $U(\mathfrak{n}^-)$ , say  $\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_N$  (the positive roots, ordered). Then  $\{Y_{\alpha_i}\}_{i \leq N}$  is a basis for  $\mathfrak{n}^-$ , and by PBW  $\{Y_{\alpha_1}^{k_1} \dots Y_{\alpha_N}^{k_N} v\}$  is a basis in  $U(\mathfrak{n}^-) \otimes_k V_\lambda M(\lambda)$ . Even though the Verma module is infinite-dimensional, it still has a weight decomposition:

$$h(Y_{\alpha_1}^{k_1} \dots Y_{\alpha_N}^{k_N} v) = (\lambda - k_1\alpha_1 - \dots)(h)(Y_{\alpha_1}^{k_1} \dots Y_{\alpha_N}^{k_N} v)$$



All the multiplicities along the edges are 1. Then you get multiplicity 2 just below  $\lambda$  because there is a linear dependence. You can check that parallel to the edges, you keep getting multiplicity 2. And in general, the multiplicity grows by 1 with every step you take from the wall.

► **Exercise 18.2.** Check this out.

We have a nice formula:

$$m_\mu = \#\{\lambda - \mu = \sum_{\alpha \in \Delta^+} k_\alpha \alpha\}$$

This is called the Kostant partition function.

Let's calculate the multiplicity circled in the previous picture. We have that  $2\alpha_3 = \alpha_3 + \alpha_2 + \alpha_1 = 2\alpha_1 + 2\alpha_2$ , so the multiplicity is 3. These calculations will prepare you for the Weyl formula.

**Claim.** A Verma module  $M(\lambda)$  has a unique proper maximal submodule  $N(\lambda)$ .

*Proof.*  $N$  being proper is equivalent to  $N \cap V_\lambda = 0$ . This property is clearly preserved under taking sums, so you get a unique maximal submodule.  $\square$

If  $V$  and  $W$  are irreducible representations that have the same highest weight, then they are both the unique irreducible quotient  $M(\lambda)/N(\lambda)$ , so they are isomorphic. Thus, the map from irreducibles to  $P^+$  is injective.

Now we need to show that if  $\lambda \in P^+$ , then the quotient  $M(\lambda)/N(\lambda)$  is finite-dimensional. The trick is to look at  $\mathfrak{sl}(2)$ . The picture of the Verma module is that you have all the weights  $\lambda - k\alpha$  with multiplicity 1. If you mod out by any submodule, you get something finite-dimensional.

*Sketch of Proof.* If  $w$  is a weight vector in a Verma module such that  $X_i w = 0$  for  $i = 1, \dots, n$ , then we claim that  $w \in N(\lambda)$ . To see this, you note that you can never get back to  $\lambda$ .

Suppose  $\langle \lambda, \check{\alpha}_i \rangle = n_i \in \mathbb{Z}_{\geq 0}$ . Then  $w = Y_i^{n_i+1} v$  and  $X_i w = 0$ . We have  $X_j Y_i^{n_i+1} v = Y_i^{n_i+1} X_j v = 0$  because  $[X_j, Y_i] = 0$  for  $i \neq j$ .

Consider the quotient  $M(\lambda)/N(\lambda) = V(\lambda)$ . Then we have that  $Y_i^{n_i+1} v = 0$ , so the  $Y_i$  act locally nilpotently on  $V(\lambda)$ . And  $X_i$  also act locally nilpotently (even on the Verma module, not just the quotient). Then  $s_i = \exp X_i \exp(-Y_i) \exp X_i$ , and we get that  $P(V(\lambda))$  is  $W$  invariant because of something. But then you can start with  $\lambda$  and look at the orbit. The weights are in the convex hull, which is bounded, so  $V(\lambda)$  is finite-dimensional.  $\square$

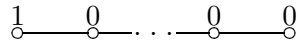
We introduced these Verma modules because we will use them later to write down the character formula.

Next time:

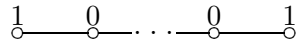
- (1) Fundamental representations, etc.
- (2) Weyl character formula

## Lecture 19 - The Weyl character formula

If  $\lambda \in P^+$  (i.e.  $(\lambda, \check{\alpha}_i) \in \mathbb{Z}_{\geq 0}$  for all  $i$ ), then we can construct a representation  $V(\lambda)$  which is a representation with  $\lambda$  as the highest weight. We define the *fundamental weights*  $\omega_1, \dots, \omega_n$  of a representation to be those weights for which  $(\omega_i, \check{\alpha}_j) = \delta_{ij}$ . It is clear that any weight  $\lambda = \lambda_1\omega_1 + \dots + \lambda_n\omega_n$  for  $\lambda_i \geq 0$ . So people often draw the Dynkin diagrams with the fundamental weights labelled:



This is  $V(\omega_1)$ , the first fundamental representation for  $\mathfrak{sl}(n)$ . Here is the adjoint representation:



Observe that if  $v \in V$  a highest vector, and  $w \in W$  another highest weight vector in another representation, then  $v \otimes w \in V \otimes W$  is a highest weight vector. It follows that every finite dimensional irreducible representation can be realized as a subrepresentation in a tensor product of fundamental representations.

**Example 19.1.** Let's calculate the fundamental weights for  $\mathfrak{sl}(n+1)$ . Recall that we have simple roots  $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_n - \varepsilon_{n+1}$ , which are also the coroots. It follows that  $\omega_i = \varepsilon_1 + \dots + \varepsilon_i$  for  $i = 1, \dots, n$ .

Consider  $\Lambda^k E$ , where  $E$  is the standard representation. Let  $e_1, \dots, e_{n+1}$  be a weight basis for  $E$  (i.e. the vectors  $e_1, \dots, e_n$  are a basis, with decreasing weights  $\varepsilon_1, \dots, \varepsilon_n$ ). We'd like to write down the weights  $P(\Lambda^k E)$ . Note that  $\Lambda^k E$  is spanned by the vectors  $e_{i_1} \wedge \dots \wedge e_{i_k}$ , which have weights  $\varepsilon_{i_1} + \dots + \varepsilon_{i_k}$ . Thus, the highest weight is  $\varepsilon_1 + \dots + \varepsilon_k = \omega_k$ , so  $\Lambda^k E = V_{\omega_k}$  is a fundamental representation. Notes also that the weights form the single orbit  $W\omega_k$ .

$\Lambda^k E = V_{\omega_k}$

Representations where the weights form a single orbit are called *minuscule*.

*Remark 19.2.* If  $\lambda$  is the highest weight of the representation  $V$ , then  $-\lambda$  is a weight of  $V^*$ . To compute the highest weight of  $V^*$ , we need to get back into  $P^+$ , so we apply the longest word in the Weyl group:  $-w(\lambda)$  is the highest weight of  $V^*$ . Thus, there is a fixed involution of the Weyl chamber (namely,  $-w$ ) which takes the highest weight of  $V$  to the highest weight of  $V^*$ . It is clear that  $-w$  preserves the set of simple roots and preserves inner products, so it is an involution of the Dynkin diagram.

In the case of  $\mathfrak{sl}(n+1)$ , the involution is . In particular, the dual of the standard representation  $V_{\omega_1}$  is  $V_{\omega_n}$ .

The bijection between  $P^+$  and irreducible representations implies

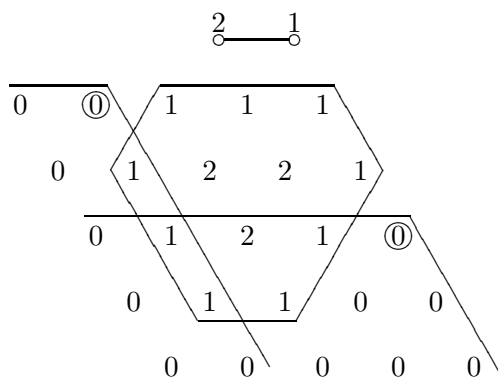
**Corollary 19.3.** *If  $V, W$  are finite dimensional representations, with  $ch V = ch W$ , then  $V \simeq W$ .*

Fundamental weights of  $\mathfrak{sl}(n+1)$

Highest weights of duals

How do you calculate  $ch V$ ? Recall that  $ch V = \sum m_\mu e^\mu$ , where  $m_\mu$  is the multiplicity of the weight  $\mu$ . You can do it with Verma modules. Remember that for Verma modules, you still get a weight decomposition.

**Example 19.4.**  $\mathfrak{g} = \mathfrak{sl}(3)$ . Consider the case  $\lambda = 2\omega_1 + \omega_2$ :



In the quotient, the shown weights disappear. What about the one below it? Its multiplicity in the Verma module is 2. The multiplicity in the Verma module centered at  $2\omega_1$  is 1, so in the quotient, you get multiplicity 1. How about right below it? You had 3 from the big Verma module, and you remove 1 and 1, so you get 1 left. Further down and to the left, you get all zeros. Similarly, we can fill in the other numbers. You get the leftover hexagon.

Explain this picture better

We have the Weyl character formula, which is nice if you have a computer:

### The character formula

First some notation:

1.  $W$  is the Weyl group.  $(-1)^w = \det(w) = (-1)^{\text{sgn}(w)}$ .
2.  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ . Note that  $r_i(\rho) = \rho - \alpha_i = \rho - (\rho, \check{\alpha}_i)\alpha_i$ , so we know that  $(\rho, \check{\alpha}_i) = 1$  for all  $i$ . Thus,  $\rho$  is the sum of all the fundamental weights.

Then the result is that

$$ch V(\lambda) = \frac{\sum_{w \in W} (-1)^w e^{w(\lambda + \rho)}}{\sum_{w \in W} (-1)^w e^{w(\rho)}} \quad (\text{Weyl Character Formula}^1)$$

<sup>1</sup> This formula may look ugly, but it is *sweet*. It says that you can compute the character of  $V_\lambda$  in the following way. Translate the Weyl vector  $\rho$  around by the Weyl group; this

note that the numerator and denominator is skew with respect to the action of  $W$ , so  $ch$  is  $W$ -symmetric. The denominator is called the *Weyl denominator*.

For those who know something about the representation theory of  $\mathfrak{gl}(n)$ , I'd like to show the connection between these two things. If  $\mathfrak{g} = \mathfrak{gl}(n) \supset \mathfrak{sl}(n)$ . Let  $z_i = e^{\varepsilon_i}$ , so  $z_1 \cdots z_n = 1$ . We know that  $W \simeq S_n$ , which acts on the  $z_i$  by permutation. We have that

$$\begin{aligned} \rho &= \frac{1}{2} \sum_{i < j} \varepsilon_i - \varepsilon_j \\ &= \frac{1}{2} ((n-1)\varepsilon_1 + (n-3)\varepsilon_2 + \cdots + (1-n)\varepsilon_n) \\ &= \end{aligned}$$

so

$$e^\rho = z_1^{n-1} z_2^{n-3} \cdots z_n^{1-n}.$$

Then

$$ch V(\lambda) = \frac{\sum_{\sigma \in S_n} (-1)^\sigma z_{\sigma(1)}^{a_1+n-1} \cdots z_{\sigma(n)}^{a_n}}{\sum_{\sigma \in S_n} (-1)^\sigma z_{\sigma(1)}^{n-1} \cdots z_{\sigma(n)}^0}$$

The denominator (call it  $\mathcal{D}$ ) is the famous Vandermonde determinant, which you should know

vandermonde

Where  $\lambda = a_1\varepsilon_1 + \cdots + a_n\varepsilon_n$ . Then  $(\lambda, \check{\alpha}_i) = a_i - a_{i+1} \geq 0$ .

Usually, the representations are encoded as Young diagrams. The marks on the dynkin diagram are the differences in consecutive rows in the young diagram.

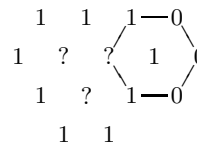
The numerator is

$$\mathcal{D}_\lambda = \det \begin{pmatrix} z_1^{a_1+n-1} & \cdots & z_n^{a_1+n-1} \\ \vdots & & \vdots \\ z_1^{a_n} & & \vdots \end{pmatrix}$$

will form some polytope. Make a piece of cardboard shaped like this polytope (ok, so maybe this is only practical for rank 2), and put  $(-1)^w$  at the vertex  $w(\rho)$ . This is your *cardboard denominator*. Now the formula tells you that when you center your cardboard denominator around any weight, and then add the multiplicities of the weights of  $V_\lambda$  at the vertices with the appropriate sign, you'll get zero (unless you centered your cardboard at  $w(\lambda + \rho)$ , in which case only one non-zero multiplicity shows up in the sum, so you'll get  $\pm 1$ ). Since we know that the highest weight has multiplicity 1, we can use this to compute the rest of the character.

For  $\mathfrak{sl}(3)$ , your cardboard denominator will be a hexagon, and one step of computing the character of  $V_{2\omega_1+\omega_2}$  might look like:

$0 = 0 - 1 + ? - 1 + 0 - 0$ , so  $? = 2$ . Since  $ch V$  is symmetric with respect to  $W$ , all three of the ?s must be 2.



So the character is the Schur polynomial.

*Proof of character formula.* Express  $ch V(\lambda)$  as a linear combination of characters of Verma modules. This is what we did with the picture for  $\mathfrak{sl}(3)$ .

Consider  $\mathfrak{sl}(2)$ . We know what the representations are:  $V(n)$ . The Verma module is infinite in the negative direction, but at  $-n - 2$ , there is another Verma module, which is also infinite. We have that  $\rho = \omega = \frac{\alpha}{2}$

picture

Thus, we claim that  $ch V(n) = ch M(n) - ch M(-n - 2)$ . To see this, we compute  $ch M(n) = e^{n\omega} + e^{(n-2)\omega} + \dots = e^{n\omega}(1 + e^{-\alpha} + e^{-2\alpha} + \dots) = \frac{e^{n\omega}}{1 - e^{-\alpha}}$ , and similarly,  $ch M(-n - 2) = \frac{e^{(-n-2)\omega}}{1 - e^{-\alpha}}$ . So we get that  $ch M(n) = \frac{e^{(n+1)\omega}}{(e^{\alpha/2} - e^{-\alpha/2})}$  and  $ch M(-n - 2) = \frac{e^{-(n+1)\omega}}{(e^{\alpha/2} - e^{-\alpha/2})}$

So we have that  $ch V(\lambda) = \frac{e^{(n+1)\omega} - e^{(-n-1)\omega}}{e^{\alpha/2} - e^{-\alpha/2}}$

Consider  $M(\lambda)$  for  $\lambda \in P$ . Then

$$ch M(\lambda) = e^\lambda \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} + \dots)$$

where

$$m_\mu = \#\{\lambda - \mu = \sum_{\alpha \in \Delta^+} n_\alpha \alpha\}$$

We claim that this character is

$$\prod_{\alpha \in \Delta^+} (1 + e^{-\alpha} + \dots) = \sum m_\gamma e^\gamma$$

where  $m_\gamma = \#\{\gamma = \sum n_\alpha \alpha\}$ . Thus, we have that

$$ch M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})} = \boxed{\frac{e^{\lambda + \rho}}{\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2})}}_1$$

where  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ .

Now recall the Casimir operator. If  $\{e_i\}$  is a basis for  $\mathfrak{g}$ , with dual basis  $\{f_i\}$ . Then the Casimir operator is  $\Omega = \sum e_i f_i \in U\mathfrak{g}$ . We showed before that  $\Omega$  is in the center of  $U\mathfrak{g}$ , so that  $\Omega g = g\Omega$  for all  $g \in \mathfrak{g}$ .

**Claim.**  $\boxed{\Omega|_{M(\lambda)} = (\lambda + 2\rho, \lambda)\text{Id}}_2$ . This is 2



We will use the identity  $(\lambda + 2\rho, \lambda) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho)$ . It is enough to show that something highest weight, then use stuff

Let  $u_i$  be an orthonormal basis for  $\mathfrak{h}$ .  $\{X_\alpha\}_{\alpha \in \Delta}, \left\{\frac{Y_\alpha}{(X_\alpha, Y_\alpha)}\right\}$  dual basis.

We can write

$$\Omega = \sum_{i=1}^n u_i^2 + \sum_{\alpha \in \Delta^+} \frac{X_\alpha Y_\alpha}{(X_\alpha, Y_\alpha)} + \frac{Y_\alpha X_\alpha}{(X_\alpha, Y_\alpha)}$$

Let  $v$  be the highest vector. then we have that

$$\Omega v = \left(\sum_1^n \lambda(u_i)^2\right)v + \sum_{\alpha \in \Delta^+} \frac{\lambda(H_\alpha)}{(X_\alpha, Y_\alpha)}$$

since  $u_i(v) = \lambda(u_i)v$ ,  $X_\alpha v = 0$ ,  $Y_\alpha X_\alpha v = 0$ ,  $X_\alpha Y_\alpha = H_\alpha + Y_\alpha X_\alpha$ ,  $X_\alpha Y_\alpha v = \lambda(H_\alpha)v$ .

We have that  $\lambda(H_\alpha) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$  and  $(X_\alpha, Y_\alpha) = \frac{1}{2}(H_\alpha, H_\alpha) = \frac{2}{(\alpha, \alpha)}$ .

We get that

$$\Omega v = (\lambda, \lambda)v + \sum_{\alpha \in \Delta^+} (\lambda, \alpha)v = \text{what we want}$$

*Remark 19.5.* Let

$$\Omega^\lambda = \{\mu \in P \mid (\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho)\}$$

geometrically, this is a sphere centered somewhere. This is a *finite set*. This is 3.

We have that  $M(\lambda)$  has a unique irreducible quotient, which we denote  $V(\lambda)$ . If  $\lambda \in P^+$ , then this is finite-dimensional, but in general, it is not. For example, sometimes the Verma module itself is irreducible (like in the  $\mathfrak{sl}(2)$  case).

Point number 4 is that there exists a finite filtration of the Verma module

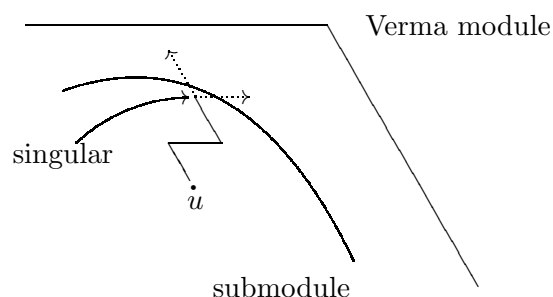
$$0 = F_0 M(\lambda) \subset F_1 M(\lambda) \subset \cdots \subset F_k M(\lambda) = M(\lambda)$$

such that the consecutive quotients are irreducible with  $F_i/F_{i-1} \simeq V(\mu)$  for some  $\mu \in \Omega^\lambda$ .

We say that a vector  $w$  is *singular of weight  $\mu$*  if  $hw = \mu(h)w$  and  $\mathfrak{n}^+ w = 0$ .

Suppose a module is generated by some highest vector, and all singular vectors are proportional to the highest vector, then the representation is

irreducible. Why is this so? The best way is to draw a picture:



Suppose you have some  $u$  in a proper submodule. Apply the positive roots to it ... you will eventually get to a place where you would kill it by applying positive roots, but that will be a singular vector.

Take a smallest singular vector  $w$  in  $M(\lambda)$ . Because it is smallest, there are no more singular vectors below it. Take  $F_1M(\lambda)$  to be the submodule generated by  $w$ . Then this submodule is irreducible and generated by the highest vector  $w$  (with some weight  $\mu$ ). Why does  $\mu$  lie in this sphere  $\Omega^\lambda$ ? Because the Casimir must act on  $\mu$  in the same way as it acted on  $\lambda$ . Now take the quotient and do the same. The filtration is finite because of point 3, that the sphere is finite. You can use each weight only as much as the multiplicity in the Verma module, and all the multiplicities are finite.

We get a nice conclusion:

$$ch M(\lambda) = \sum_{\substack{\mu \leq \lambda \\ \mu \in \Omega^\lambda}} b_\mu ch(V(\mu))$$

[[box this too]] where  $\lambda - \mu = \sum n_\alpha \alpha$ ,  $n_\alpha \in \mathbb{Z}_{\geq 0}$ . We have that  $b_\lambda = 1$ .

Now you can invert this kind of equation to get

$$ch V(\lambda) = \sum_{\substack{\mu \leq \lambda \\ \mu \in \Omega^\lambda}} c_\mu ch(M(\mu)) \quad (*)$$

[[more box maybe ... this is the fact we need]] with  $c_\lambda = 1$ . If you like, you get the  $c$ 's by inverting the matrix of  $b$ 's.

Now

$$ch V(\lambda) = \frac{\sum c_\mu e^{\mu+\rho}}{\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

call the denominator  $\Gamma$ , so

$$\Gamma ch(V(\lambda)) = \sum c_\mu e^{\mu+\rho}$$

---

You have an action of  $W$  on the LHS. You can check that  $w(\Gamma) = (-1)^w \Gamma$ , so the RHS must also be skew-symmetric with respect to the action of  $W$ . Because of this, you can combine the terms by orbits of the Weyl group

Try to finish this yourself. Calculate the  $c_\mu$ . □

Next time, we'll finish, and talk about the relation to compact groups.

## Lecture 20

This is Serganova's last lecture.

We have that

$$ch V(\lambda) = \sum_{\substack{\mu \leq \lambda \\ b_\lambda = 1}} c_\mu ch M(\mu) \quad (*)$$

which we get from

$$ch M(\lambda) = \sum_{\substack{\mu \leq \lambda \\ b_\lambda = 1}} ch V(\mu)$$

where  $\mu \in P$ , but not in  $P^+$ . The  $b_\mu$  are Kazhdan-Lusztig multiplicities, and are hard to compute. But we can compute the  $c_\mu$  when  $\lambda \in P^+$ .

There was once a conjecture that all the  $c_\mu$  were  $\pm 1$ , but this is false, and leads to some interesting stuff.

Since we know the characters of all the Verma modules, we can rewrite (\*) as

$$\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2}) ch V(\lambda) = \sum_{\mu \in \Omega^\lambda, \mu \leq \lambda} c_\mu e^{\mu+\lambda}$$

we know that  $w(LHS) = (-1)^w LHS$ , so the same must be true of the RHS:

$$w\left(\sum c_\mu e^{\mu+\rho}\right) = (-1)^w \sum c_\mu e^{\mu+\rho}$$

This is equivalent to saying that  $c_{w(\mu+\rho)-\rho} = (-1)^w c_\mu$ . We also know that  $c_\lambda = 1$ , so you get

$$\sum (-1)^w e^{w(\lambda+\rho)} + \sum_{\substack{\mu < \lambda \\ \mu+\rho \in P^+}} \left( \sum_{w \in W} (-1)^w e^{w(\mu+\rho)} \right)$$

$W$  acts in such a way that every orbit contains an element of  $P^+$ , so we can take  $\mu + \rho \in P^+$ . We want to show that  $c_\mu = 0$  for  $\mu < \lambda$ .  $\mu \in \Omega^\lambda$ . We have that  $(\mu + \rho, \mu + \rho) = (\lambda + \rho, \lambda + \rho)$  and

$$\lambda - \mu = \sum n_i \alpha_i$$

with  $n_i \in \mathbb{Z}_{\geq 0}$ . So we get that  $(\lambda - \mu, \lambda + \mu + 2\rho) = 0$ , so  $(\sum n_i \alpha_i, \lambda + \mu + 2\rho) = 0$ , so for  $\lambda + \mu + \rho \in P^+$ , we have that

$$(\lambda + \mu + \rho, \alpha_i) \geq 0$$

$(\rho, \alpha_i) > 0$ , recall that  $(\rho, \check{\alpha}_i) = 1$ .

$$\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2}) ch V(\lambda) = \sum_{w \in W} (-1)^w e^{w(\lambda+\rho)}$$

$\lambda = 0$ ,  $ch V(0) = e^0 = 1$ . gives the Weyl denominator identity:

$$\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w \in W} (-1)^w e^{w(\rho)} = \mathcal{D}$$

So we get the formula

$$ch V(\lambda) = \frac{1}{\mathcal{D}} \sum_{w \in W} (-1)^w e^{w(\lambda + \rho)}$$

Dimension formula: we can get

$$\dim V(\lambda) = \prod_{\alpha \in \Delta^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}$$

In the  $\mathfrak{sl}(n)$  case, you get  $\varepsilon_1 + \dots + \varepsilon_n = 0$ , and you get  $\rho = stuff$ , so you can compute dimension as

$$\prod_{\alpha \in \Delta^+} (\rho, \alpha) = (n-1)!(n-2)! \dots = (n-1)!!$$

If  $\lambda = \sum a_i \varepsilon_i$ , then we get

$$\prod_{i < j} (a_i + a_j + j - i)$$

We can prove this from the character formula? We have to take a limit as  $e \rightarrow 1$  (since we get 0/0 if we take  $e = 1$ ).

$$\lim_{u \rightarrow 0} \frac{\sum (-1)^w e^{(w(\lambda + \rho), u)}}{\sum (-1)^w e^{(w(\rho), u)}}$$

Let  $u = \rho t$  for  $t \rightarrow 0$ , then we get

$$\frac{\prod_{\alpha \in \Delta^+} e^{t(\lambda + \rho, \alpha/2)} - e^{-t(\lambda + \rho, \alpha/2)}}{\prod_{\alpha \in \Delta^+} e^{t(\rho, \alpha/2)} - e^{-t(\rho, \alpha/2)}}$$

using  $(w(\lambda + \rho), u) = (\lambda + \rho, w^{-1}(u))$ , and from here it is just an exercise in L'hospital.

You can keep playing with this formula to get some nice things. It is nice for calculating  $V(\lambda) \otimes V(\mu) = \sum b_{\lambda\mu}^{\nu} V(\nu)$ . As for finite groups, we do this by looking that  $ch V(\lambda) \cdot ch V(\mu)$  and writing it as  $\sum b_{\lambda\mu}^{\nu} ch V(\nu)$ . In the finite group case, we used orthogonality relations among the characters.

We have a natural inner product  $\langle e^{\lambda}, e^{\mu} \rangle = \delta_{\lambda + \mu, 0}$ , but this isn't good for some reason. The inner product we use is

$$(\phi, \psi) = \frac{1}{|W|} (D\phi, \bar{D}\psi)$$

where  $e^{\bar{\lambda}} = e^{-\lambda}$ . It is a calculation to check that characters of irreducible representations form an orthonormal basis with respect to this inner product.

## Compact Groups

So far we classified semi-simple Lie algebras over a characteristic 0 algebraically closed field. How is this connected to compact groups. Let  $G$  be a compact connected group (over  $\mathbb{R}$ ), like  $SU(2)$ , which is topologically  $S^3$ .

$$SU(n) = \{X \in GL(n, \mathbb{C}) \mid X^t X = \text{Id}, \det X = 1\}$$

► **Exercise 20.1.** If  $G$  is abelian, then it is a product of circles, so it is  $\mathbb{T}^n$ .

There exists the  $G$ -invariant volume form satisfying

$$(1) \int_G \omega = 1$$

$$(2) \int_G f(gh)\omega = \int_G f(hg)\omega = \int_G f(g)\omega$$

It is a nice exercise to construct this ... you take a volume near the identity and translate it around. So you pick  $\omega_e \in \Lambda^{\text{top}}(T_e G)^*$  and define  $\omega_g = L_{g^{-1}}^* \omega_e$ . Check for yourself that this form is right invariant.

From this you get a nice theorem:

**Theorem 20.1.** *If  $V$  is a finite dimensional representation of  $G$  (over  $\mathbb{R}$ ), then there exists a positive definite  $G$ -invariant inner product on  $V$ :  $(gv, gw) = (v, w)$ .*

*Proof.* Start with some positive definite inner product  $Q(v, w)$ , and define

$$(v, w) = \int_G Q(gv, gw)\omega$$

which is invariant. □

It follows that any finite dimensional representation of a compact group  $G$  is completely reducible (i.e. splits into a direct sum of irreducibles). By the way, representations of Lie groups are always taken to be smooth.

In particular, look at  $Ad : G \rightarrow GL(\mathfrak{g})$ , which gives  $ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ , which must split into irreducible ideals:  $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \mathfrak{a}$ , each of which is one-dimensional or simple (dump all the one-dimensional guys into  $\mathfrak{a}$ , which is then the center). Thus, the Lie algebra of a compact group is the direct sum of its center and a semi-simple Lie algebra. Such a Lie algebra is called *reductive*.

If  $G$  is simply connected, then I claim that  $\mathfrak{a}$  is trivial. This is because the simply connected group connected to  $\mathfrak{a}$  must be a torus, so a center gives you some fundamental group. Thus, if  $G$  is simply connected, then  $\mathfrak{g}$  is semi-simple.

Criterion: If the Killing form  $B$  on  $\mathfrak{g}$  is negative definite, then the corresponding group is compact.

If you have  $\mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ , and you know that  $\mathfrak{g}$  has an invariant positive definite product, so it lies in  $\mathfrak{so}(\mathfrak{g})$ . Here you have  $A^t = -A$ , so you have to check that  $\text{tr}(A^2) < 0$ . It is not hard to check that the eigenvalues of  $A$  are imaginary (as soon as  $A^t = -A$ ), so we have that the trace of the square is negative.

$\mathfrak{g}$  is a Lie algebra over  $\mathbb{R}$

How to classify compact Lie algebras? We know the classification over  $\mathbb{C}$ , so we can always take  $\mathfrak{g} \rightsquigarrow \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ , which remains semi-simple. However, this process might not be injective. For example, take  $\mathfrak{su}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & a \end{pmatrix} \mid a \in \mathbb{R}, b \in \mathbb{C} \right\}$  and  $\mathfrak{sl}(2, \mathbb{R})$ , then they complexify to the same thing.

$\mathfrak{g}$  in this case is called a real form of  $\mathfrak{g}_{\mathbb{C}}$ . So you can start with  $\mathfrak{g}_{\mathbb{C}}$  and classify all real forms.

**Theorem 20.2** (Cartan?). *Every semi-simple Lie algebra has exactly one (up to isomorphism) compact real form.*

For example, for  $\mathfrak{sl}(2)$  it is  $\mathfrak{su}(2)$ .

Classical Lie groups:  $SL(n, \mathbb{C}), SO(n, \mathbb{C})$  ( $SO$  has lots of real forms of this, because in the real case, you get a signature of a form; in the complex case, all forms are isomorphic),  $Sp(2n, \mathbb{C})$ . What are the corresponding compact simple Lie groups?

Compact real forms:  $SU(n)$  = the group of linear operators on  $\mathbb{C}^n$  preserving a positive definite Hermitian form.  $SL(n)$  = the group of linear operators on  $\mathbb{R}^n$  preserving a positive definite symmetric bilinear form.  $Sp(2n)$  = the group of linear operators on  $\mathbb{H}^n$  preserving a positive definite Hermitian form

We're not going to prove this theorem because we don't have time, but let's show existence.

*Proof of existence.* Suppose  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$ . Then you can construct  $\sigma : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{g}_{\mathbb{C}}$  "complex conjugation". Then  $\sigma$  preserves the commutator, but it is an anti-linear involution. Classifying real forms amounts to classifying all anti-linear involutions. There should be one that corresponds to the compact algebra. Take  $X_1, \dots, X_n, H_1, \dots, H_n, Y_1, \dots, Y_n$  generators for the algebra. Then we just need to define  $\sigma$  on the generators:  $\sigma(X_i) = -Y_i, \sigma(Y_i) = -X_i, \sigma(H_i) = -H_i$ , and extend anti-linearly. This particular  $\sigma$  is called the *Cartan involution*.

Now we claim that  $\mathfrak{g} = (\mathfrak{g}_{\mathbb{C}})^{\sigma} = \{X \mid \sigma(X) = X\}$  is a compact simple Lie algebra. We just have to check that the Killing form is negative definite. If you take  $h \in \mathfrak{h}$ , written as  $h = \sum a_i H_i$ , then  $\sigma(h) = h$  implies that all the  $a_i$  are purely imaginary. This implies that the eigenvalues of  $h$  are imaginary, which implies that  $B(h, h) < 0$ . You also have to check it on  $X_i, Y_i$ . The

fixed things will be of the form  $(aX_i - \bar{a}Y_i) \in \mathfrak{g}$ . The Weyl group action shows that  $B$  is negative on all of the root space.

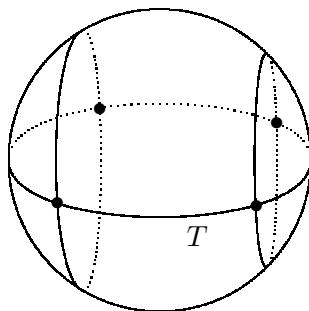
Look at  $\exp \mathfrak{h}^\sigma \subset G$  (simply connected), which is called the maximal torus  $T$ . I'm going to tell you several facts now. You can always think of  $T$  as  $\mathbb{R}^n/L$ . The point is that  $\mathbb{R}^n$  can be identified with  $\mathfrak{h}_{re}$ , and  $\mathfrak{h}_{re}^*$  has two natural lattices:  $Q$  (the root lattice) and  $P$  (the weight lattice). So one can identify  $T = \mathbb{R}^n/L = \mathfrak{h}_{re}/\check{P}$ , where  $\check{P}$  is the natural dual lattice to  $P$ , the set of  $h \in \mathfrak{h}$  such that  $\langle \omega, h \rangle \in \mathbb{Z}$  for all  $\omega \in P$ .  $G$  is simply connected, and when you quotient by the center, you get  $AdG$ , and all other groups with the same algebra are in between.  $AdT = \mathfrak{h}_{re}/\check{Q}$ . We have the sequence  $\{1\} \rightarrow Z(G) \rightarrow T \rightarrow AdT \rightarrow \{1\}$ . You can check that any element is semi-simple in a compact group, so the center of  $G$  is the quotient  $P/Q \simeq \check{Q}/\check{P}$ . Observe that  $|P/Q| =$  the determinant of the Cartan matrix. For example, if  $\mathfrak{g} = \mathfrak{sl}(3)$ , then we have  $\det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3$ , and the center of  $SU(3)$  is the set of all elements of the form  $diag(\omega, \omega, \omega)$  where  $\omega^3 = 1$ .

$G_2$  has only one real form because the det is 1? □

Orthogonality relations for compact groups:

$$\int_G \chi(g) \bar{\psi}(g^{-1}) \omega = \delta_{\chi, \psi}$$

where  $\chi$  and  $\psi$  are characters of irreducible representations. You know that the character is constant on conjugacy classes, so you can integrate over the conjugacy classes. There is a nice picture for  $SU(2)$ .



The integral can be written as

$$\frac{1}{|W|} \int_T \chi(t) \bar{\psi}(t) Vol(C(t)) dt$$

And  $Vol(C(t)) = \mathcal{D}(t) \bar{\mathcal{D}}(t)$ . You divide by  $|W|$  because that is how many times each class hits  $T$ .

You know that  $ch V(\lambda) = e^\lambda + \text{lower term}$



## Lecture 21

The new lecturer is Borchers.

This is the last third of the course: Examples and calculations. The (unofficial) goal is to prove no theorems. We'll try to do

1. Clifford algebras and spin groups.
2. Construction of all Lie groups and all representations. You might say this is impossible, so let's just try to do all simple ones, and in particular  $E_8, E_7, E_6$ .
3. Representations of  $SL_2(\mathbb{R})$ .

Today we'll give an overview of Lie algebras and Lie groups. We'll compare with algebraic groups (matrix groups), and finite groups.

### What does a general Lie group look like?

**Example 21.1.** Let  $G$  be the group of all shape-preserving (translations, rotations, reflections, dilations) transformations of  $\mathbb{R}^4$ , sometimes called  $\mathbb{R}^4 \cdot GO_4(\mathbb{R})$ .<sup>1</sup> This is pretty much the smallest possible Lie group with all the properties of a general Lie group.

- (1)  $G$  has a connected component  $G_{\text{conn}}$ , which is a normal subgroup, and  $G/G_{\text{conn}}$  is a discrete group.

Any Lie group can be built from a connected Lie group and a discrete group (0-dimensional group). Classifying discrete groups is completely hopeless.

- (2) We let  $G_{\text{sol}}$  be the maximal normal connected solvable subgroup of  $G_{\text{conn}}$ . Recall that solvable means that there is a chain  $G_{\text{sol}} \supseteq G'_{\text{sol}} \supseteq \cdots \supseteq 1$  where all consecutive quotients are abelian. So you build solvable groups out of abelian groups.

$G_{\text{conn}}/G_{\text{sol}}$  is connected and has no solvable normal subgroups. It turns out that it is much easier to deal with the group if you kill off all the solvable subgroups like this. Groups like this are semi-simple. They are almost a product of simple groups. A semi-simple group  $G$  modulo its discrete center is a product of simple groups.

The nice thing is that connected simple Lie groups are all completely classified. The list is quite long, and it has a lot of infinite series and a lot of exceptional guys. Some examples are  $PSU(n)$ ,  $PSL_n(\mathbb{R})$ ,  $PSL_n(\mathbb{C})$ . You can go from simple Lie groups to simple Lie algebras

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<sup>1</sup> $\mathbb{R}^4$  is translations,  $G$  is multiplication by scalars,  $O$  is reflections and rotations

over  $\mathbb{R}$ , then complexify to get Lie algebras over  $\mathbb{C}$ . This last step is finite-to-one, and they can be classified fairly easily. For example, the Lie algebras of  $\mathfrak{sl}_2(\mathbb{R})$  and  $\mathfrak{su}_2(\mathbb{R})$  are different, but they have the same complexification,  $\mathfrak{sl}_2(\mathbb{C})$ , which, by the way, is actually simple as a *real* Lie algebra. Its complexification is not simple, it is  $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ .

Suppose  $G_{\text{sol}}$  is solvable. The only thing you can say is that its derived subgroup  $G'_{\text{sol}}$  is nilpotent. That is,  $G' \supseteq [G', G'] \supseteq [G', [G', G']] \supseteq \dots \supseteq 1$ , or rather there is a chain such that  $G_i/G_{i+1} \subseteq$  the center of  $G_1/G_{i+1}$ . Thus, nilpotent groups can be built up by taking central extensions. Nilpotent groups are in practice not classifiable. The simply connected ones are all isomorphic to subgroups of upper-triangular matrices

$$\left\{ \left( \begin{array}{ccc} 1 & & * \\ & \ddots & \\ 0 & & 1 & \\ & & & 1 \end{array} \right) \right\}$$

If you put arbitrary things on the diagonal, that is what the solvable groups look like.

We have the picture

$$\text{all connected} \left\{ \begin{array}{l} G \\ | \\ G_{\text{conn}} \\ | \\ G_* \\ | \\ G_{\text{sol}} \\ | \\ G_{\text{nil}} \\ | \\ 1 \end{array} \right. \left. \begin{array}{l} \text{) discrete; classification hopeless} \\ \text{) } \prod \text{ simples; classified} \\ \text{) abelian discrete} \\ \text{) abelian} \\ \text{) nilpotent; classification a mess} \end{array} \right\} \text{classification trivial}$$

In our case,  $G$  is  $\mathbb{R}^4 \cdot GO_4(\mathbb{R})$ . The connected component  $G_{\text{conn}}$  is  $\mathbb{R}^4 \cdot GO_4^+(\mathbb{R})$ , which is the stuff that preserves orientation.  $G_{\text{sol}}$  is  $\mathbb{R}^4 \cdot \mathbb{R}^+$  (rescaling).  $G_{\text{nil}}$  is  $\mathbb{R}^4$ .  $G_{\text{conn}}/G_{\text{sol}}$  is  $SO_4(\mathbb{R})$ ,  $G_{\text{sol}}/G_{\text{nil}}$  is  $\mathbb{R}^+$ .  $G_{\text{conn}}/G_*$  is  $PSO_4(\mathbb{R}) \simeq SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$ .  $G_*/G_{\text{sol}}$  is  $\mathbb{Z}/2\mathbb{Z}$ .

Note: What is  $O_4(\mathbb{R})$ ? It almost is the product  $O_3(\mathbb{R}) \times O_3(\mathbb{R})$ . To see this is to look at the quaternions  $\mathbb{H}$ , which is an associative, but not commutative algebra. Notice that  $q\bar{q} = a^2 + b^2 + c^2 + d^2$ , which is  $> 0$  if  $q \neq 0$ , so any non-zero quaternion has an inverse. Thus,  $\mathbb{H}$  is a division algebra. Put  $\mathbb{R}^4$  to be the quaternions, and  $S^3$  is the unit sphere in this  $\mathbb{R}^4$

consisting of the quaternions such that  $\|q\| = q\bar{q} = 1$ . It is easy to check that  $\|pq\| = \|p\| \|q\|$ . So left or right multiplication by an element of  $S^3$  give rotations of  $\mathbb{R}^4$ . Thus, we have that  $S^3 \times S^3 \rightarrow O_4(\mathbb{R})$ , and the image is  $SO_4(\mathbb{R})$ . This is not quite injective because the kernel is  $\{(1, 1), (-1, -1)\}$ . So you get  $S^3 \times S^3 / (-1, -1) \simeq SO_4(\mathbb{R})$ .

Let  $\mathbb{R}^3$  be the purely imaginary quaternions. Then  $q \in S^3$  acts by conjugation on  $\mathbb{R}^3$ , so we get  $S^3 \rightarrow O_3(\mathbb{R})$ , and again it is not quite injective or surjective. The image is  $SO_3(\mathbb{R})$  and the kernel is  $\{1, -1\}$ , so  $S^3 / \{1, -1\} \simeq SO_3(\mathbb{R})$ . So we almost have that  $SO_4$  isomorphic to  $SO_3 \times SO_3$ . If you take a double cover of  $SO_4(\mathbb{R})$ , it is the product of two copies of a double cover of  $SO_3(\mathbb{R})$ . Orthogonal groups in dimension 4 have a strong tendency to split up like this.

## Comparison of Lie groups and Lie algebras

Let  $\mathfrak{g}$  be a Lie algebra. We can set  $\mathfrak{g}_{\text{sol}}$  to be the maximal solvable ideal (normal subalgebra), and  $\mathfrak{g}_{\text{nil}} = \mathfrak{g}'_{\text{sol}}$  and you get a chain

$$\begin{array}{c} \mathfrak{g} \\ | \\ \mathfrak{g}_{\text{sol}} \\ | \\ \mathfrak{g}_{\text{nil}} \\ | \\ 0 \end{array} \begin{array}{l} ) \prod \text{ simples; classification known} \\ ) \text{ abelian; easy to classify} \\ ) \text{ nilpotent; classification a mess} \end{array}$$

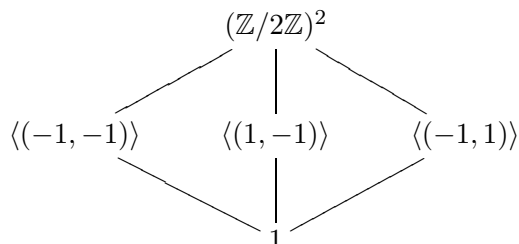
We have a map from Lie groups to Lie algebras. How much structure does this detect? This is an “isomorphism” (equivalence of categories) from simply connected Lie groups to Lie algebras. It cannot detect

- (1) non-trivial components of  $G$ .  $SO_4$  and  $O_4$  have the same Lie algebra
- (2) the (discrete) center of  $G$ . If  $Z$  is any discrete subgroup in the center of  $G$ , then  $G$  and  $G/Z$  have the same Lie algebra. For example  $SU(2)$  and  $PSU(2)$  have the same Lie algebra.

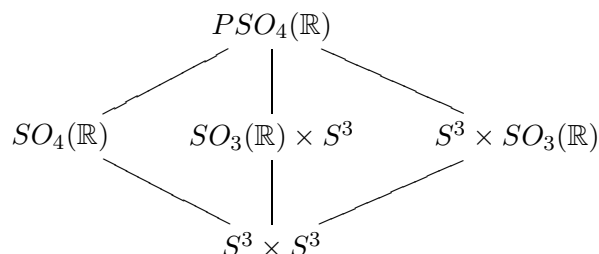
If  $G$  is connected, with universal cover  $\tilde{G}$ , which is simply connected, then  $G = \tilde{G} / \text{discrete subgroup of the center}$ . So if you know the universal cover, then you can read off all the other connected groups with the same Lie algebra.

Let's find all the groups with the algebra  $\mathfrak{o}_4(\mathbb{R})$ . First let's find a simply connected group with this Lie algebra. You might guess  $SO_4(\mathbb{R})$ , but that isn't simply connected. The simply connected one is  $S^3 \times S^3$  as we saw before (it is a product of two simply connected groups, so it is simply connected).

What is the center? The center of  $S^3$  is generated by  $-1$ , so the center of  $S^3 \times S^3$  is  $(\mathbb{Z}/2\mathbb{Z})^2$ , the Klein four group. There are three subgroups of order 2



There are 5 groups with this Lie algebra therefore:



These orthogonal groups tend to have double covers, which are important if you want to study fermions.

## Finite groups and Lie groups

Analogies:

- (1) The classification of finite groups resembles the classification of simply Lie groups when  $n \geq 2$ .

For example,  $PSL_n(\mathbb{R})$  is a simple Lie group, and  $PSL_n(\mathbb{F}_q)$  is a simple finite group except for  $n = 2, q \leq 3$ . Simple finite groups form about 18 series similar to Lie groups, and 26 or 27 exceptions, called sporadic groups, which don't seem to have any analogues for Lie groups.

- (2) Finite groups and Lie groups are both built up from simple and abelian groups. However, the way that finite groups are built is much more complicated than the way Lie groups are built. Finite groups can contain simple subgroups in very complicated ways; not just as direct factors. For example, there are wreath products. Let  $G$  be finite simple, and  $H$  be simple, acting on a set of  $n$  points. Then form  $G^n$ , acted on by  $H$  in the obvious way. Then you can form the semi-direct product  $G^n \cdot H$ . There is no analogue for (finite-dimensional) Lie groups (There is an analogue for infinite-dimensional Lie groups, which is why there isn't much theory there.).

- (3) Solvable finite groups need not have nilpotent group as a derived subgroup. Example:  $S^4$ , with derived group  $A^4$ , which is not nilpotent.

### Algebraic groups (over $\mathbb{R}$ ) and Lie groups

Any algebraic group gives you a Lie group. By algebraic group, we mean an algebraic variety which is also a group, such as  $GL_n(\mathbb{R})$ . Probably all the Lie groups you've come across have been algebraic groups. Since they are so similar, we'll list differences.

Differences between algebraic groups and Lie groups:

- (1) Unipotent and semi-simple abelian algebraic groups are totally different, but for Lie groups they are nearly the same.  $\mathbb{R} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  is unipotent and  $\mathbb{R}^\times \simeq \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$  is semi-simple. The homomorphism  $\exp : \mathbb{R} \rightarrow \mathbb{R}^\times$  is not rational, so you need something transcendental to see that they are almost the same as Lie groups.
- (2) Abelian varieties are different from affine algebraic groups.

**Example 21.2.** Take the elliptic curve  $y^2 = x^3 + x$ , and look at the circle group of matrices  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$  with  $a^2 + b^2 = 1$ . These are both  $S^1$  as Lie groups, but completely different as algebraic groups because of the different algebraic structures (complete or affine varieties).

- (3) Some Lie groups do not correspond to ANY algebraic groups.
- (a) The Heisenberg group: the group of symmetries of  $L^2(\mathbb{R})$  generated by translations ( $f(x) \mapsto f(x+y)$ ), multiplication by  $e^{ixy}$  ( $f(x) \mapsto e^{ixy}f(x)$ ), and multiplication by  $e^{iy}$  ( $f(x) \mapsto e^{iy}f(x)$ ). This can also be modeled as

$$\left\{ \left( \begin{array}{ccc} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array} \right) \right\} / \left\{ \left( \begin{array}{ccc} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \mid n \in \mathbb{Z} \right\}$$

It has the property that in any finite dimensional representation, the center acts trivially, so it is not isomorphic to any algebraic group.

- (b) The metaplectic group. Look at  $SL_2(\mathbb{R})$ , with algebra  $\mathfrak{sl}_2(\mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+d=0 \}$ . Find all groups with this Lie algebra. There are two obvious ones:  $SL_2(\mathbb{R})$  and  $PSL_2(\mathbb{R})$ . There aren't any other ones that can be represented as groups of finite-dimensional matrices. However, if you look at  $SL_2(\mathbb{R})$ , you'll find that it is not simply connected.

Iwasawa decomposition:  $SL_2(\mathbb{R}) \simeq K \times A \times N$  as a topological space, where  $K$  is the complex subgroup of things of the form  $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ , which is  $S^1$ ,  $A$  is  $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ , and  $N$  is  $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ , so it has fundamental group  $\mathbb{Z}$ .

So we take the universal cover  $\widetilde{SL_2(\mathbb{R})}$ , which has center  $\mathbb{Z}$ . Any finite-dimensional representation of  $\widetilde{SL_2(\mathbb{R})}$  factors through  $SL_2(\mathbb{R})$ . So  $\widetilde{SL_2(\mathbb{R})}$  cannot be written as a group of finite-dimensional matrices. Representing this group is a real pain.

Most important case: the metaplectic group  $Mp_2(\mathbb{R})$  is the connected double cover of  $SL_2(\mathbb{R})$ . It turns up in the theory of modular forms of half-integral weight and has a representation called the metaplectic representation.

next class we'll do Clifford algebras and spin groups.

## Lecture 22 - Clifford algebras and Spin groups

Today we'll mainly talk about spin groups and Clifford algebras. First let's look at the most "important" Lie groups.

Dimension 1: There are just  $\mathbb{R}$  and  $S^1 = \mathbb{R}/\mathbb{Z}$ .

Dimension 2: You get the abelian groups, which are all the quotients of  $\mathbb{R}^2$  by some discrete subgroup (of the center, which is all of  $\mathbb{R}^2$ ). You get three cases:  $\mathbb{R}^2$ ,  $\mathbb{R}^2/\mathbb{Z} = \mathbb{R} \times S^1$ , and  $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$ . There is one other group in 2 dimensions, which is the smallest non-abelian solvable Lie group: the set of all matrices of the form  $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ , where  $a > 0$ . The Lie algebra is the subalgebra of  $2 \times 2$  matrices of the form  $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ , which is generated by two elements as  $\langle X, Y \rangle$ , with  $[X, Y] = X$ .

Dimension 3: You get the group  $SL_2(\mathbb{R})$ , which is the most important Lie group of all! There are some variations, like the quotient  $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\text{center}$  ( $P$  always means that you've quotiented out by the center); the center of  $SL_2(\mathbb{R})$  is just  $\pm 1$ . As we saw before,  $SL_2(\mathbb{R})$  is not simply connected ... it has the double cover  $Mp_2(\mathbb{R})$ , which is also not simply connected.  $PSL_2(\mathbb{R})$  is familiar because it acts on the upper half plane by fractional linear transformations. There is a compact version of this:  $SU_2$  ( $S$  means determinant 1,  $U$  stands for unitary, which means it preserves the unitary form  $z_1\bar{z}_1 + z_2\bar{z}_2$  on  $\mathbb{C}^2$ ).  $SU_2 \simeq S^3$ , and it surjects onto  $SO_3(\mathbb{R})$ , with kernel  $\pm 1$  ( $O$  means it preserves an inner product). We also call  $SU_2$   $\text{Spin}_3(\mathbb{R})$ . The Lie algebras  $\mathfrak{sl}_2(\mathbb{R})$  and  $\mathfrak{su}_2(\mathbb{R})$  are different over the reals, but when you complexify, they become equal.

There is another interesting 3-dimensional algebra: the Heisenberg algebra: generated by  $X, Y, Z$ , with  $[X, Y] = Z$ , and  $Z$  central. You can think of this as strictly upper triangular matrices.

There are some less interesting algebras and groups. There are abelian groups, which is boring, and there are lots of boring solvable ones, such as  $\mathbb{R}^1$ +some 2-dimensional solvable or  $\mathbb{R}^2 \cdot \mathbb{R}^1$  with some action of  $\mathbb{R}^1$  on  $\mathbb{R}^2$ .

As you go up in dimension, the number of boring solvable algebras goes up a lot, and nobody can do anything about them, so you forget about them.

Dimension 6: (nothing interesting happens in dimensions 4,5) You get  $SL_2(\mathbb{C}) \cong \text{Spin}_{1,3}(\mathbb{R})$

Dimension 8:  $SU_3(\mathbb{R})$  and  $SL_3(\mathbb{R})$ , which is the first one with a non-trivial root system.

Dimension 14:  $G_2$ , which we might discuss.

Dimension 248:  $E_8$ , which we will discuss because it is my favorite.

This class is mostly about finite-dimensional algebras, but let's mention some infinite-dimensional Lie groups or Lie algebras.

1.  $\text{Aut}(\text{Hilbert space})$ .
2.  $\text{Diff}(\text{Manifold}) \dots$  physics stuff.

3. Gauge groups: (continuous, smooth, analytic, or whatever) maps from a manifold  $M$  to a group  $G$ .
4. Virasoro algebra:  $[L_i, L_j] = (i - j)L_{i+j} + \frac{i^3 - j^3}{12}c$ . If you forget about the  $c$ , you get vector fields on  $S^1$ , so this is some central extension of  $\text{Vect}(S^1)$ .
5. Affine Kac-Moody algebras, which are more or less central extensions of certain gauge groups over the circle.

### Clifford Algebras and Spin Groups

With Lie algebras of small dimensions, there are accidental isomorphisms. Almost all of these can be explained with Clifford algebras and Spin groups.

Motivational examples:

1.  $SO_2(\mathbb{R}) = S^1$ : you can double cover  $S^1$  by itself
2.  $SO_3(\mathbb{R})$ : has a simply connected double cover  $S^3$
3.  $SO_4(\mathbb{R})$ : has a simply connected double cover  $S^3 \times S^3$
4.  $SO_5(\mathbb{C})$ : Look at  $Sp_4(\mathbb{C})$ , which acts on  $\mathbb{C}^4$  and on  $\Lambda^2(\mathbb{C}^4)$ , which is 6-dimensional, and decomposes as  $5 \oplus 1$ .  $\Lambda^2(\mathbb{C}^4)$  has a symmetric bilinear form given by  $\Lambda^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{C}^4) \rightarrow \Lambda^4(\mathbb{C}^4) \simeq \mathbb{C}$ , and  $Sp_4(\mathbb{C})$  preserves this form. You get that  $Sp_4(\mathbb{C})$  acts on  $\mathbb{C}^5$ , preserving a symmetric bilinear form, so it maps to  $SO_5(\mathbb{C})$ . You can check that the kernel is  $\pm 1$ . So  $Sp_4(\mathbb{C})$  is a double cover of  $SO_5(\mathbb{C})$
5.  $SL_4(\mathbb{C})$ : acts on  $\mathbb{C}^4$ , and we still have our 6-dimensional  $\Lambda^2(\mathbb{C}^4)$ , with a symmetric bilinear form. From this you get

$$\pm 1 \rightarrow SL_4(\mathbb{C}) \rightarrow SO_6(\mathbb{C})$$

So we have double covers  $S^1$ ,  $S^3$ ,  $S^3 \times S^3$ ,  $Sp_4(\mathbb{C})$ ,  $SL_4(\mathbb{C})$  of the orthogonal groups in dimensions 2,3,4,5, and 6, respectively. What is the next one? All of these look completely unrelated.

Examples of Clifford algebras:

1.  $\mathbb{C}$ : generated by  $\mathbb{R}$ , together with  $i$ , with  $i^2 = -1$
2.  $\mathbb{H}$ : generated by  $\mathbb{R}$ , together with  $i, j$ , each squaring to  $-1$ , with  $ij + ji = 0$ .



3. Dirac wanted a square root for the operator  $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}$  (the wave operator in 4 dimensions). What he tried doing was supposing that the square root is of the form  $A = \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} + \gamma_4 \frac{\partial}{\partial t}$ . Then set  $A^2 = \nabla$  and compare coefficients, and you find that  $\gamma_1^2 = 1$ ,  $\gamma_2^2 = 1$ ,  $\gamma_3^2 = 1$ ,  $\gamma_4^2 = -1$ , and  $\gamma_i \gamma_j + \gamma_j \gamma_i = 0$  for  $i \neq j$ .


Dirac solved this by taking the  $\gamma_i$  to be  $4 \times 4$  complex matrices. The operator  $A$  acts on vector-valued functions on space-time.

**Definition 22.1.** A general *Clifford algebra* over  $\mathbb{R}$  should be generated by elements  $\gamma_1, \dots, \gamma_n$  such that  $\gamma_i^2$  is some given real, and  $\gamma_i \gamma_j + \gamma_j \gamma_i = 0$  for  $i \neq j$ .

**Definition 22.2** (better (invariant) definition). Suppose we have some vector space  $V$  over some field  $K$  (usually  $\mathbb{R}$ ), with some quadratic form  $N : V \rightarrow K$ . Then the *Clifford algebra* over  $K$  is generated by the vector space  $V$ , with relations  $v^2 = N(v)$ .

Note that  $N(\lambda v) = \lambda^2 N(v)$ , and the expression  $(a, b) := N(a + b) - N(a) - N(b)$  is bilinear, so if the characteristic of  $K$  is not 2, we have  $N(a) = \frac{(a, a)}{2}$ . Thus, you can work with the bilinear form instead of the quadratic form. We'll use quadratic forms so that everything works in characteristic 2.

**Example 22.3.** Take  $V = \mathbb{R}^2$  with basis  $i, j$ , with  $N(xi + yj) = -x^2 - y^2$ . Then the relations are  $(xi + yj)^2 = -x^2 - y^2$ , which gives you exactly the relations for the quaternions.

 **Warning 22.4.** A few authors (mainly in index theory) use the relations  $v^2 = -N(v)$ . Some people add a factor of 2, which usually doesn't matter, but is wrong in characteristic 2.

Let's work out a few more small examples:

$C_{m,n}(\mathbb{R})$  is the Clifford algebra over  $\mathbb{R}$  of the quadratic form  $x_1^2 + \dots + x_m^2 - x_{m+1}^2 - \dots - x_{m+n}^2$  ( $m$  pluses and  $n$  minuses).

Problem: Find the structure of all the  $C_{m,n}(\mathbb{R})$ .

We know that  $C_{0,0}(\mathbb{R})$  is  $\mathbb{R}$

$C_{1,0}(\mathbb{R})$  is  $\mathbb{R}[\varepsilon]/(\varepsilon^2 - 1) = \mathbb{R} \oplus \mathbb{R}$

$C_{0,1}(\mathbb{R})$  is  $\mathbb{R}[i]/(i^2 + 1) = \mathbb{C}$

$C_{2,0}(\mathbb{R})$  is  $\mathbb{R}[\alpha, \beta]/(\alpha^2 - 1, \beta^2 - 1, \alpha\beta + \beta\alpha)$ , which we can model by  $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

We have a homomorphism  $C_{2,0}(\mathbb{R}) \rightarrow M_2(\mathbb{R})$ , which is onto (because the two matrices generate  $M_2(\mathbb{R})$ ); the dimension of the later is 4, and the dimension of  $C_{2,0}(\mathbb{R})$  is at most 4 because it is spanned by  $1, \alpha, \beta, \alpha\beta$ . So this map is actually an isomorphism. Thus  $C_{2,0}(\mathbb{R}) \simeq M_2(\mathbb{R})$ . Note that the

same argument shows that  $C_V(K)$  has dimension at most  $2^{\dim V}$ . The tough part of Clifford algebras is to show that it cannot be smaller than that.

$C_{1,1}(\mathbb{R})$  is  $\mathbb{R}[\alpha, \beta]/(\alpha^2 - 1, \beta^2 + 1, \alpha\beta + \beta\alpha)$ , which we model by  $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

	0	1	2	3	4
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{M}_2(\mathbb{H})$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{M}_2(\mathbb{R})$	$\mathbb{M}_2(\mathbb{C})$		
2	$\mathbb{M}_2(\mathbb{R})$	$\mathbb{M}_2(\mathbb{R}) \oplus \mathbb{M}_2(\mathbb{R})$			
3	$\mathbb{M}_2(\mathbb{C})$				

What is  $C_{U \oplus V}$  in terms of  $C_U$  and  $C_V$ . You might guess  $C_{U \oplus V} \cong C_U \otimes C_V$ .

Answer 1: No for obvious definition of  $\otimes$ :  $C_{1,1} \neq C_{1,0} \otimes C_{0,1}$ .

Answer 2: Yes for the correct definition of  $\otimes$ : Clifford algebras are  $\mathbb{Z}/2\mathbb{Z}$ -graded.  $C_V(K) = C_V^0(K) \oplus C_V^1(K)$  where  $C_V^0(K)$  is spanned by the products of even numbers of elements of  $V$ . Superalgebra  $\otimes$  of  $\mathbb{Z}/2\mathbb{Z}$ -graded algebras is  $(a \otimes b)(c \otimes d) = (-1)^{\deg b \cdot \deg c} ac \otimes bd$ .

What is  $C_{U \oplus V}(K)$ ? Let's take  $\dim U$  to be even for now. Take  $\alpha_1, \dots, \alpha_m$  to be an orthogonal basis for  $U$  and let  $\beta_1, \dots, \beta_n$  to be an orthogonal basis for  $V$ . Then set  $\gamma_i = \alpha_1 \alpha_2 \cdots \alpha_m \beta_i$ . What are the relations between the  $\alpha_i$  and the  $\gamma_j$ ?

$$\begin{aligned} \alpha_i \gamma_j &= \alpha_i \alpha_1 \alpha_2 \cdots \alpha_m \beta_i \\ &= \alpha_1 \alpha_2 \cdots \alpha_m \beta_i \alpha_i \end{aligned}$$

(since  $\dim U$  even,  $\alpha_i$  anti-commutes with everything)

$$\begin{aligned} \gamma_i \gamma_j &= \gamma_i \alpha_1 \cdots \alpha_m \beta_j \\ &= \alpha_1 \cdots \alpha_m \gamma_i \beta_j \\ &= \alpha_1 \cdots \alpha_m \alpha_1 \cdots \alpha_m \beta_i \beta_j \\ &= -\gamma_j \gamma_i \end{aligned}$$

Finally, you can compute

$$\begin{aligned} \gamma_i^2 &= \alpha_1 \cdots \alpha_m \alpha_1 \cdots \alpha_m \beta_i \beta_i \\ &= \pm \alpha_1^2 \cdots \alpha_m^2 \beta_i^2 \end{aligned}$$

which is some real number. So the  $\gamma_i$ 's satisfy the relations of some Clifford algebra and commute with the  $\alpha_i$ . The reason for throwing in all those  $\alpha_i$ 's, you get some things that commute with the  $\alpha_i$ 's, but satisfy the relations of a different Clifford algebra.

We see that  $C_{U \oplus V}(K) \cong C_U(K) \otimes C_W(K)$ , where  $W$  is  $V$  with the quadratic form multiplied by  $\pm \alpha_1^2 \cdots \alpha_m^2$ . If you work it out carefully, you get that the  $\pm \alpha_1^2 \cdots \alpha_m^2$  is  $(-1)^{\dim U/2} \cdot \text{discriminant}(U)$ .

Take  $\dim U = 2$  and see what we get. We find that  $C_{m+2,n}(\mathbb{R}) \cong \mathbb{M}_2(\mathbb{R}) \otimes C_{n,m}(\mathbb{R})$ , where the indices have reversed because we have a factor of  $-1$ , so we must change the sign of the form, which means that we change the subscripts. Similarly, we find that  $C_{m+1,n+1}(\mathbb{R}) \cong \mathbb{M}_2(\mathbb{R}) \otimes C_{m,n}(\mathbb{R})$ , where the indices are in the right order. Finally, we have that  $C_{m,n+2}(\mathbb{R}) \cong \mathbb{H} \otimes C_{n,m}(\mathbb{R})$ . Using these formulas, we can reduce any Clifford algebra to tensor products of things like  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$ , and  $\mathbb{M}_2(\mathbb{R})$ .

Recall the rules for taking tensor products of matrix algebras:

1.  $\mathbb{R} \otimes X \cong X$
2.  $\mathbb{C} \otimes \mathbb{H} \cong \mathbb{M}_2(\mathbb{C})$  because all 2-dimensional quadratic forms look the same over  $\mathbb{C}$
3.  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$
4.  $\mathbb{H} \otimes_{\mathbb{C}} \mathbb{H} \cong \mathbb{M}_4(\mathbb{R})$ , which you can see by thinking of the  $\mathbb{H}$ 's by acting by left and right multiplication on  $\mathbb{R}^4$ .
5.  $\mathbb{M}_m(\mathbb{M}_n(X)) \cong \mathbb{M}_{mn}(X)$
6.  $\mathbb{M}_m(A) \otimes \mathbb{M}_n(B) \cong \mathbb{M}_{mn}(A \otimes B)$

Now we can work out every possible real Clifford algebra: see the table<sup>1</sup>.

You don't need to fill in the middle because you just take  $\mathbb{M}_n$  to  $\mathbb{M}_{2n}$ . Then you find that  $C_{8+m,n} \cong C_{m,n} \otimes \mathbb{M}_{16}(\mathbb{R})$ . You also get a mod 8 periodicity going down.

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<sup>1</sup>The table will look all wrong in the dvi; look at the pdf or ps instead.

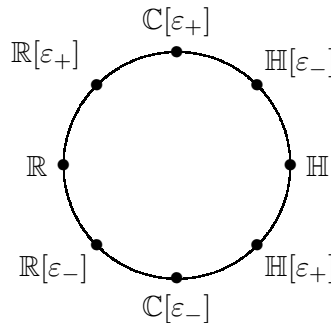
$\mathfrak{n}$	0	1	2	3	4	5	6	7	8
$\mathfrak{m}$									
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_{16}(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_{16}(\mathbb{R})$
1	$\mathbb{R} \oplus \mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$						
2	$M_2(\mathbb{R})$	$M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$	etc.						
3	$M_2(\mathbb{C})$								
4	$M_2(\mathbb{H})$								
5	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$								
6	$M_4(\mathbb{H})$								
7									
8									

## Lecture 23

Last time we defined the Clifford algebra  $C_V(K)$ , where  $V$  a vector space over  $K$  with a quadratic form  $N$ .  $C_V(K)$  is generated by  $V$  with  $x^2 = N(x)$ .  $C_{m,n}(\mathbb{R})$  uses the form  $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$ . We found that the structure depends heavily on  $m - n \pmod 8$ .

*Remark 23.1.* This mod 8 periodicity turns up in several other places:

- (1) Clifford algebras
- (2) Bott periodicity, which says that stable homotopy groups of orthogonal groups are periodic mod 8.
- (3) Real  $K$ -theory, which also has a mod 8 periodicity.
- (4) Even unimodular lattices (such as the  $E_8$  lattice) in  $\mathbb{R}^{m,n}$  exist if and only if  $m - n \equiv 0 \pmod 8$ .
- (5) The Super Brauer group of  $\mathbb{R}$  is  $\mathbb{Z}/8\mathbb{Z}$ . The Super Brauer group consists of super division algebras over  $\mathbb{R}$



where  $\varepsilon_{\pm}$  are odd and  $\varepsilon_{\pm}^2 = \pm 1$  and  $\varepsilon_{\pm}z = \bar{z}\varepsilon_{\pm}$  for  $z \in \mathbb{C}$ .

Recall that  $C_V(\mathbb{R}) = C_V^0(\mathbb{R}) \oplus C_V^1(\mathbb{R})$ , where  $C_V^1(\mathbb{R})$  is the odd part and  $C_V^0(\mathbb{R})$  is the even part. It turns out that we will need to know the structure of  $C_{m,n}^0(\mathbb{R})$ . Fortunately, this is easily reduced to the previous case. Look at  $C_{U \oplus V}^0(\mathbb{R})$  with  $\dim U = 1$ , with  $\gamma$  a basis for  $U$  and  $\gamma_1, \dots, \gamma_n$  form a basis for  $V$ . Then  $C_{U \oplus V}^0(\mathbb{R})$  is generated by  $\gamma\gamma_1, \dots, \gamma\gamma_n$ . What are the relations? we have that

$$\gamma\gamma_i \cdot \gamma\gamma_j = -\gamma\gamma_j\gamma\gamma_i$$

for  $i \neq j$ , and

$$(\gamma\gamma_i)^2 = (-\gamma^2)\gamma_i^2$$

So the generators  $\gamma\gamma_i$  of  $C_{U \oplus V}^0(\mathbb{R})$  also generate a Clifford algebra,  $C_{V(-disc)}$ , where we've multiplied the form on  $V$  by  $-\gamma^2$ . Over  $\mathbb{R}$ , we find that

$C_{m+1,n}^0(\mathbb{R}) \cong C_{n,m}(\mathbb{R})$  and  $C_{m,n+1}^1(\mathbb{R}) \cong C_{m,n}(\mathbb{R})$ . So we've worked out all the real algebras and all the even and odd parts of them.

*Remark 23.2.* For Complex Clifford algebras, it is similar, but easier. You find that  $C_{2m}(\mathbb{C}) \cong \mathbb{M}_{2m}(\mathbb{C})$  and  $C_{2m+1}(\mathbb{C}) \cong \mathbb{M}_{2m}(\mathbb{C}) \oplus \mathbb{M}_{2m}(\mathbb{C})$ . Notice that we have mod 2 periodicity now. Notice also that Bott periodicity for the unitary group is mod 2. You could figure these out by tensoring the real algebras with  $\mathbb{C}$  if you wanted.

## Clifford Groups

Usually the Clifford groups are denoted  $\Gamma_V(K)$ . We will show that  $\Gamma_V(K)$  is a subgroup of  $C_V(K)^\times$  and we'll find an exact sequence

$$1 \rightarrow K^\times \xrightarrow{\text{central}} \Gamma_V(K) \rightarrow O_V(K) \rightarrow 1$$

**Definition 23.3.**  $\Gamma_V(K) = \{ \text{homogeneous (either degree 1 or degree 0, not really needed) invertible elements } x \text{ of } C_V(K) \text{ such that } xV\alpha(x)^{-1} \subseteq V \}$  (recall that  $V \subseteq C_V(K)$ ), where  $\alpha$  is the automorphism of  $C_V(K)$  induced by  $-1$  on  $V$ .

Many books leave out the  $\alpha$ , which is a mistake (though not a serious one). Why do we use  $xV\alpha(x)^{-1}$  instead of  $xVx^{-1}$ , which is also possible? Reasons:

- (1) It is the correct "superalgebra" sign. The superalgebra convention says that whenever you exchange two elements of odd degree, you put in a  $-1$ , and you think of  $V$  and  $x$  as odd.
- (2) Putting  $\alpha$  in makes the theory much cleaner in ODD dimensions. For example, we will get a map  $C_V(K) \rightarrow O_V(K)$  which is onto if we use  $\alpha$ , but not if we do not. (You get  $SO_V(K)$  without the  $\alpha$ , which isn't too bad, but is still annoying.)

**Lemma 23.4.**<sup>1</sup> (*V non-degenerate*) *The elements of  $\Gamma_V(K)$  acting trivially on  $V$  are the elements of  $K^\times \subseteq \Gamma_V(K) \subseteq C_V(K)$ .*

*Proof.* Suppose  $a_0 + a_1 \in \Gamma_V^0(K) + \Gamma_V^1(K)$  acts trivially on  $V$ . Then  $(a_0 + a_1)v = v(a_0 - a_1)$  (the minus sign comes because we're using  $\alpha$ ), so  $a_0v = va_0$  and  $a_1v = -va_1$ . Choose an orthogonal basis  $\gamma_1, \dots, \gamma_n$  for  $V$ .<sup>2</sup> Write  $a_0$  as a linear combination of monomials in this basis, so that

$$a_0 = m + \gamma_1 n$$

<sup>1</sup>I promised no Lemmas or Theorems, but I was lying to you.

<sup>2</sup>All these results are true in characteristic 2, but you have to work harder ... you can't go around choosing orthogonal bases.

where  $m \in C_V^0(K)$  and  $n \in C_V^1(K)$  and neither  $m$  nor  $n$  contain a factor of  $\gamma_1$ . Apply the relations

$$a_0v = va_0 \quad \text{and} \quad a_1v = -va_1$$

with  $v = \gamma_1$  and use orthogonality to conclude that  $a_0$  contains no monomial with a factor  $\gamma_1$ . Repeat with  $v$  equal to the other basis elements. Similarly, write  $a_1$  as a linear combination of monomials in the basis, and apply the relations to  $v$  varying among the basis elements.

► **Exercise 23.1.** Check that you must have  $a_0 \in K$  and  $a_1 = 0$ . (If you don't use  $\alpha$ ,  $a_1$  could be  $\gamma_1 \cdots \gamma_n$  when  $n$  is odd.)

So  $a_0 + a_1 = a_0 \in K \cap \Gamma_V(K) = K^*$ . □

Now we define the anti-automorphism  $T$  on  $C_V(K)$  to be 1 on  $V$ , and extend to  $C_V(K)$  (anti means that  $(ab)^T = b^T a^T$ ). Do not confuse  $a \mapsto \alpha(a)$  (automorphism),  $a \mapsto a^T$  (anti-automorphism), and  $a \mapsto \alpha(a^T)$  (anti-automorphism).

Now we define the spinor norm  $N$  by  $N(a) = aa^T$ ,  $N^\alpha(a) = a\alpha(a)^T$  for  $a \in C_V(K)$ . Main properties: (1)  $N$  is a homomorphism  $\Gamma_V(K) \rightarrow C_V(K)$  with image in  $K^\times$  ( $N$  is a mess on the rest of  $C_V(K)$ ) and (2) the image of  $\Gamma_V(K)$  in  $\text{Aut } V$  lies in  $O_V(K)$ . Notice that on  $V$ ,  $N$  coincides with the quadratic form  $N$  (many authors seem not to have noticed this, and use different letters ... sometimes they use a stupid sign convention which makes them different).

*Proof.* We show that if  $a \in \Gamma_V(K)$ , then  $N^\alpha(a)$  acts trivially on  $V$ . Calculate

$$\begin{aligned} \alpha(N^\alpha(a))v(N^\alpha(a))^{-1} &= \alpha(a)a^T v \alpha(a^T)^{-1} a^{-1} \\ &= \alpha(a)\alpha(a^{-1})v a a^{-1} && \text{(apply } T; \text{ use } T|_V = 1) \\ &= v \end{aligned}$$

So by the previous lemma,  $N^\alpha(a) \in K^\times$ . □

This implies that  $N^\alpha$  is a homomorphism on  $\Gamma_V(K)$  because

$$\begin{aligned} N^\alpha(a)N^\alpha(b) &= a\alpha(a)^T N^\alpha(b) \\ &= aN^\alpha(b)\alpha(a)^T && (N^\alpha(b) \text{ is central}) \\ &= ab\alpha(b)^T \alpha(a)^T \\ &= (ab)\alpha(ab)^T = N^\alpha(ab) \end{aligned}$$

After all this work with  $N^\alpha$ , what we're really interested is  $N$ .  $N = N^\alpha$  on  $\Gamma_V^0(K)$  and  $N = -N^\alpha$  on  $\Gamma_V^1(K)$ . Since  $\Gamma_V(K) = \Gamma_V^0(K) \cup \Gamma_V^1(K)$ , which

shows that  $N$  is also a homomorphism. Observe that the Clifford action on  $V$  preserves  $N$  on  $V$  because  $N$  is a homomorphism, but  $N|_V$  is just the quadratic form on  $V$ . So we get a homomorphism from  $C_V(K) \rightarrow O_V(K)$ . What is its kernel, and is it surjective? The kernel is easy to figure out, because we did it a few minutes ago. The kernel is the elements of  $\Gamma_V(K)$  acting trivially on  $V$ , which is just the elements of  $K^\times$ . Is it onto? To see this, we should write some explicit elements. Say  $r \in V$  and  $N(r) \neq 0$ . Then  $r \in \Gamma_V(K)$  (we'll see this in a second). What is its action on  $V$ ?

$$\begin{aligned} rv\alpha(r)^{-1} &= -rv \frac{r}{N(r)} \\ &= -r && \text{if } v=r \\ &= v && \text{if } (v, r) = 0 \end{aligned}$$

So it acts by reflection through the hyperplane  $r^\perp$ . This is another good reason for putting the  $\alpha$  ... you would get something more annoying otherwise. You might think that we'll conclude that the map is surjective because  $O_V(K)$  is generated by reflections. That's wrong! Orthogonal groups need not be generated by reflections.

Here is a counterexample. Take the field  $\mathbb{F}_2$  and  $V = \mathbb{F}_2^4 = H \oplus H$ , where  $H$  is the quadratic form  $x^2 + y^2 + xy$  (each  $H$  has three elements of norm 1). Then the reflections generate the group  $S_3 \times S_3$  of order 36 ... there is no way to flip the two  $H$ 's by reflections.

► **Exercise 23.2.** check this

But the orthogonal group has order 72. It turns out that this is the ONLY counterexample. Any other field, or any other non-degenerate form on any other space works just fine.

In fact, the map  $C_V(K) \rightarrow O_V(K)$  is always onto even in the example above. For groups over  $\mathbb{R}$  (which is what we are interested in in this course), this follows easily from the fact that  $O_V(\mathbb{R})$  is indeed generated by reflections.

Summarizing: We have the exact sequence

$$1 \rightarrow K^\times \rightarrow \Gamma_V(K) \rightarrow O_V(K) \rightarrow 1$$

where  $K^\times$  lands in the center of  $\Gamma_V(K)$ .

$$\begin{array}{ccccccc} 1 & \longrightarrow & K^\times & \longrightarrow & \Gamma_V(K) & \longrightarrow & O_V(K) \longrightarrow 1 \\ & & \parallel & & \downarrow N & & \downarrow N \\ 1 & \longrightarrow & K^\times & \xrightarrow{x \mapsto x^2} & K^\times & \longrightarrow & K^\times / (K^\times)^2 \longrightarrow 1 \end{array}$$



**Definition 23.5.**  $\text{Pin}_V(K) = \{ \text{elements of } \Gamma_V(K) \text{ of spinor norm } 1 \}$ , and  $\text{Spin}_V(K) = \{ \text{even elements of } \text{Pin}_V(K) = \text{Pin}_V^0(K) \cup \text{Pin}_V^1(K) \}$ .

Note that the kernel of  $N$  on  $K^\times$  is the elements  $x$  with  $x^2 = 1$ , so it is elements  $= \pm 1$ . By taking elements of norm 1, we get sequences

$$1 \rightarrow \pm 1 \rightarrow \text{Pin}_V(K) \rightarrow O_V(K)$$

$$1 \rightarrow \pm 1 \rightarrow \text{Spin}_V(K) \rightarrow SO_V(K)$$

These maps are NOT always onto, but there are many important cases when they are, like when  $V$  has a positive definite quadratic form. The image is the set of elements of  $O_V(K)$  or  $SO_V(K)$  which have spinor norm 1 in  $K^\times / (K^\times)^2$ .

What is  $N : O_V(K) \rightarrow K^\times / (K^\times)^2$ ? It is the UNIQUE homomorphism such that  $N(a) = N(r)$  if  $a$  is reflection in  $r^\perp$ , and  $r$  is a vector of norm  $N(r)$ .

**Example 23.6.** Take  $V$  to be a positive definite vector space over  $\mathbb{R}$ . Then  $N$  maps to 1 in  $\mathbb{R}^\times / (\mathbb{R}^\times)^2 = \pm 1$  (because  $N$  is positive definite). So the spinor norm on  $O_V(\mathbb{R})$  is TRIVIAL.

So if  $V$  is positive definite, we get double covers

$$1 \rightarrow \pm 1 \rightarrow \text{Pin}_V(\mathbb{R}) \rightarrow O_V(\mathbb{R}) \rightarrow 1$$

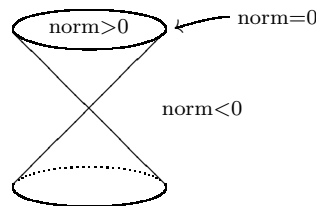
$$1 \rightarrow \pm 1 \rightarrow \text{Spin}_V(\mathbb{R}) \rightarrow SO_V(\mathbb{R}) \rightarrow 1$$

This will account for the weird double covers we saw before.

What if  $V$  is negative definite. Every reflection now has image  $-1$  in  $\mathbb{R}^\times / (\mathbb{R}^\times)^2$ , so the spinor norm  $N$  is the same as the determinant map  $O_V(\mathbb{R}) \rightarrow \pm 1$ .

So in order to find interesting examples of the spinor norm, you have to look at cases that are neither positive definite nor negative definite.

Let's look at Lorentz space:  $\mathbb{R}^{1,3}$ .



Reflection through a vector of norm  $< 0$  (spacelike vector,  $P$ : parity reversal) has spinor norm  $-1$ ,  $\det -1$  and reflection through a vector of norm  $> 0$  (timelike vector,  $T$ : time reversal) has spinor norm  $+1$ ,  $\det -1$ . So  $O_{1,3}(\mathbb{R})$  has 4 components (it is not hard to check that these are all the components), usually called 1,  $P$ ,  $T$ , and  $PT$ .

*Remark 23.7.* For those who know Galois cohomology. We get an exact sequence of algebraic groups

$$1 \rightarrow GL_1 \rightarrow \Gamma_V \rightarrow O_V \rightarrow 1$$

(algebraic group means you don't put a field). You do not necessarily get an exact sequence when you put in a field.

If

$$1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$$

is exact,

$$1 \rightarrow A(K) \rightarrow B(K) \rightarrow C(K)$$

is exact. What you really get is

$$1 \rightarrow H^0(\text{Gal}(\bar{K}/K), A) \rightarrow H^0(\text{Gal}(\bar{K}/K), B) \rightarrow H^0(\text{Gal}(\bar{K}/K), C) \rightarrow \\ \rightarrow H^1(\text{Gal}(\bar{K}/K), A) \rightarrow \dots$$

It turns out that  $H^1(\text{Gal}(\bar{K}/K), GL_1) = 1$ . However,  $H^1(\text{Gal}(\bar{K}/K), \pm 1) = K^\times / (K^\times)^2$ .

So from

$$1 \rightarrow GL_1 \rightarrow \Gamma_V \rightarrow O_V \rightarrow 1$$

you get

$$1 \rightarrow K^\times \rightarrow \Gamma_V(K) \rightarrow O_V(K) \rightarrow 1 = H^1(\text{Gal}(\bar{K}/K), GL_1)$$


However, taking

$$1 \rightarrow \mu_2 \rightarrow \text{Spin}_V \rightarrow SO_V \rightarrow 1$$

you get

$$1 \rightarrow \pm 1 \rightarrow \text{Spin}_V(K) \rightarrow SO_V(K) \xrightarrow{N} K^\times / (K^\times)^2 = H^1(\bar{K}/K, \mu_2)$$

so the non-surjectivity of  $N$  is some kind of higher Galois cohomology.

 *Warning 23.8.*  $\text{Spin}_V \rightarrow SO_V$  is onto as a map of ALGEBRAIC GROUPS, but  $\text{Spin}_V(K) \rightarrow SO_V(K)$  need NOT be onto.

**Example 23.9.** Take  $O_3(\mathbb{R}) \cong SO_3(\mathbb{R}) \times \{\pm 1\}$  as 3 is odd (in general  $O_{2n+1}(\mathbb{R}) \cong SO_{2n+1}(\mathbb{R}) \times \{\pm 1\}$ ). So we have a sequence

$$1 \rightarrow \pm 1 \rightarrow \text{Spin}_3(\mathbb{R}) \rightarrow SO_3(\mathbb{R}) \rightarrow 1.$$

Notice that  $\text{Spin}_3(\mathbb{R}) \subseteq C_3^0(\mathbb{R}) \cong \mathbb{H}$ , so  $\text{Spin}_3(\mathbb{R}) \subseteq \mathbb{H}^\times$ , and in fact we saw that it is  $S^3$ .

## Lecture 24

Last time we constructed the sequences

$$\begin{aligned}
 1 &\rightarrow K^\times \rightarrow \Gamma_V(K) \rightarrow O_V(K) \rightarrow 1 \\
 1 &\rightarrow \pm 1 \rightarrow \text{Pin}_V(K) \rightarrow O_V(K) \xrightarrow{N} K^\times / (K^\times)^2 \\
 1 &\rightarrow \pm 1 \rightarrow \text{Spin}_V(K) \rightarrow SO_V(K) \xrightarrow{N} K^\times / (K^\times)^2
 \end{aligned}$$

### Spin representations of Spin and Pin groups

Notice that  $\text{Pin}_V(K) \subseteq C_V(K)^\times$ , so any module over  $C_V(K)$  gives a representation of  $\text{Pin}_V(K)$ . We already figured out that  $C_V(K)$  are direct sums of matrix algebras over  $\mathbb{R}, \mathbb{C}$ , and  $\mathbb{H}$ .

What are the representations (modules) of complex Clifford algebras? Recall that  $C_{2n}(\mathbb{C}) \cong \mathbb{M}_{2n}(\mathbb{C})$ , which has a representations of dimension  $2^n$ , which is called the spin representation of  $\text{Pin}_V(K)$  and  $C_{2n+1}(\mathbb{C}) \cong \mathbb{M}_{2n}(\mathbb{C}) \times \mathbb{M}_{2n}(\mathbb{C})$ , which has 2 representations, called the spin representations of  $\text{Pin}_{2n+1}(K)$ .

What happens if we restrict these to  $\text{Spin}_V(\mathbb{C}) \subseteq \text{Pin}_V(\mathbb{C})$ ? To do that, we have to recall that  $C_{2n}^0(\mathbb{C}) \cong \mathbb{M}_{2^{n-1}}(\mathbb{C}) \times \mathbb{M}_{2^{n-1}}(\mathbb{C})$  and  $C_{2n+1}^0(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C})$ . So in EVEN dimensions  $\text{Pin}_{2n}(\mathbb{C})$  has 1 spin representation of dimension  $2^n$  splitting into 2 HALF SPIN representations of dimension  $2^{n-1}$  and in ODD dimensions,  $\text{Pin}_{2n+1}(\mathbb{C})$  has 2 spin representations of dimension  $2^n$  which become the same on restriction to  $\text{Spin}_V(\mathbb{C})$ .

Now we'll give a second description of spin representations. We'll just do the even-dimensional case (odd is similar). Say  $\dim V = 2n$ , and say we're over  $\mathbb{C}$ . Choose an orthonormal basis  $\gamma_1, \dots, \gamma_{2n}$  for  $V$ , so that  $\gamma_i^2 = 1$  and  $\gamma_i \gamma_j = -\gamma_j \gamma_i$ . Now look at the group  $G$  generated by  $\gamma_1, \dots, \gamma_{2n}$ , which is finite, with order  $2^{1+2n}$  (you can write all its elements explicitly). You can see that representations of  $C_V(\mathbb{C})$  correspond to representations of  $G$ , with  $-1$  acting as  $-1$  (as opposed to acting as 1). So another way to look at representations of the Clifford algebra, you can look at representations of  $G$ .

Let's look at the structure of  $G$ :

- (1) The center is  $\pm 1$ . This uses the fact that we are in even dimensions, lest  $\gamma_1 \cdots \gamma_{2n}$  also be central.
- (2) The conjugacy classes: 2 of size 1 (1 and  $-1$ ),  $2^{2n} - 1$  of size 2 ( $\pm \gamma_{i_1} \cdots \gamma_{i_n}$ ), so we have a total of  $2^{2n} + 1$  conjugacy classes, so we should have that many representations.  $G/\text{center}$  is abelian, isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{2n}$ , which gives us  $2^{2n}$  representations of dimension 1,

so there is only one more left to find! We can figure out its dimension by recalling that the sum of the squares of the dimensions of irreducible representations gives us the order of  $G$ , which is  $2^{2n+1}$ . So  $2^{2n} \times 1^1 + 1 \times d^2 = 2^{2n+1}$ , where  $d$  is the dimension of the mystery representation. Thus,  $d = \pm 2^n$ , so  $d = 2^n$ . Thus,  $G$ , and therefore  $C_V(\mathbb{C})$ , has an irreducible representation of dimension  $2^n$  (as we found earlier in another way).

**Example 24.1.** Consider  $O_{2,1}(\mathbb{R})$ . As before,  $O_{2,1}(\mathbb{R}) \cong SO_{2,1}(\mathbb{R}) \times (\pm 1)$ , and  $SO_{2,1}(\mathbb{R})$  is not connected: it has two components, separated by the spinor norm  $N$ . We have maps

$$1 \rightarrow \pm 1 \rightarrow \text{Spin}_{2,1}(\mathbb{R}) \rightarrow SO_{2,1}(\mathbb{R}) \xrightarrow{N} \pm 1.$$

$\text{Spin}_{2,1}(\mathbb{R}) \subseteq C_{2,1}^*(\mathbb{R}) \cong M_2(\mathbb{R})$ , so  $\text{Spin}_{2,1}(\mathbb{R})$  has one 2-dimensional spin representation. So there is a map  $\text{Spin}_{2,1}(\mathbb{R}) \rightarrow SL_2(\mathbb{R})$ ; by counting dimensions and such, you can show it is an isomorphism. So  $\text{Spin}_{2,1}(\mathbb{R}) \cong SL_2(\mathbb{R})$ .

Now let's look at some 4-dimensional orthogonal groups

**Example 24.2.** Look at  $SO_4(\mathbb{R})$ , which is compact. It has a complex spin representation of dimension  $2^{4/2} = 4$ , which splits into two half spin representations of dimension 2. We have the sequence

$$1 \rightarrow \pm 1 \rightarrow \text{Spin}_4(\mathbb{R}) \rightarrow SO_4(\mathbb{R}) \rightarrow 1 \quad (N = 1)$$

$\text{Spin}_4(\mathbb{R})$  is also compact, so the image in any complex representation is contained in some unitary group. So we get two maps  $\text{Spin}_4(\mathbb{R}) \rightarrow SU(2) \times SU(2)$ , and both sides have dimension 6 and centers of order 4. Thus, we find that  $\text{Spin}_4(\mathbb{R}) \cong SU(2) \times SU(2) \cong S^3 \times S^3$ , which give you the two half spin representations.

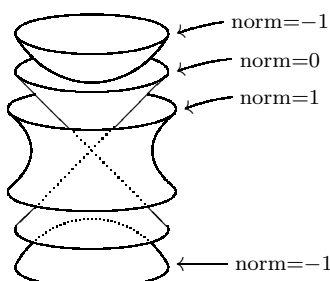
So now we've done the positive definite case.

**Example 24.3.** Look at  $SO_{3,1}(\mathbb{R})$ . Notice that  $O_{3,1}(\mathbb{R})$  has four components distinguished by the maps  $\det, N \rightarrow \pm 1$ . So we get

$$1 \rightarrow \pm 1 \rightarrow \text{Spin}_{3,1}(\mathbb{R}) \rightarrow SO_{3,1}(\mathbb{R}) \xrightarrow{N} \pm 1 \rightarrow 1$$

We expect 2 half spin representations, which give us two homomorphisms  $\text{Spin}_{3,1}(\mathbb{R}) \rightarrow SL_2(\mathbb{C})$ . This time, each of these homomorphisms is an isomorphism (I can't think of why right now). The  $SL_2(\mathbb{C})$ s are double covers of simple groups. Here, we don't get the splitting into a product as in the positive definite case. This isomorphism is heavily used in quantum field theory because  $\text{Spin}_{3,1}(\mathbb{R})$  is a double cover of the connected component of

the Lorentz group (and  $SL_2(\mathbb{C})$  is easy to work with). Note also that the center of  $\text{Spin}_{3,1}(\mathbb{R})$  has order 2, not 4, as for  $\text{Spin}_{4,0}(\mathbb{R})$ . Also note that the group  $PSL_2(\mathbb{C})$  acts on the compactified  $\mathbb{C} \cup \{\infty\}$  by  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau+b}{c\tau+d}$ . Subgroups of this group are called KLEINIAN groups. On the other hand, the group  $SO_{3,1}(\mathbb{R})^+$  (identity component) acts on  $\mathbb{H}^3$  (three-dimensional hyperbolic space). To see this, look at



One sheet of norm  $-1$  hyperboloid is isomorphic to  $\mathbb{H}^3$  under the induced metric. In fact, we'll define hyperbolic space that way. If you're a topologist, you're very interested in hyperbolic 3-manifolds, which are  $\mathbb{H}^3/(\text{discrete subgroup of } SO_{3,1}(\mathbb{R}))$ . If you use the fact that  $SO_{3,1}(\mathbb{R}) \cong PSL_2(\mathbb{R})$ , then you see that these discrete subgroups are in fact Klienian groups.

There are lots of exceptional isomorphisms in small dimension, all of which are very interesting, and almost all of them can be explained by spin groups.

**Example 24.4.**  $O_{2,2}(\mathbb{R})$  has 4 components (given by  $\det, N$ );  $C_{2,2}^0(\mathbb{R}) \cong \mathbb{M}_2(\mathbb{R}) \times \mathbb{M}_2(\mathbb{R})$ , which induces an isomorphism  $\text{Spin}_{2,2}(\mathbb{R}) \rightarrow SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ , which give you the two half spin representations. Both sides have dimension 6 with centers of order 4. So this time we get two non-compact groups. Let's look at the fundamental group of  $SL_2(\mathbb{R})$ , which is  $\mathbb{Z}$ , so the fundamental group of  $\text{Spin}_{2,2}(\mathbb{R})$  is  $\mathbb{Z} \oplus \mathbb{Z}$ . As we recall,  $\text{Spin}_{4,0}(\mathbb{R})$  and  $\text{Spin}_{3,1}(\mathbb{R})$  were both simply connected. This shows that SPIN GROUPS NEED NOT BE SIMPLY CONNECTED. So we can take covers of it. What do the corresponding covers (e.g. the universal cover) of  $\text{Spin}_{2,2}(\mathbb{R})$  look like? This is hard to describe because for FINITE dimensional complex representations, you get finite dimensional representations of the lie algebra  $L$ , which correspond to the finite-dimensional representations of  $L \otimes \mathbb{C}$ , which correspond to the finite-dimensional representations of  $L' = \text{Lie algebra of } \text{Spin}_{4,0}(\mathbb{R})$ , which correspond to the finite-dimensional representations of  $\text{Spin}_{4,0}(\mathbb{R})$ , which has no covers because it is simply connected. This means that any finite-dimensional representation of a cover of  $\text{Spin}_{2,2}(\mathbb{R})$  actually factors through  $\text{Spin}_{2,2}(\mathbb{R})$ . So there is no way you can talk about

these things with finite matrices, and infinite-dimensional representations are hard.

To summarize, the ALGEBRAIC GROUP  $\text{Spin}_{2,2}$  is simply connected (as an algebraic group) (think of an algebraic group as a functor from rings to groups), which means that it has no algebraic central extensions. However, the LIE GROUP  $\text{Spin}_{2,2}(\mathbb{R})$  is NOT simply connected; it has fundamental group  $\mathbb{Z} \oplus \mathbb{Z}$ . This problem does not happen for COMPACT Lie groups (where every finite cover is algebraic).

We've done  $O_{4,0}$ ,  $O_{3,1}$ , and  $O_{2,2}$ , from which we can obviously get  $O_{1,3}$  and  $O_{0,4}$ . Note that  $O_{4,0}(\mathbb{R}) \cong O_{0,4}(\mathbb{R})$ ,  $SO_{4,0}(\mathbb{R}) \cong SO_{0,4}(\mathbb{R})$ ,  $\text{Spin}_{4,0}(\mathbb{R}) \cong \text{Spin}_{0,4}(\mathbb{R})$ . However,  $\text{Pin}_{4,0}(\mathbb{R}) \not\cong \text{Pin}_{0,4}(\mathbb{R})$ . These two are hard to distinguish. We have

$$\begin{array}{ccc} \text{Pin}_{4,0}(\mathbb{R}) & & \text{Pin}_{0,4}(\mathbb{R}) \\ \downarrow & & \downarrow \\ O_{4,0}(\mathbb{R}) & = & O_{0,4}(\mathbb{R}) \end{array}$$

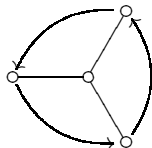
Take a reflection (of order 2) in  $O_{4,0}(\mathbb{R})$ , and lift it to the Pin groups. What is the order of the lift? The reflection vector  $v$ , with  $v^2 = \pm 1$  lifts to the element  $v \in \Gamma_V(\mathbb{R}) \subseteq C_V^*(\mathbb{R})$ . Notice that  $v^2 = 1$  in the case of  $\mathbb{R}^{4,0}$  and  $v^2 = -1$  in the case of  $\mathbb{R}^{0,4}$ , so in  $\text{Pin}_{4,0}(\mathbb{R})$ , the reflection lifts to something of order 2, but in  $\text{Pin}_{0,4}(\mathbb{R})$ , you get an element of order 4!. So these two groups are different.

Two groups are *isoclinic* if they are confusingly similar. A similar phenomenon is common for groups of the form  $2 \cdot G \cdot 2$ , which means it has a center of order 2, then some group  $G$ , and the abelianization has order 2. Watch out.

► **Exercise 24.1.**  $\text{Spin}_{3,3}(\mathbb{R}) \cong SL_4(\mathbb{R})$ .

## Triality

This is a special property of 8-dimensional orthogonal groups. Recall that  $O_8(\mathbb{C})$  has the Dynkin diagram  $D_4$ , which has a symmetry of order three:



But  $O_8(\mathbb{C})$  and  $SO_8(\mathbb{C})$  do NOT have corresponding symmetries of order three. The thing that does have the symmetry of order three is the spin group! The group  $\text{Spin}_8(\mathbb{R})$  DOES have “extra” order three symmetry. You can see it as follows. Look at the half spin representations of

$\text{Spin}_8(\mathbb{R})$ . Since this is a spin group in even dimension, there are two.  $C_{8,0}(\mathbb{R}) \cong \mathbb{M}_{2^{8/2-1}}(\mathbb{R}) \times \mathbb{M}_{2^{8/2-1}}(\mathbb{R}) \cong \mathbb{M}_8(\mathbb{R}) \times \mathbb{M}_8(\mathbb{R})$ . So  $\text{Spin}_8(\mathbb{R})$  has two 8-dimensional real half spin representations. But the spin group is compact, so it preserves some quadratic form, so you get 2 homomorphisms  $\text{Spin}_8(\mathbb{R}) \rightarrow SO_8(\mathbb{R})$ . So  $\text{Spin}_8(\mathbb{R})$  has THREE 8-dimensional representations: the half spins, and the one from the map to  $SO_8(\mathbb{R})$ . These maps  $\text{Spin}_8(\mathbb{R}) \rightarrow SO_8(\mathbb{R})$  lift to Triality automorphisms  $\text{Spin}_8(\mathbb{R}) \rightarrow \text{Spin}_8(\mathbb{R})$ . The center of  $\text{Spin}_8(\mathbb{R})$  is  $(\mathbb{Z}/2) + (\mathbb{Z}/2)$  because the center of the Clifford group is  $\pm 1, \pm \gamma_1 \cdots \gamma_8$ . There are 3 non-trivial elements of the center, and quotienting by any of these gives you something isomorphic to  $SO_8(\mathbb{R})$ . This is special to 8 dimensions.

### More about Orthogonal groups

Is  $O_V(K)$  a simple group? NO, for the following reasons:

- (1) There is a determinant map  $O_V(K) \rightarrow \pm 1$ , which is usually onto, so it can't be simple.
- (2) There is a spinor norm map  $O_V(K) \rightarrow K^\times / (K^\times)^2$
- (3)  $-1 \in \text{center of } O_V(K)$ .
- (4)  $SO_V(K)$  tends to split if  $\dim V = 4$ , abelian if  $\dim V = 2$ , and trivial if  $\dim V = 1$ .

It turns out that they are usually simple apart from these four reasons why they're not. Let's mod out by the determinant, to get to  $SO$ , then look at  $\text{Spin}_V(K)$ , then quotient by the center, and assume that  $\dim V \geq 5$ . Then this is usually simple. The center tends to have order 1, 2, or 4. If  $K$  is a FINITE field, then this gives many finite simple groups.

Note that  $SO_V(K)$  is NOT a subgroup of  $O_V(K)$ , elements of determinant 1 in general, it is the image of  $\Gamma_V^0(K) \subseteq \Gamma_V(K) \rightarrow O_V(K)$ , which is the correct definition. Let's look at why this is right and the definition you know is wrong. There is a homomorphism  $\Gamma_V(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$ , which takes  $\Gamma_V^0(K)$  to 0 and  $\Gamma_V^1(K)$  to 1 (called the DICKSON INVARIANT). It is easy to check that  $\det(v) = (-1)^{\text{dickson invariant}(v)}$ . So if the characteristic of  $K$  is not 2,  $\det = 1$  is equivalent to dickson = 0, but in characteristic 2, determinant is the wrong invariant (because determinant is always 1).

Special properties of  $O_{1,n}(\mathbb{R})$  and  $O_{2,n}(\mathbb{R})$ .  $O_{1,n}(\mathbb{R})$  acts on hyperbolic space  $\mathbb{H}^n$ , which is a component of norm  $-1$  vectors in  $\mathbb{R}^{n,1}$ .  $O_{2,n}(\mathbb{R})$  acts on the "Hermitian symmetric space" (Hermitian means it has a complex structure, and symmetric means really nice). There are three ways to construct this space:

- (1) It is the set of positive definite 2-dimensional subspaces of  $\mathbb{R}^{2,n}$
- (2) It is the norm 0 vectors  $\omega$  of  $\mathbb{P}\mathbb{C}^{2,n}$  with  $(\omega, \bar{\omega}) = 0$ .
- (3) It is the vectors  $x + iy \in \mathbb{R}^{1,n-1}$  with  $y \in C$ , where the cone  $C$  is the interior of the norm 0 cone.

► **Exercise 24.2.** Show that these are the same.

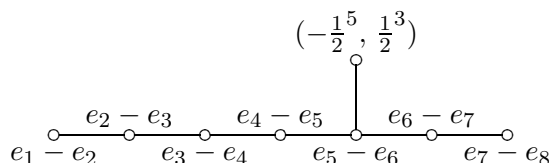
Next week, we'll mess around with  $E_8$ .



## Lecture 25 - $E_8$

In this lecture we use a vector notation in which powers represent repetitions: so  $(1^8) = (1, 1, 1, 1, 1, 1, 1, 1)$  and  $(\pm\frac{1}{2}^{12}, 0^6) = (\pm\frac{1}{2}, \pm\frac{1}{2}, 0, 0, 0, 0, 0, 0)$ .

Recall that  $E_8$  has the Dynkin diagram



where each vertex is a root  $r$  with  $(r, r) = 2$ ;  $(r, s) = 0$  when  $r$  and  $s$  are not joined, and  $(r, s) = -1$  when  $r$  and  $s$  are joined. We choose an orthonormal basis  $e_1, \dots, e_8$ , in which the roots are as given.

We want to figure out what the root lattice  $L$  of  $E_8$  is (this is the lattice generated by the roots). If you take  $\{e_i - e_{i+1}\} \cup (-1^5, 1^3)$  (all the  $A_7$  vectors plus twice the strange vector), they generate the  $D_8$  lattice  $= \{(x_1, \dots, x_8) | x_i \in \mathbb{Z}, \sum x_i \text{ even}\}$ . So the  $E_8$  lattice consists of two cosets of this lattice, where the other coset is  $\{(x_1, \dots, x_8) | x_i \in \mathbb{Z} + \frac{1}{2}, \sum x_i \text{ odd}\}$ .

Alternative version: If you reflect this lattice through the hyperplane  $e_1^\perp$ , then you get the same thing except that  $\sum x_i$  is always even. We will freely use both characterizations, depending on which is more convenient for the calculation at hand.

We should also work out the weight lattice, which is the vectors  $s$  such that  $(r, r)/2$  divides  $(r, s)$  for all roots  $r$ . Notice that the weight lattice of  $E_8$  is contained in the weight lattice of  $D_8$ , which is the union of four cosets of  $D_8$ :  $D_8$ ,  $D_8 + (1, 0^7)$ ,  $D_8 + (\frac{1}{2}^8)$  and  $D_8 + (-\frac{1}{2}, \frac{1}{2}^7)$ . Which of these have integral inner product with the vector  $(-\frac{1}{2}^5, \frac{1}{2}^3)$ ? They are the first and the last, so the weight lattice of  $E_8$  is  $D_8 \cup D_8 + (-\frac{1}{2}, \frac{1}{2}^7)$ , which is equal to the root lattice of  $E_8$ .

In other words, the  $E_8$  lattice  $L$  is UNIMODULAR (equal to its dual  $L'$ ), where the dual is the lattice of vectors having integral inner product with all lattice vectors. This is also true of  $G_2$  and  $F_4$ , but is not in general true of Lie algebra lattices.

The  $E_8$  lattice is EVEN, which means that the inner product of any vector with itself is always even.

Even unimodular lattices in  $\mathbb{R}^n$  only exist if  $8|n$  (this 8 is the same 8 that shows up in the periodicity of Clifford groups). The  $E_8$  lattice is the only example in dimension equal to 8 (up to isomorphism, of course). There are two in dimension 16 (one of which is  $L \oplus L$ , the other is  $D_{16} \cup$  some coset). There are 24 in dimension 24, which are the Niemeier lattices. In 32 dimensions, there are more than a billion!

The Weyl group of  $E_8$  is generated by the reflections through  $s^\perp$  where  $s \in L$  and  $(s, s) = 2$  (these are called roots). First, let's find all the roots:  $(x_1, \dots, x_8)$  such that  $\sum x_i^2 = 2$  with  $x_i \in \mathbb{Z}$  or  $\mathbb{Z} + \frac{1}{2}$  and  $\sum x_i$  even. If  $x_i \in \mathbb{Z}$ , obviously the only solutions are permutations of  $(\pm 1, \pm 1, 0^6)$ , of which there are  $\binom{8}{2} \times 2^2 = 112$  choices. In the  $\mathbb{Z} + \frac{1}{2}$  case, you can choose the first 7 places to be  $\pm \frac{1}{2}$ , and the last coordinate is forced, so there are  $2^7$  choices. Thus, you get 240 roots.

Let's find the orbits of the roots under the action of the Weyl group. We don't yet know what the Weyl group looks like, but we can find a large subgroup that is easy to work with. Let's use the Weyl group of  $D_8$ , which consists of the following: we can apply all permutations of the coordinates, or we can change the sign of an even number of coordinates (e.g., reflection in  $(1, -1, 0^6)$  swaps the first two coordinates, and reflection in  $(1, -1, 0^6)$  followed by reflection in  $(1, 1, 0^6)$  changes the sign of the first two coordinates.)

Notice that under the Weyl group of  $D_8$ , the roots form two orbits: the set which is all permutations of  $(\pm 1^2, 0^6)$ , and the set  $(\pm \frac{1}{2}^8)$ . Do these become the same orbit under the Weyl group of  $E_8$ ? Yes; to show this, we just need one element of the Weyl group of  $E_8$  taking some element of the first orbit to the second orbit. Take reflection in  $(\frac{1}{2}^8)^\perp$  and apply it to  $(1^2, 0^6)$ : you get  $(\frac{1}{2}^2, -\frac{1}{2}^6)$ , which is in the second orbit. So there is just one orbit of roots under the Weyl group.

What do orbits of  $W(E_8)$  on other vectors look like? We're interested in this because we might want to do representation theory. The character of a representation is a map from weights to integers, which is  $W(E_8)$ -invariant. Let's look at vectors of norm 4 for example. So  $\sum x_i^2 = 4$ ,  $\sum x_i$  even, and  $x_i \in \mathbb{Z}$  or  $x_i \in \mathbb{Z} + \frac{1}{2}$ . There are  $8 \times 2$  possibilities which are permutations of  $(\pm 2, 0^7)$ . There are  $\binom{8}{4} \times 2^4$  permutations of  $(\pm 1^4, 0^4)$ , and there are  $8 \times 2^7$  permutations of  $(\pm \frac{3}{2}, \pm \frac{1}{2}^7)$ . So there are a total of  $240 \times 9$  of these vectors. There are 3 orbits under  $W(D_8)$ , and as before, they are all one orbit under the action of  $W(E_8)$ . Just reflect  $(2, 0^7)$  and  $(1^3, -1, 0^4)$  through  $(\frac{1}{2}^8)$ .

► **Exercise 25.1.** Show that the number of norm 6 vectors is  $240 \times 28$ , and they form one orbit

(If you've seen a course on modular forms, you'll know that the number of vectors of norm  $2n$  is given by  $240 \times \sum_{d|n} d^3$ . If you let call these  $c_n$ , then  $\sum c_n q^n$  is a modular form of level 1 ( $E_8$  even, unimodular), weight 4 ( $\dim E_8/2$ ).

For norm 8 there are two orbits, because you have vectors that are twice a norm 2 vector, and vectors that aren't. As the norm gets bigger, you'll get a large number of orbits.

What is the order of the Weyl group of  $E_8$ ? We'll do this by 4 different methods, which illustrate the different techniques for this kind of thing:

- (1) This is a good one as a mnemonic. The order of  $E_8$  is given by

$$\begin{aligned} |W(E_8)| &= 8! \times \prod \left( \begin{array}{c} \text{numbers on the} \\ \text{affine } E_8 \text{ diagram}^1 \end{array} \right) \times \frac{\text{Weight lattice of } E_8}{\text{Root lattice of } E_8} \\ &= 8! \times \left( \begin{array}{cccccccc} & & & & 3 & & & \\ & & & & \circ & & & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{array} \right) \times 1 \\ &= 2^{14} \times 3^5 \times 5^2 \times 7 \end{aligned}$$

We can do the same thing for any other Lie algebra, for example,

$$\begin{aligned} |W(F_4)| &= 4! \times \left( \begin{array}{ccccc} 1 & 2 & 3 & 4 & 2 \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & & & & & \end{array} \right) \times 1 \\ &= 2^7 \times 3^2 \end{aligned}$$

- (2) The order of a reflection group is equal to the products of degrees of the fundamental invariants. For  $E_8$ , the fundamental invariants are of degrees 2,8,12,14,18,20,24,30 (primes +1).
- (3) This one is actually an honest method (without quoting weird facts). The only fact we will use is the following: suppose  $G$  acts transitively on a set  $X$  with  $H =$  the group fixing some point; then  $|G| = |H| \cdot |X|$ .

This is a general purpose method for working out the orders of groups. First, we need a set acted on by the Weyl group of  $E_8$ . Let's take the root vectors (vectors of norm 2). This set has 240 elements, and the Weyl group of  $E_8$  acts transitively on it. So  $|W(E_8)| = 240 \times |\text{subgroup fixing } (1, -1, 0^6)|$ . But what is the order of this subgroup (call it  $G_1$ )? Let's find a set acted on by this group. It acts on the set of norm 2 vectors, but the action is NOT transitive. What are the orbits?  $G_1$  fixes  $s = (1, -1, 0^6)$ . For other roots  $r$ ,  $G_1$  obviously fixes  $(r, s)$ . So how many roots are there with a given inner product with  $s$ ?

$(s, r)$	number	choices
2	1	$s$
1	56	$(1, 0, \pm 1^6), (0, -1, \pm 1^6), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}^6)$
0	126	
-1	56	
-2	1	$-s$

<sup>1</sup>These are the numbers giving highest root.

So there are at least 5 orbits under  $G_1$ . In fact, each of these sets is a single orbit under  $G_1$ . How can we see this? Find a large subgroup of  $G_1$ . Take  $W(D_6)$ , which is all permutations of the last 6 coordinates and all even sign changes of the last 6 coordinates. It is generated by reflections associated to the roots orthogonal to  $e_1$  and  $e_2$  (those that start with two 0s). The three cases with inner product 1 are three orbits under  $W(D_6)$ . To see that there is a single orbit under  $G_1$ , we just need some reflections that mess up these orbits. If you take a vector  $(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}^6)$  and reflect norm 2 vectors through it, you will get exactly 5 orbits. So  $G_1$  acts transitively on these orbits.

We'll use the orbit of vectors  $r$  with  $(r, s) = -1$ . Let  $G_2$  be the vectors fixing  $s$  and  $r$ :  $\overset{s}{\circ} \text{---} \overset{r}{\circ}$ . We have that  $|G_1| = |G_2| \cdot 56$ .

Keep going ... it gets tedious, but here are the answers up to the last step:

Our plan is to chose vectors acted on by  $G_i$ , fixed by  $G_{i+1}$  which give us the Dynkin diagram of  $E_8$ . So the next step is to try to find vectors  $t$  that give us the picture  $\overset{s}{\circ} \text{---} \overset{r}{\circ} \text{---} \dots \text{---} \overset{t}{\circ}$ , i.e, they have inner product  $-1$  with  $r$  and 0 with  $s$ . The possibilities for  $t$  are  $(-1, -1, 0, 0^5)$  (one of these),  $(0, 0, 1, \pm 1, 0^4)$  and permutations of its last five coordinates (10 of these), and  $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}^5)$  (there are 16 of these), so we get 27 total. Then we could check that they form one orbit, which is boring.

Next find vectors which go next to  $t$  in our picture:

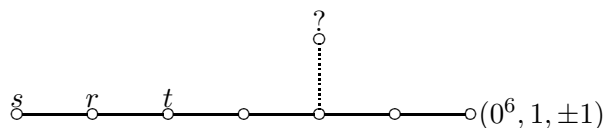
$\overset{s}{\circ} \text{---} \overset{r}{\circ} \text{---} \overset{t}{\circ} \text{---} \dots \text{---} \circ$ , i.e., whose inner product is  $-1$  with  $t$  and zero with  $r, s$ . The possibilities are permutations of the last four coords of  $(0, 0, 0, 1, \pm 1, 0^3)$  (8 of these) and  $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}^4)$  (8 of these), so there are 16 total. Again check transitivity.

Find a fifth vector; the possibilities are  $(0^4, 1, \pm 1, 0^2)$  and perms of the last three coords (6 of these), and  $(-\frac{1}{2}^4, \frac{1}{2}, \pm \frac{1}{2}^3)$  (4 of these) for a total of 10.

For the sixth vector, we can have  $(0^5, 1, \pm 1, 0)$  or  $(0^5, 1, 0, \pm 1)$  (4 possibilities) or  $(-\frac{1}{2}^5, \frac{1}{2}, \pm \frac{1}{2}^2)$  (2 possibilities), so we get 6 total.

NEXT CASE IS TRICKY: finding the seventh one, the possibilities are  $(0^6, 1, \pm 1)$  (2 of these) and  $((-\frac{1}{2})^6, \frac{1}{2}, \frac{1}{2})$  (just 1). The proof of transitivity fails at this point. The group we're using by now doesn't even act transitively on the pair (you can't get between them by changing an even number of signs). What elements of  $W(E_8)$  fix all of these first 6 points  $\overset{s}{\circ} \text{---} \overset{r}{\circ} \text{---} \overset{t}{\circ} \text{---} \circ \text{---} \circ \text{---} \circ$ ? We want to find roots perpendicular to all of these vectors, and the only possibility is  $(\frac{1}{2})^8$ . How does reflection in this root act on the three vectors above?

$(0^6, 1^2) \mapsto ((-\frac{1}{2})^6, \frac{1}{2}^2)$  and  $(0^6, 1, -1)$  maps to itself. Is this last vector in the same orbit? In fact they are in different orbits. To see this, look for vectors



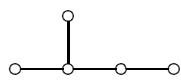
completing the  $E_8$  diagram. In the  $(0^6, 1, 1)$  case, you can take the vector  $((-\frac{1}{2})^5, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ . But in the other case, you can show that there are no possibilities. So these really are different orbits.

Use the orbit with 2 elements, and you get

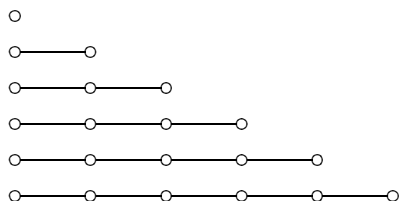
$$|W(E_8)| = 240 \times \underbrace{56 \times 27 \times 16 \times 10 \times 6 \times 2 \times 1}_{\text{order of } W(E_7)}^{\text{order of } W(E_6)}$$

because the group fixing all 8 vectors must be trivial. You also get that

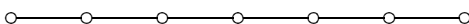
$$|W(\text{"}E_5\text{"})| = 16 \times 10 \times \underbrace{6 \times 2 \times 1}_{|W(A_4)|}^{|W(A_2 \times A_1)|}$$

where "E<sub>5</sub>" is the algebra with diagram  (that is,  $D_5$ ). Similarly,  $E_4$  is  $A_4$  and  $E_3$  is  $A_2 \times A_1$ .

We got some other information. We found that the Weyl group of  $E_8$  acts transitively on all the configurations



but not on



(4) We'll slip this in to next lecture

Also, next time we'll construct the Lie algebra  $E_8$ .

## Lecture 26

Today we'll finish looking at  $W(E_8)$ , then we'll construct  $E_8$ .

Remember that we still have a fourth method of finding the order of  $W(E_8)$ . Let  $L$  be the  $E_8$  lattice. Look at  $L/2L$ , which has 256 elements. Look at this as a set acted on by  $W(E_8)$ . There is an orbit of size 1 (represented by 0). There is an orbit of size  $240/2 = 120$ , which are the roots (a root is congruent mod  $2L$  to its negative). Left over are 135 elements. Let's look at norm 4 vectors. Each norm 4 vector,  $r$ , satisfies  $r \equiv -r \pmod{2}$ , and there are  $240 \cdot 9$  of them, which is a lot, so norm 4 vectors must be congruent to a bunch of stuff. Let's look at  $r = (2, 0, 0, 0, 0, 0, 0, 0)$ . Notice that it is congruent to vectors of the form  $(0 \cdots \pm 2 \cdots 0)$ , of which there are 16. It is easy to check that these are the only norm 4 vectors congruent to  $r \pmod{2}$ . So we can partition the norm 4 vectors into  $240 \cdot 9/16 = 135$  subsets of 16 elements. So  $L/2L$  has  $1+120+135$  elements, where 1 is the zero, 120 is represented by 2 elements of norm 2, and 135 is represented by 16 elements of norm 4. A set of 16 elements of norm 4 which are all congruent is called a FRAME. It consists of elements  $\pm e_1, \dots, \pm e_8$ , where  $e_i^2 = 4$  and  $(e_i, e_j) = 1$  for  $i \neq j$ , so up to sign it is an orthogonal basis.

Then we have

$$|W(E_8)| = (\# \text{ frames}) \times |\text{subgroup fixing a frame}|$$

because we know that  $W(E_8)$  acts transitively on frames. So we need to know what the automorphisms of an orthogonal base are. A frame is 8 subsets of the form  $(r, -r)$ , and isometries of a frame form the group  $(\mathbb{Z}/2\mathbb{Z})^8 \cdot S_8$ , but these are not all in the Weyl group. In the Weyl group, we found a  $(\mathbb{Z}/2\mathbb{Z})^7 \cdot S_8$ , where the first part is the group of sign changes of an EVEN number of coordinates. So the subgroup fixing a frame must be in between these two groups, and since these groups differ by a factor of 2, it must be one of them. Observe that changing an odd number of signs doesn't preserve the  $E_8$  lattice, so it must be the group  $(\mathbb{Z}/2\mathbb{Z})^7 \cdot S_8$ , which has order  $2^7 \cdot 8!$ . So the order of the Weyl group is

$$135 \cdot 2^7 \cdot 8! = |2^7 \cdot S_8| \times \frac{\# \text{ norm 4 elements}}{2 \times \dim L}$$

*Remark 26.1.* Similarly, if  $\Lambda$  is the Leech lattice, you actually get the order of Conway's group to be

$$|2^{12} \cdot M_{24}| \cdot \frac{\# \text{ norm 8 elements}}{2 \times \dim \Lambda}$$

where  $M_{24}$  is the Mathieu group (one of the sporadic simple groups). The Leech lattice seems very much to be trying to be the root lattice of the

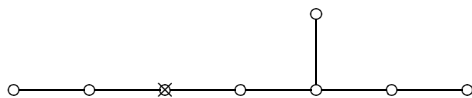
monster group, or something like that. There are a lot of analogies, but nobody can make sense of it.

$W(E_8)$  acts on  $(\mathbb{Z}/2\mathbb{Z})^8$ , which is a vector space over  $\mathbb{F}_2$ , with quadratic form  $N(a) = \frac{(a,a)}{2} \pmod 2$ , so you get a map

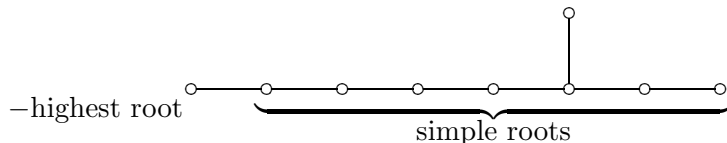
$$\pm 1 \rightarrow W(E_8) \rightarrow O_8^+(\mathbb{F}_2)$$

which has kernel  $\pm 1$  and is surjective.  $O_8^+$  is one of the 8-dimensional orthogonal groups over  $\mathbb{F}_2$ . So the Weyl group is very close to being an orthogonal group of a vector space over  $\mathbb{F}_2$ .

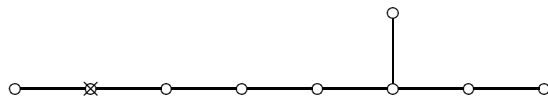
What is inside the root lattice/Lie algebra/Lie group  $E_8$ ? One obvious way to find things inside is to cover nodes of the  $E_8$  diagram:



If we remove the shown node, you see that  $E_8$  contains  $A_2 \times D_5$ . We can do better by showing that we can embed the affine  $\tilde{E}_8$  in the  $E_8$  lattice.

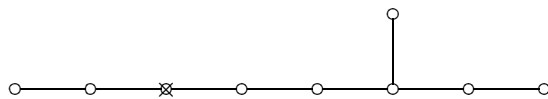


Now you can remove nodes here and get some bigger sub-diagrams. For example, if we cover



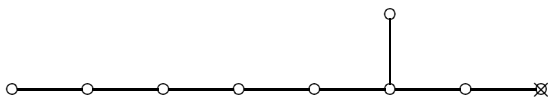
you get that an  $A_1 \times E_7$  in  $E_8$ . The  $E_7$  consisted of 126 roots orthogonal to a given root. This gives an easy construction of  $E_7$  root system, as all the elements of the  $E_8$  lattice perpendicular to  $(1, -1, 0 \dots)$

We can cover



Then we get an  $A_2 \times E_6$ , where the  $E_6$  are all the vectors with the first 3 coordinates equal. So we get the  $E_6$  lattice for free too.

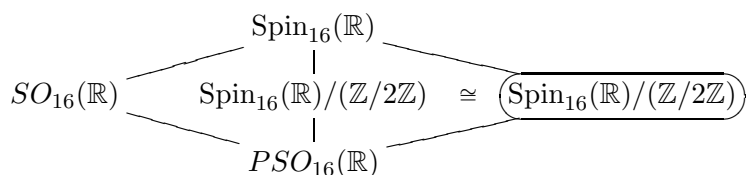
If you cover



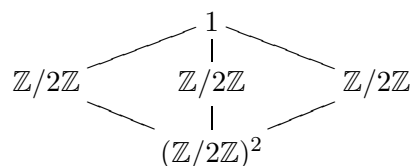
you see that there is a  $D_8$  in  $E_8$ , which is all vectors of the  $E_8$  lattice with integer coordinates. We sort of constructed the  $E_8$  lattice this way in the first place.

We can ask questions like: What is the  $E_8$  Lie algebra as a representation of  $D_8$ ? To answer this, we look at the weights of the  $E_8$  algebra, considered as a module over  $D_8$ , which are the 112 roots of the form  $(\dots \pm 1 \dots \pm 1 \dots)$  and the 128 roots of the form  $(\pm 1/2, \dots)$  and 1 vector 0, with multiplicity 8. These give you the Lie algebra of  $D_8$ . Recall that  $D_8$  is the Lie algebra of  $SO_{16}$ . The double cover has a half spin representation of dimension  $2^{16/2-1} = 128$ . So  $E_8$  decomposes as a representation of  $D_8$  as the adjoint representation (of dimension 120) plus a half spin representation of dimension 128. This is often used to construct the Lie algebra  $E_8$ . We'll do a better construction in a little while.

We've found that the Lie algebra of  $D_8$ , which is the Lie algebra of  $SO_{16}$ , is contained in the Lie algebra of  $E_8$ . Which *group* is contained in the the compact form of the  $E_8$ ? We found that there were groups



corresponding to subgroups of the center  $(\mathbb{Z}/2\mathbb{Z})^2$ :



We have a homomorphism  $\text{Spin}_{16}(\mathbb{R}) \rightarrow E_8(\text{compact})$ . What is the kernel? The kernel are elements which act trivially on the Lie algebra of  $E_8$ , which is equal to the Lie algebra  $D_8$  plus the half spin representation. On the Lie algebra of  $D_8$ , everything in the center is trivial, and on the half spin representation, one of the elements of order 2 is trivial. So the subgroup that you get is the circled one.

► **Exercise 26.1.** Show  $SU(2) \times E_7(\text{compact})/(-1, -1)$  is a subgroup of  $E_8(\text{compact})$ . Similarly, show that  $SU(9)/(\mathbb{Z}/3\mathbb{Z})$  is also. These are similar to the example above.

### Construction of $E_8$

Earlier in the course, we had some constructions:



1. using the Serre relations, but you don't really have an idea of what it looks like
2. Take  $D_8$  plus a half spin representation

Today, we'll try to find a natural map from root lattices to Lie algebras. The idea is as follows: Take a basis element  $e^\alpha$  (as a formal symbol) for each root  $\alpha$ ; then take the Lie algebra to be the direct sum of 1-dimensional spaces generated by each  $e^\alpha$  and  $L$  ( $L$  root lattice  $\cong$  Cartan subalgebra). Then we have to define the Lie bracket by setting  $[e^\alpha, e^\beta] = e^{\alpha+\beta}$ , but then we have a sign problem because  $[e^\alpha, e^\beta] \neq -[e^\beta, e^\alpha]$ . Is there some way to resolve the sign problem? The answer is that there is no good way to solve this problem (not true, but whatever). Suppose we had a nice functor from root lattices to Lie algebras. Then we would get that the automorphism group of the lattice has to be contained in the automorphism group of the Lie algebra (which is contained in the Lie group), and the automorphism group of the Lattice contains the Weyl group of the lattice. But the Weyl group is NOT usually a subgroup of the Lie group.

We can see this going wrong even in the case of  $\mathfrak{sl}_2(\mathbb{R})$ . Remember that the Weyl group is  $N(T)/T$  where  $T = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$  and  $N(T) = T \cup \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$ , and this second part is stuff having order 4, so you cannot possibly write this as a semi-direct product of  $T$  and the Weyl group.

So the Weyl group is not usually a subgroup of  $N(T)$ . The best we can do is to find a group of the form  $2^n \cdot W \subseteq N(T)$  where  $n$  is the rank. For example, let's do it for  $SL(n+1, \mathbb{R})$ . Then  $T = \text{diag}(a_1, \dots, a_n)$  with  $a_1 \cdots a_n = 1$ . Then we take the normalizer of the torus to be  $N(T) =$  all permutation matrices with  $\pm 1$ 's with determinant 1, so this is  $2^n \cdot S_n$ , and it does not split. The problem we had with signs can be traced back to the fact that this group doesn't split.

We can construct the Lie algebra from something acted on by  $2^n \cdot W$  (but not from something acted on by  $W$ ). We take a CENTRAL EXTENSION of the lattice by a group of order 2. Notation is a pain because the lattice is written additively and the extension is nonabelian, so you want it to be written multiplicatively. Write elements of the lattice in the form  $e^\alpha$  formally, so we have converted the lattice operation to multiplication. We will use the central extension

$$1 \rightarrow \pm 1 \rightarrow \hat{e}^L \rightarrow \underbrace{e^L}_{\cong L} \rightarrow 1$$

We want  $\hat{e}^L$  to have the property that  $\hat{e}^\alpha \hat{e}^\beta = (-1)^{(\alpha, \beta)} \hat{e}^\beta \hat{e}^\alpha$ , where  $\hat{e}^\alpha$  is something mapping to  $e^\alpha$ . What do the automorphisms of  $\hat{e}^L$  look like? We

get

$$1 \rightarrow \underbrace{(L/2L)}_{(\mathbb{Z}/2)^{\text{rank}(L)}} \rightarrow \text{Aut}(\hat{e}^L) \rightarrow \text{Aut}(e^L)$$

for  $\alpha \in L/2L$ , we get the map  $\hat{e}^\beta \rightarrow (-1)^{(\alpha, \beta)} \hat{e}^\beta$ . The map turns out to be onto, and the group  $\text{Aut}(e^L)$  contains the reflection group of the lattice. This extension is usually non-split.

Now the Lie algebra is  $L \oplus \{1\}$ -dimensional spaces spanned by  $(\hat{e}^\alpha, -\hat{e}^\alpha)$  for  $\alpha^2 = 2$  with the convention that  $-\hat{e}^\alpha$  ( $-1$  in the vector space) is  $-\hat{e}^\alpha$  ( $-1$  in the group  $\hat{e}^L$ ). Now define a Lie bracket by the “obvious rules”  $[\alpha, \beta] = 0$  for  $\alpha, \beta \in L$  (the Cartan is abelian),  $[\alpha, \hat{e}^\beta] = (\alpha, \beta)\hat{e}^\beta$  ( $\hat{e}^\beta$  is in the root space of  $\beta$ ), and  $[\hat{e}^\alpha, \hat{e}^\beta] = 0$  if  $(\alpha, \beta) \geq 0$  (since  $(\alpha + \beta)^2 > 2$ ),  $[\hat{e}^\alpha, \hat{e}^\beta] = \hat{e}^\alpha \hat{e}^\beta$  if  $(\alpha, \beta) < 0$  (product in the group  $\hat{e}^L$ ), and  $[\hat{e}^\alpha, (\hat{e}^\alpha)^{-1}] = \alpha$ .

**Theorem 26.2.** *Assume  $L$  is positive definite. Then this Lie bracket forms a Lie algebra (so it is skew and satisfies Jacobi).*

*Proof.* Easy but tiresome, because there are a lot of cases; let’s do them (or most of them).

We check the Jacobi identity: We want  $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$

1. all of  $a, b, c$  in  $L$ . Trivial because all brackets are zero.
2. two of  $a, b, c$  in  $L$ . Say  $\alpha, \beta, e^\gamma$

$$\underbrace{[[\alpha, \beta], e^\gamma]}_0 + \underbrace{[[\beta, e^\gamma], \alpha]}_{(\beta, \alpha)(-\alpha, \beta)e^\gamma} + [[e^\gamma, \alpha], \beta]$$

and similar for the third term, giving a sum of 0.

3. one of  $a, b, c$  in  $L$ .  $\alpha, e^\beta, e^\gamma$ .  $e^\beta$  has weight  $\beta$  and  $e^\gamma$  has weight  $\gamma$  and  $e^\beta e^\gamma$  has weight  $\beta + \gamma$ . So check the cases, and you get Jacobi:

$$\begin{aligned} [[\alpha, e^\beta], e^\gamma] &= (\alpha, \beta)[e^\beta, e^\gamma] \\ [[e^\beta, e^\gamma], \alpha] &= -[\alpha, [e^\beta, e^\gamma]] = -(\alpha, \beta + \gamma)[e^\beta, e^\gamma] \\ [[e^\gamma, \alpha], e^\beta] &= -[[\alpha, e^\gamma], e^\beta] = (\alpha, \gamma)[e^\beta, e^\gamma], \end{aligned}$$

so the sum is zero.

4. none of  $a, b, c$  in  $L$ . This is the really tiresome one,  $e^\alpha, e^\beta, e^\gamma$ . The main point of going through this is to show that it isn’t as tiresome as you might think. You can reduce it to two or three cases. Let’s make our cases depending on  $(\alpha, \beta)$ ,  $(\alpha, \gamma)$ ,  $(\beta, \gamma)$ .

- (a) if 2 of these are 0, then all the  $[[*, *], *]$  are zero.

- (b)  $\alpha = -\beta$ . By case a,  $\gamma$  cannot be orthogonal to them, so say  $(\alpha, \gamma) = 1$   $(\gamma, \beta) = -1$ ; adjust so that  $e^\alpha e^\beta = 1$ , then calculate

$$\begin{aligned} & [[e^\gamma, e^\beta], e^\alpha] - [[e^\alpha, e^\beta], e^\gamma] + [[e^\alpha, e^\gamma], e^\beta] \\ &= e^\alpha e^\beta e^\gamma - (\alpha, \gamma) e^\gamma + 0 \\ &= e^\gamma - e^\gamma = 0. \end{aligned}$$

- (c)  $\alpha = -\beta = \gamma$ , easy because  $[e^\alpha, e^\gamma] = 0$  and  $[[e^\alpha, e^\beta], e^\gamma] = -[[e^\gamma, e^\beta], e^\alpha]$
- (d) We have that each of the inner products is 1, 0 or  $-1$ . If some  $(\alpha, \beta) = 1$ , all brackets are 0.

This leaves two cases, which we'll do next time

□

## Lecture 27

Last week we talked about  $\hat{e}^L$ , which was a double cover of  $e^L$ .  $L$  is the root lattice of  $E_8$ . We had the sequence

$$1 \rightarrow \pm 1 \rightarrow \hat{e}^L \rightarrow e^L \rightarrow 1.$$

The Lie algebra structure on  $\hat{e}^L$  was given by

$$\begin{aligned} [\alpha, \beta] &= 0 \\ [\alpha, e^\beta] &= (\alpha, \beta)e^\beta \\ [e^\alpha, e^\beta] &= \begin{cases} 0 & \text{if } (\alpha, \beta) \geq 0 \\ e^\alpha e^\beta & \text{if } (\alpha, \beta) = -1 \\ \alpha & \text{if } (\alpha, \beta) = -2 \end{cases} \end{aligned}$$

The Lie algebra is  $L \oplus \bigoplus_{\alpha^2=2} \hat{e}^\alpha$ .

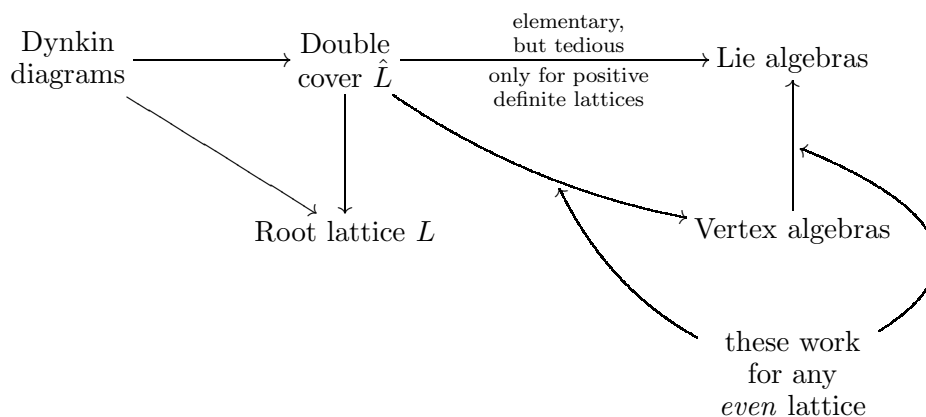
Let's finish checking the Jacobi identity. We had two cases left:

$$[[e^\alpha, e^\beta], e^\gamma] + [[e^\beta, e^\gamma], e^\alpha] + [[e^\gamma, e^\alpha], e^\beta] = 0$$

- $(\alpha, \beta) = (\beta, \gamma) = (\gamma, \alpha) = -1$ , in which case  $\alpha + \beta + \gamma = 0$ . then  $[[e^\alpha, e^\beta], e^\gamma] = [e^\alpha e^\beta, e^\gamma] = \alpha + \beta$ . By symmetry, the other two terms are  $\beta + \gamma$  and  $\gamma + \alpha$ ; the sum of all three terms is  $2(\alpha + \beta + \gamma) = 0$ .
- $(\alpha, \beta) = (\beta, \gamma) = -1$ ,  $(\alpha, \gamma) = 0$ , in which case  $[e^\alpha, e^\gamma] = 0$ . We check that  $[[e^\alpha, e^\beta], e^\alpha] = [e^\alpha e^\beta, e^\alpha] = e^\alpha e^\beta e^\alpha$  (since  $(\alpha + \beta, \alpha) = -1$ ). Similarly, we have  $[[e^\beta, e^\gamma], e^\alpha] = [e^\beta e^\gamma, e^\alpha] = e^\beta e^\gamma e^\alpha$ . We notice that  $e^\alpha e^\beta = -e^\beta e^\alpha$  and  $e^\gamma e^\alpha = e^\alpha e^\gamma$  so  $e^\alpha e^\beta e^\gamma = -e^\beta e^\gamma e^\alpha$ ; again, the sum of all three terms in the Jacobi identity is 0.

This concludes the verification of the Jacobi identity, so we have a Lie algebra.

Is there a proof avoiding case-by-case check? Good news: yes! Bad news: it's actually more work. We really have functors as follows:



where  $\hat{L}$  is generated by  $\hat{e}^{\alpha_i}$  (the  $i$ 's are the dots in your Dynkin diagram), with  $\hat{e}^{\alpha_i} \hat{e}^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j)} \hat{e}^{\alpha_j} \hat{e}^{\alpha_i}$ , and  $-1$  is central of order 2.

Unfortunately, you have to spend several weeks learning vertex algebras. In fact, the construction we did was the vertex algebra approach, with all the vertex algebras removed. So there is a more general construction which gives a much larger class of infinite-dimensional Lie algebras.

Now we should study the double cover  $\hat{L}$ , and in particular prove its existence. Given a Dynkin diagram, we can construct  $\hat{L}$  as generated by the elements  $e^{\alpha_i}$  for  $\alpha_i$  simple roots with the given relations. It is easy to check that we get a surjective homomorphism  $\hat{L} \rightarrow L$  with kernel generated by  $z$  with  $z^2 = 1$ . What's a little harder to show is that  $z \neq 1$  (i.e., show that  $\hat{L} \neq L$ ). The easiest way to do it is to use cohomology of groups, but since we have such an explicit case, we'll do it bare hands:

Problem: Given  $Z, H$  groups with  $Z$  abelian, construct central extensions

$$1 \rightarrow Z \rightarrow G \rightarrow H \rightarrow 1$$

(where  $Z$  lands in the center of  $G$ ). Let  $G$  be the set of pairs  $(z, h)$ , and set the product  $(z_1, h_1)(z_2, h_2) = (z_1 z_2 c(h_1, h_2), h_1 h_2)$ , where  $c(h_1, h_2) \in Z$  ( $c(h_1, h_2)$  will be a cocycle in group cohomology). We obviously get a homomorphism by mapping  $(z, h) \mapsto h$ . If  $c(1, h) = c(h, 1) = 1$  (normalization), then  $z \mapsto (z, 1)$  is a homomorphism mapping  $Z$  to the center of  $G$ . In particular,  $(1, 1)$  is the identity. We'll leave it as an exercise to figure out what the inverses are. When is this thing *associative*? Let's just write everything out:

$$\begin{aligned} ((z_1, h_1)(z_2, h_2))(z_3, h_3) &= (z_1 z_2 z_3 c(h_1, h_2) c(h_1 h_2, h_3), h_1 h_2 h_3) \\ (z_1, h_1)((z_2, h_2)(z_3, h_3)) &= (z_1 z_2 z_3 c(h_1, h_2 h_3) c(h_2, h_3), h_1 h_2 h_3) \end{aligned}$$

so we must have

$$c(h_1, h_2) c(h_1 h_2, h_3) = c(h_1 h_2, h_3) c(h_2, h_3).$$

This identity is actually very easy to satisfy in one particular case: when  $c$  is bimultiplicative:  $c(h_1, h_2 h_3) = c(h_1, h_2) c(h_1, h_3)$  and  $c(h_1 h_2, h_3) = c(h_1, h_3) c(h_2, h_3)$ . That is, we have a map  $H \times H \rightarrow Z$ . Not all cocycles come from such maps, but this is the case we care about.

To construct the double cover, let  $Z = \pm 1$  and  $H = L$  (free abelian). If we write  $H$  additively, we want  $c$  to be a bilinear map  $L \times L \rightarrow \pm 1$ . It is really easy to construct bilinear maps on free abelian groups. Just take any basis  $\alpha_1, \dots, \alpha_n$  of  $L$ , choose  $c(\alpha_i, \alpha_j)$  arbitrarily for each  $i, j$  and extend  $c$  via bilinearity to  $L \times L$ . In our case, we want to find a double cover  $\hat{L}$  satisfying  $\hat{e}^\alpha \hat{e}^\beta = (-1)^{(\alpha, \beta)} \hat{e}^\beta \hat{e}^\alpha$  where  $\hat{e}^\alpha$  is a lift of  $e^\alpha$ . This just

means that  $c(\alpha, \beta) = (-1)^{(\alpha, \beta)} c(\beta, \alpha)$ . To satisfy this, just choose  $c(\alpha_i, \alpha_j)$  on the basis  $\{\alpha_i\}$  so that  $c(\alpha_i, \alpha_j) = (-1)^{(\alpha_i, \alpha_j)} c(\alpha_j, \alpha_i)$ . This is trivial to do as  $(-1)^{(\alpha_i, \alpha_i)} = 1$ . Notice that this uses the fact that the lattice is even. There is no canonical way to choose this 2-cocycle (otherwise, the central extension would split as a product), but all the different double covers are isomorphic because we can specify  $\hat{L}$  by generators and relations. Thus, we have constructed  $\hat{L}$  (or rather, verified that the kernel of  $\hat{L} \rightarrow L$  has order 2, not 1).

Let's now look at lifts of automorphisms of  $L$  to  $\hat{L}$ .

► **Exercise 27.1.** Any automorphism of  $L$  preserving  $(, )$  lifts to an automorphism of  $\hat{L}$

There are two special cases:

1.  $-1$  is an automorphism of  $L$ , and we want to lift it to  $\hat{L}$  explicitly. First attempt: try sending  $\hat{e}^\alpha$  to  $\hat{e}^{-\alpha} := (\hat{e}^\alpha)^{-1}$ , which doesn't work because  $a \mapsto a^{-1}$  is not an automorphism on non-abelian groups.

Better:  $\omega : \hat{e}^\alpha \mapsto (-1)^{\alpha^2/2} (\hat{e}^\alpha)^{-1}$  is an automorphism of  $\hat{L}$ . To see this, check

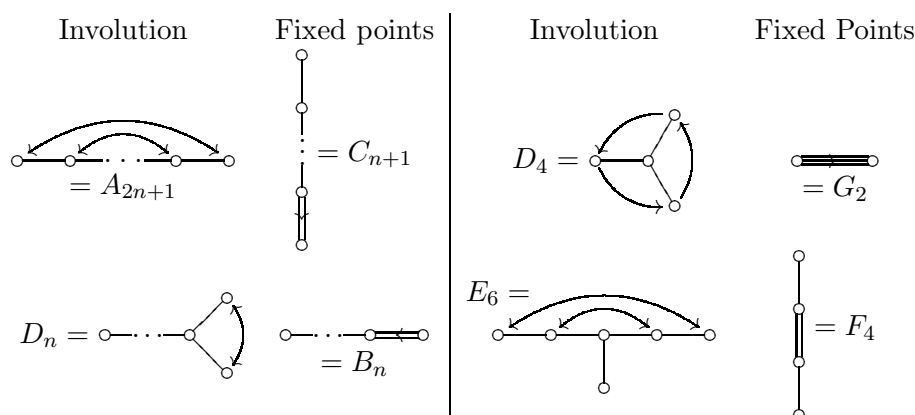
$$\begin{aligned}\omega(\hat{e}^\alpha)\omega(\hat{e}^\beta) &= (-1)^{(\alpha^2+\beta^2)/2} (\hat{e}^\alpha)^{-1} (\hat{e}^\beta)^{-1} \\ \omega(\hat{e}^\alpha \hat{e}^\beta) &= (-1)^{(\alpha+\beta)^2/2} (\hat{e}^\beta)^{-1} (\hat{e}^\alpha)^{-1}\end{aligned}$$

which work out just right

2. If  $r^2 = 2$ , then  $\alpha \mapsto \alpha - (\alpha, r)r$  is an automorphism of  $L$  (reflection through  $r^\perp$ ). You can lift this by  $\hat{e}^\alpha \mapsto \hat{e}^\alpha (\hat{e}^r)^{-(\alpha, r)} \times (-1)^{\binom{(\alpha, r)}{2}}$ . This is a homomorphism (check it!) of order (usually) 4!

*Remark 27.1.* Although automorphisms of  $L$  lift to automorphisms of  $\hat{L}$ , the lift might have larger order.

This construction works for the root lattices of  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$ , and  $E_8$ ; these are the lattices which are even, positive definite, and generated by vectors of norm 2 (in fact, all such lattices are sums of the given ones). What about  $B_n$ ,  $C_n$ ,  $F_4$  and  $G_2$ ? The reason the construction doesn't work for these cases is because there are roots of different lengths. These all occur as fixed points of diagram automorphisms of  $A_n$ ,  $D_n$  and  $E_6$ . In fact, we have a *functor* from Dynkin diagrams to Lie algebras, so an automorphism of the diagram gives an automorphism of the algebra




$A_{2n}$  doesn't really give you a new algebra: it corresponds to some super-algebra stuff.

### Construction of the Lie group of $E_8$

It is the group of automorphisms of the Lie algebra generated by the elements  $\exp(\lambda Ad(\hat{e}^\alpha))$ , where  $\lambda$  is some real number,  $\hat{e}^\alpha$  is one of the basis elements of the Lie algebra corresponding to the root  $\alpha$ , and  $Ad(\hat{e}^\alpha)(a) = [\hat{e}^\alpha, a]$ . In other words,

$$\exp(\lambda Ad(\hat{e}^\alpha))(a) = 1 + \lambda[\hat{e}^\alpha, a] + \frac{\lambda^2}{2}[\hat{e}^\alpha, [\hat{e}^\alpha, a]].$$

and all the higher terms are zero. To see that  $Ad(\hat{e}^\alpha)^3 = 0$ , note that if  $\beta$  is a root, then  $\beta + 3\alpha$  is not a root (or 0).

 **Warning 27.2.** In general, the group generated by these automorphisms is NOT the whole automorphism group of the Lie algebra. There might be extra diagram automorphisms, for example.

We get some other things from this construction. We can get simple groups over finite fields: note that the construction of a Lie algebra above works over any commutative ring (e.g. over  $\mathbb{Z}$ ). The only place we used division is in  $\exp(\lambda Ad(\hat{e}^\alpha))$  (where we divided by 2). The only time this term is non-zero is when we apply  $\exp(\lambda Ad(\hat{e}^\alpha))$  to  $\hat{e}^{-\alpha}$ , in which case we find that  $[\hat{e}^\alpha, [\hat{e}^\alpha, \hat{e}^{-\alpha}]] = [\hat{e}^\alpha, \alpha] = -(\alpha, \alpha)\hat{e}^\alpha$ , and the fact that  $(\alpha, \alpha) = 2$  cancels the division by 2. So we can in fact construct the  $E_8$  group over *any* commutative ring. You can mumble something about group schemes over  $\mathbb{Z}$  at this point. In particular, we have groups of type  $E_8$  over *finite fields*, which are actually finite simple groups (these are called Chevalley groups; it takes work to show that they are simple, there is a book by Carter called *Finite Simple Groups* which you can look at).

## Real forms

So we've constructed the Lie group and Lie algebra of type  $E_8$ . There are in fact several *different* groups of type  $E_8$ . There is one *complex* Lie algebra of type  $E_8$ , which corresponds to several different real Lie algebras of type  $E_8$ .

Let's look at some smaller groups:

**Example 27.3.**  $\mathfrak{sl}_2(\mathbb{R}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $a, b, c, d$  real  $a + d = 0$ ; this is not compact. On the other hand,  $\mathfrak{su}_2(\mathbb{R}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $d = -a$  imaginary  $b = -\bar{c}$ , is compact. These have the same Lie algebra over  $\mathbb{C}$ .

Let's look at what happens for  $E_8$ . In general, suppose  $L$  is a Lie algebra with complexification  $L \otimes \mathbb{C}$ . How can we find another Lie algebra  $M$  with the same complexification?  $L \otimes \mathbb{C}$  has an anti-linear involution  $\omega_L : l \otimes z \mapsto l \otimes \bar{z}$ . Similarly, it has an anti-linear involution  $\omega_M$ . Notice that  $\omega_L \omega_M$  is a linear involution of  $L \otimes \mathbb{C}$ . Conversely, if we know this involution, we can reconstruct  $M$  from it. Given an involution  $\omega$  of  $L \otimes \mathbb{C}$ , we can get  $M$  as the fixed points of the map  $a \mapsto \omega_L \omega(a) = \overline{\omega(a)}$ . Another way is to put  $L = L^+ \oplus L^-$ , which are the  $+1$  and  $-1$  eigenspaces, then  $M = L^+ \oplus iL^-$ .

Thus, to find other real forms, we have to study the involutions of the complexification of  $L$ . The exact relation is kind of subtle, but this is a good way to go.

**Example 27.4.** Let  $L = \mathfrak{sl}_2(\mathbb{R})$ . It has an involution  $\omega(m) = -m^T$ .  $\mathfrak{su}_2(\mathbb{R})$  is the set of fixed points of the involution  $\omega$  times complex conjugation on  $\mathfrak{sl}_2(\mathbb{C})$ , by definition.

So to construct real forms of  $E_8$ , we want some involutions of the Lie algebra  $E_8$  which we constructed. What involutions do we know about? There are two obvious ways to construct involutions:

1. Lift  $-1$  on  $L$  to  $\hat{e}^\alpha \mapsto (-1)^{\alpha^2/2} (\hat{e}^\alpha)^{-1}$ , which induces an involution on the Lie algebra.
2. Take  $\beta \in L/2L$ , and look at the involution  $\hat{e}^\alpha \mapsto (-1)^{(\alpha, \beta)} \hat{e}^\alpha$ .

(2) gives nothing new ... you get the Lie algebra you started with. (1) only gives you one real form. To get all real forms, you multiply these two kinds of involutions together.

Recall that  $L/2L$  has 3 orbits under the action of the Weyl group, of size 1, 120, and 135. These will correspond to the three real forms of  $E_8$ . How do we distinguish different real forms? The answer was found by Cartan: look at the signature of an invariant quadratic form on the Lie algebra!

A bilinear form  $(\ , \ )$  on a Lie algebra is called *invariant* if  $([a, b], c) + (b[a, c]) = 0$  for all  $a, b, c$ . This is called invariant because it corresponds to



the form being invariant under the corresponding group action. Now we can construct an invariant bilinear form on  $E_8$  as follows:

1.  $(\alpha, \beta)_{\text{in the Lie algebra}} = (\alpha, \beta)_{\text{in the lattice}}$
2.  $(\hat{e}^\alpha, (\hat{e}^\alpha)^{-1}) = 1$
3.  $(a, b) = 0$  if  $a$  and  $b$  are in root spaces  $\alpha$  and  $\beta$  with  $\alpha + \beta \neq 0$ .

This gives an invariant inner product on  $E_8$ , which you prove by case-by-case check

► **Exercise 27.2.** do these checks

Next time, we'll use this to produce bilinear forms on all the real forms and then we'll calculate the signatures.

## Lecture 28

Last time, we constructed a Lie algebra of type  $E_8$ , which was  $L \oplus \bigoplus \hat{e}^\alpha$ , where  $L$  is the root lattice and  $\alpha^2 = 2$ . This gives a double cover of the root lattice:

$$1 \rightarrow \pm 1 \rightarrow \hat{e}^L \rightarrow e^L \rightarrow 1.$$

We had a lift for  $\omega(\alpha) = -\alpha$ , given by  $\omega(\hat{e}^\alpha) = (-1)^{(\alpha^2/2)}(\hat{e}^\alpha)^{-1}$ . So  $\omega$  becomes an automorphism of order 2 on the Lie algebra.  $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha$  is also an automorphism of the Lie algebra.

Suppose  $\sigma$  is an automorphism of order 2 of the real Lie algebra  $L = L^+ + L^-$  (eigenspaces of  $\sigma$ ). We saw that you can construct another real form given by  $L^+ + iL^-$ . Thus, we have a map from conjugacy classes of automorphisms with  $\sigma^2 = 1$  to real forms of  $L$ . This is not in general an isomorphism.

Today we'll construct some more real forms of  $E_8$ .  $E_8$  has an invariant symmetric bilinear form  $(e^\alpha, (e^\alpha)^{-1}) = 1$ ,  $(\alpha, \beta) = (\beta, \alpha)$ . The form is unique up to multiplication by a constant since  $E_8$  is an irreducible representation of  $E_8$ . So the *absolute value of the signature* is an invariant of the Lie algebra.

For the split form of  $E_8$ , what is the signature of the invariant bilinear form (the split form is the one we just constructed)? On the Cartan subalgebra  $L$ ,  $(, )$  is positive definite, so we get +8 contribution to the signature. On  $\{e^\alpha, (e^\alpha)^{-1}\}$ , the form is  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , so it has signature  $0 \cdot 120$ . Thus, the signature is 8. So if we find any real form with a different signature, we'll have found a new Lie algebra.

Let's first try involutions  $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha$ . But this doesn't change the signature.  $L$  is still positive definite, and you still have  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$  on the other parts. These Lie algebras actually turn out to be isomorphic to what we started with (though we haven't shown that they are isomorphic).

Now try  $\omega : e^\alpha \mapsto (-1)^{\alpha^2/2}(e^\alpha)^{-1}$ ,  $\alpha \mapsto -\alpha$ . What is the signature of the form? Let's write down the + and - eigenspaces of  $\omega$ . The + eigenspace will be spanned by  $e^\alpha - e^{-\alpha}$ , and these vectors have norm -2 and are orthogonal. The - eigenspace will be spanned by  $e^\alpha + e^{-\alpha}$  and  $L$ , which have norm 2 and are orthogonal, and  $L$  is positive definite. What is the Lie algebra corresponding to the involution  $\omega$ ? It will be spanned by  $e^\alpha - e^{-\alpha}$  where  $\alpha^2 = 2$  (norm -2), and  $i(e^\alpha + e^{-\alpha})$  (norm -2), and  $iL$  (which is now negative definite). So the bilinear form is *negative definite*, with signature  $-248 (\neq \pm 8)$ .

With some more work, you can actually show that this is the Lie algebra of the *compact* form of  $E_8$ . This is because the automorphism group of  $E_8$  preserves the invariant bilinear form, so it is contained in  $O_{0,248}(\mathbb{R})$ , which is compact.

Now let's look at involutions of the form  $e^\alpha \mapsto (-1)^{(\alpha, \beta)} \omega(e^\alpha)$ . Notice that  $\omega$  commutes with  $e^\alpha \mapsto (-1)^{(\alpha, \beta)} e^\alpha$ . The  $\beta$ 's in  $(\alpha, \beta)$  correspond to  $L/2L$  modulo the action of the Weyl group  $W(E_8)$ . Remember this has three orbits, with 1 norm 0 vector, 120 norm 2 vectors, and 135 norm 4 vectors. The norm 0 vector gives us the compact form. Let's look at the other cases and see what we get.

Suppose  $V$  has a negative definite symmetric inner product  $(\cdot, \cdot)$ , and suppose  $\sigma$  is an involution of  $V = V_+ \oplus V_-$  (eigenspaces of  $\sigma$ ). What is the signature of the invariant inner product on  $V_+ \oplus iV_-$ ? On  $V_+$ , it is negative definite, and on  $iV_-$  it is positive definite. Thus, the signature is  $\dim V_- - \dim V_+ = -\text{tr}(\sigma)$ . So we want to work out the traces of these involutions.

Given some  $\beta \in L/2L$ , what is  $\text{tr}(e^\alpha \mapsto (-1)^{(\alpha, \beta)} e^\alpha)$ ? If  $\beta = 0$ , the traces is obviously 248 because we just have the identity map. If  $\beta^2 = 2$ , we need to figure how many roots have a given inner product with  $\beta$ . Recall that this was determined before:

$(\alpha, \beta)$	# of roots $\alpha$ with given inner product	eigenvalue
2	1	1
1	56	-1
0	126	1
-1	56	-1
-2	1	1

Thus, the trace is  $1 - 56 + 126 - 56 + 1 + 8 = 24$  (the 8 is from the Cartan subalgebra). So the signature of the corresponding form on the Lie algebra is  $-24$ . We've found a third Lie algebra.

If we also look at the case when  $\beta^2 = 4$ , what happens? How many  $\alpha$  with  $\alpha^2 = 2$  and with given  $(\alpha, \beta)$  are there? In this case, we have:

$(\alpha, \beta)$	# of roots $\alpha$ with given inner product	eigenvalue
2	14	1
1	64	-1
0	84	1
-1	64	-1
-2	14	1

The trace will be  $14 - 64 + 84 - 64 + 14 + 8 = -8$ . This is just the split form again.

Summary: We've found 3 forms of  $E_8$ , corresponding to 3 classes in  $L/2L$ , with signatures 8,  $-24$ ,  $-248$ , corresponding to involutions  $e^\alpha \mapsto (-1)^{(\alpha, \beta)} e^{-\alpha}$  of the *compact* form. If  $L$  is the *compact* form of a simple Lie algebra, then Cartan showed that the other forms correspond exactly to the conjugacy classes of involutions in the automorphism group of  $L$  (this

doesn't work if you don't start with the compact form — so always start with the compact form).

In fact, these three are the *only* forms of  $E_8$ , but we won't prove that.

### Working with simple Lie groups

As an example of how to work with simple Lie groups, we will look at the general question: Given a simple Lie group, what is its homotopy type? Answer:  $G$  has a unique conjugacy class of maximal compact subgroups  $K$ , and  $G$  is homotopy equivalent to  $K$ .

*Proof for  $GL_n(\mathbb{R})$ .* First pretend  $GL_n(\mathbb{R})$  is simple, even though it isn't; whatever. There is an obvious compact subgroup:  $O_n(\mathbb{R})$ . Suppose  $K$  is *any* compact subgroup of  $GL_n(\mathbb{R})$ . Choose any positive definite form  $(\ , \ )$  on  $\mathbb{R}^n$ . This will probably not be invariant under  $K$ , but since  $K$  is compact, we can average it over  $K$  get one that is: define a new form  $(a, b)_{\text{new}} = \int_K (ka, kb) dk$ . This gives an invariant positive definite bilinear form (since integral of something positive definite is positive definite). Thus, any compact subgroup preserves some positive definite form. But the subgroup fixing some positive definite bilinear form is conjugate to a subgroup of  $O_n(\mathbb{R})$  (to see this, diagonalize the form). So  $K$  is contained in a conjugate of  $O_n(\mathbb{R})$ .

Next we want to show that  $G = GL_n(\mathbb{R})$  is homotopy equivalent to  $O_n(\mathbb{R}) = K$ . We will show that  $G = KAN$ , where  $K$  is  $O_n$ ,  $A$  is all diagonal matrices with positive coefficients, and  $N$  is matrices which are upper triangular with 1s on the diagonal. This is the *Iwasawa decomposition*. In general, we get  $K$  compact,  $A$  semi-simple abelian, and  $N$  is unipotent. The proof of this you saw before was called the Gram-Schmidt process for orthonormalizing a basis. Suppose  $v_1, \dots, v_n$  is any basis for  $\mathbb{R}^n$ .

1. Make it orthogonal by subtracting some stuff, you'll get  $v_1, v_2 - *v_1, v_3 - *v_2 - *v_1, \dots$
2. Normalize by multiplying each basis vector so that it has norm 1. Now we have an orthonormal basis.

This is just another way to say that  $GL_n$  can be written as  $KAN$ . Making things orthogonal is just multiplying by something in  $N$ , and normalizing is just multiplication by some diagonal matrix with positive entries. An orthonormal basis is an element of  $O_n$ . Tada! This decomposition is just a topological one, not a decomposition as groups. Uniqueness is easy to check.

Now we can get at the homotopy type of  $GL_n$ .  $N \cong \mathbb{R}^{n(n-1)/2}$ , and  $A \cong (\mathbb{R}^+)^n$ , which are contractible. Thus,  $GL_n(\mathbb{R})$  has the same homotopy type as  $O_n(\mathbb{R})$ , its maximal compact subgroup.  $\square$

If you wanted to know  $\pi_1(GL_3(\mathbb{R}))$ , you could calculate  $\pi_1(O_3(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ , so  $GL_3(\mathbb{R})$  has a double cover. Nobody has shown you this double cover because it is *not algebraic*.

**Example 28.1.** Let's go back to various forms of  $E_8$  and figure out (guess) the fundamental groups. We need to know the maximal compact subgroups.

1. One of them is easy: the compact form is its own maximal compact subgroup. What is the fundamental group? Remember or quote the fact that for compact simple groups,  $\pi_1 \cong \frac{\text{weight lattice}}{\text{root lattice}}$ , which is 1. So this form is simply connected.
2.  $\beta^2 = 2$  case (signature  $-24$ ). Recall that there were 1, 56, 126, 56, and 1 roots  $\alpha$  with  $(\alpha, \beta) = 2, 1, 0, -1$ , and  $-2$  respectively, and there are another 8 dimensions for the Cartan subalgebra. On the 1, 126, 1, 8 parts, the form is negative definite. The sum of these root spaces gives a Lie algebra of type  $E_7A_1$  with a negative definite bilinear form (the 126 gives you the roots of an  $E_7$ , and the 1s are the roots of an  $A_1$ ). So it a reasonable guess that the maximal compact subgroup has something to do with  $E_7A_1$ .  $E_7$  and  $A_1$  are not simply connected: the compact form of  $E_7$  has  $\pi_1 = \mathbb{Z}/2$  and the compact form of  $A_1$  also has  $\pi_1 = \mathbb{Z}/2$ . So the universal cover of  $E_7A_1$  has center  $(\mathbb{Z}/2)^2$ . Which part of this acts trivially on  $E_8$ ? We look at the  $E_8$  Lie algebra as a representation of  $E_7 \times A_1$ . You can read off how it splits from the picture above:  $E_8 \cong E_7 \oplus A_1 \oplus 56 \otimes 2$ , where 56 and 2 are irreducible, and the centers of  $E_7$  and  $A_1$  both act as  $-1$  on them. So the maximal compact subgroup of this form of  $E_8$  is the simply connected compact form of  $E_7 \times A_1 / (-1, -1)$ . This means that  $\pi_1(E_8)$  is the same as  $\pi_1$  of the compact subgroup, which is  $(\mathbb{Z}/2)^2 / (-1, -1) \cong \mathbb{Z}/2$ . So this simple group has a nontrivial double cover (which is non-algebraic).
3. For the other (split) form of  $E_8$  with signature 8, the maximal compact subgroup is  $\text{Spin}_{16}(\mathbb{R}) / (\mathbb{Z}/2)$ , and  $\pi_1(E_8)$  is  $\mathbb{Z}/2$ .

You can compute any other homotopy invariants with this method.

Let's look at the 56-dimensional representation of  $E_7$  in more detail. We had the picture

$(\alpha, \beta)$	# of $\alpha$ 's
2	1
1	56
0	126
-1	56
-2	1

The Lie algebra  $E_7$  fixes these 5 spaces of  $E_8$  of dimensions 1, 56, 126 + 8, 56, 1. From this we can get some representations of  $E_7$ . The 126 + 8 splits as 1 + (126 + 7). But we also get a 56-dimensional representation of  $E_7$ . Let's show that this is actually an irreducible representation. Recall that in calculating  $W(E_8)$ , we showed that  $W(E_7)$  acts transitively on this set of 56 roots of  $E_8$ , which can be considered as weights of  $E_7$ .

An irreducible representation is called *minuscule* if the Weyl group acts transitively on the weights. This kind of representation is particularly easy to work with. It is really easy to work out the character for example: just translate the 1 at the highest weight around, so every weight has multiplicity 1.

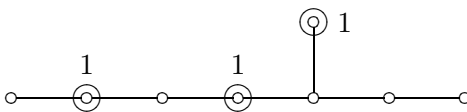
So the 56-dimensional representation of  $E_7$  must actually be the irreducible representation with whatever highest weight corresponds to one of the vectors.

### Every possible simple Lie group

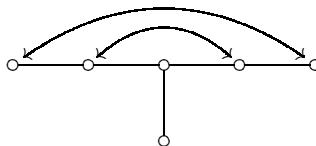
We will construct them as follows: Take an involution  $\sigma$  of the compact form  $L = L^+ + L^-$  of the Lie algebra, and form  $L^+ + iL^-$ . The way we constructed these was to first construct  $A_n$ ,  $D_n$ ,  $E_6$ , and  $E_7$  as for  $E_8$ . Then construct the involution  $\omega : e^\alpha \mapsto -e^{-\alpha}$ . We get  $B_n$ ,  $C_n$ ,  $F_4$ , and  $G_2$  as fixed points of the involution  $\omega$ .

Kac classified all automorphisms of finite order of any compact simple Lie group. The method we'll use to classify involutions is extracted from his method. We can construct lots of involutions as follows:

1. Take any Dynkin diagram, say  $E_8$ , and select some of its vertices, corresponding to simple roots. Get an involution by taking  $e^\alpha \mapsto \pm e^\alpha$  where the sign depends on whether  $\alpha$  is one of the simple roots we've selected. However, this is not a great method. For one thing, you get a lot of repeats (recall that there are only 3, and we've found  $2^8$  this way).



2. Take any diagram automorphism of order 2, such as



This gives you more involutions.

Next time, we'll see how to cut down this set of involutions.

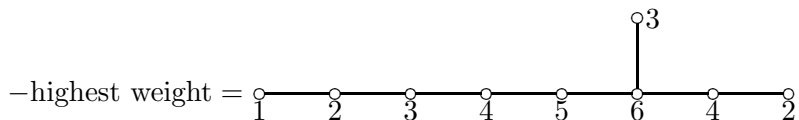
## Lecture 29

Split form of Lie algebra (we did this for  $A_n, D_n, E_6, E_7, E_8$ ):  $A = \bigoplus \hat{e}^\alpha \oplus L$ . Compact form  $A^+ + iA^-$ , where  $A^\pm$  eigenspaces of  $\omega : \hat{e}^\alpha \mapsto (-1)^{\alpha^2/2} \hat{e}^{-\alpha}$ .

We talked about other involutions of the compact form. You get all the other forms this way.

The idea now is to find ALL real simple Lie algebras by listing all involutions of the compact form. We will construct all of them, but we won't prove that we have all of them.

We'll use Kac's method for classifying all automorphisms of order  $N$  of a compact Lie algebra (and we'll only use the case  $N = 2$ ). First let's look at inner automorphisms. Write down the AFFINE Dynkin diagram



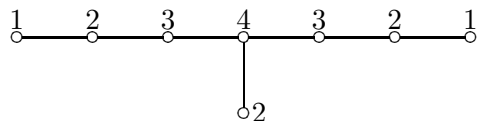
Choose  $n_i$  with  $\sum n_i m_i = N$  where the  $m_i$  are the numbers on the diagram. We have an automorphism  $e^{\alpha_j} \mapsto e^{2\pi i n_j / N} e^{\alpha_j}$  induces an automorphism of order dividing  $N$ . This is obvious. The point of Kac's theorem is that all inner automorphisms of order dividing  $N$  are obtained this way and are conjugate if and only if they are conjugate by an automorphism of the Dynkin diagram. We won't actually prove Kac's theorem because we just want to get a bunch of examples. See [Kac90] or [Hel01].

**Example 29.1.** Real forms of  $E_8$ . We've already found three, and it took us a long time. We can now do it fast. We need to solve  $\sum n_i m_i = 2$  where  $n_i \geq 0$ ; there are only a few possibilities:

$\sum n_i m_i = 2$	# of ways	how to do it	maximal compact subgroup $K$
$2 \times 1$	one way		$E_8$ (compact form)
$1 \times 2$	two ways		$A_1 E_7$
			$D_8$ (split form)
$1 \times 1 + 1 \times 1$	no ways		

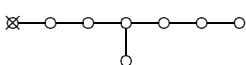
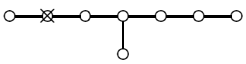
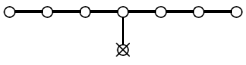
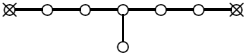
The points NOT crossed off form the Dynkin diagram of the maximal compact subgroup. Thus, by just looking at the diagram, we can see what all the real forms are!

**Example 29.2.** Let's do  $E_7$ . Write down the affine diagram:



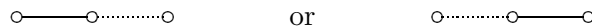


We get the possibilities

$\sum n_i m_i = 2$	# of ways	how to do it	maximal compact subgroup $K$
$2 \times 1$	one way*		$E_7$ (compact form)
$1 \times 2$	two ways*		$A_1 D_6$
			$A_7$ (split form)**
$1 \times 1 + 1 \times 1$	one way		$E_6 \oplus \mathbb{R}$ ***

- (\*) The number of ways is counted up to automorphisms of the diagram.
- (\*\*) In the split real form, the maximal compact subgroup has dimension equal to half the number of roots. The roots of  $A_7$  look like  $\varepsilon_i - \varepsilon_j$  for  $i, j \leq 8$  and  $i \neq j$ , so the dimension is  $8 \cdot 7 + 7 = 56 = \frac{112}{2}$ .
- (\*\*\*) The maximal compact subgroup is  $E_6 \oplus \mathbb{R}$  because the fixed subalgebra contains the whole Cartan, and the  $E_6$  only accounts for 6 of the 7 dimensions. You can use this to construct some interesting representations of  $E_6$  (the minuscule ones). How does the algebra  $E_7$  decompose as a representation of the algebra  $E_6 \oplus \mathbb{R}$ ?

We can decompose it according to the eigenvalues of  $\mathbb{R}$ . The  $E_6 \oplus \mathbb{R}$  is the zero eigenvalue of  $\mathbb{R}$  [why?], and the rest is 54-dimensional. The easy way to see the decomposition is to look at the roots. Remember when we computed the Weyl group we looked for vectors like



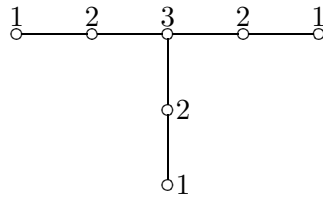
The 27 possibilities (for each) form the weights of a 27-dimensional representation of  $E_6$ . The orthogonal complement of the two nodes is an  $E_6$  root system whose Weyl group acts transitively on these 27 vectors (we showed that these form a single orbit, remember?). Vectors of the  $E_7$  root system are the vectors of the  $E_6$  root system plus these 27 vectors plus the other 27 vectors. This splits up the  $E_7$  explicitly. The two 27s form single orbits, so they are irreducible. Thus,  $E_7 \cong E_6 \oplus \mathbb{R} \oplus 27 \oplus 27$ , and the 27s are minuscule.

Let  $K$  be a maximal compact subgroup, with Lie algebra  $\mathbb{R} + E_6$ . The factor of  $\mathbb{R}$  means that  $K$  has an  $S^1$  in its center. Now look at the space  $G/K$ , where  $G$  is the Lie group of type  $E_7$ , and  $K$  is the maximal compact subgroup. It is a *Hermitian symmetric space*. Symmetric space means that it is a (simply connected) Riemannian manifold  $M$  such that for each point  $p \in M$ , there is an automorphism fixing  $p$  and acting as  $-1$  on the tangent space. This looks weird, but it turns out that all kinds of nice objects you know about are symmetric spaces. Typical examples you may have seen:

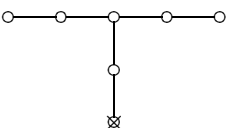
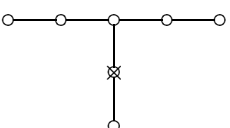
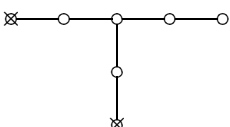
spheres  $S^n$ , hyperbolic space  $\mathbb{H}^n$ , and Euclidean space  $\mathbb{R}^n$ . Roughly speaking, symmetric spaces have nice properties of these spaces. Cartan classified all symmetric spaces: they are non-compact simple Lie groups modulo the maximal compact subgroup (more or less ... depending on simply connectedness hypotheses 'n such). Historically, Cartan classified simple Lie groups, and then later classified symmetric spaces, and was surprised to find the same result. Hermitian symmetric spaces are just symmetric spaces with a complex structure. A standard example of this is the upper half plane  $\{x + iy | y > 0\}$ . It is acted on by  $SL_2(\mathbb{R})$ , which acts by  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}$ .

Let's go back to this  $G/K$  and try to explain why we get a Hermitian symmetric space from it. We'll be rather sketchy here. First of all, to make it a symmetric space, we have to find a nice invariant Riemannian metric on it. It is sufficient to find a positive definite bilinear form on the tangent space at  $p$  which is invariant under  $K$  ... then you can translate it around. We can do this as  $K$  is compact (so you have the averaging trick). Why is it Hermitian? We'll show that there is an almost complex structure. We have  $S^1$  acting on the tangent space of each point because we have an  $S^1$  in the center of the stabilizer of any given point. Identify this  $S^1$  with complex numbers of absolute value 1. This gives an invariant almost complex structure on  $G/K$ . That is, each tangent space is a complex vector space. Almost complex structures don't always come from complex structures, but this one does (it is integrable). Notice that it is a little unexpected that  $G/K$  has a complex structure ( $G$  and  $K$  are odd-dimensional in the case of  $G = E_7$ ,  $K = E_6 \oplus \mathbb{R}$ , so they have no hope of having a complex structure).

**Example 29.3.** Let's look at  $E_6$ , with affine Dynkin diagram



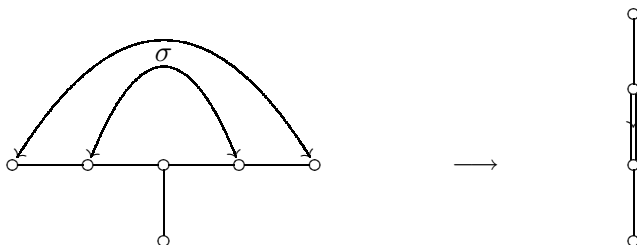
We get the possibilities

$\sum n_i m_i = 2$	# of ways	how to do it	maximal compact subgroup $K$
$2 \times 1$	one way		$E_6$ (compact form)
$1 \times 2$	one way		$A_1 A_5$
$1 \times 1 + 1 \times 1$	one way		$D_5 \oplus \mathbb{R}$

In the last one, the maximal compact subalgebra is  $D_5 \oplus \mathbb{R}$ . Just as before, we get a Hermitian symmetric space. Let's compute its dimension (over  $\mathbb{C}$ ). The dimension will be the dimension of  $E_6$  minus the dimension of  $D_5 \oplus \mathbb{R}$ , all divided by 2 (because we want complex dimension), which is  $(78 - 46)/2 = 16$ .

So we have found two non-compact simply connected Hermitian symmetric spaces of dimensions 16 and 27. These are the only "exceptional" cases; all the others fall into infinite families!

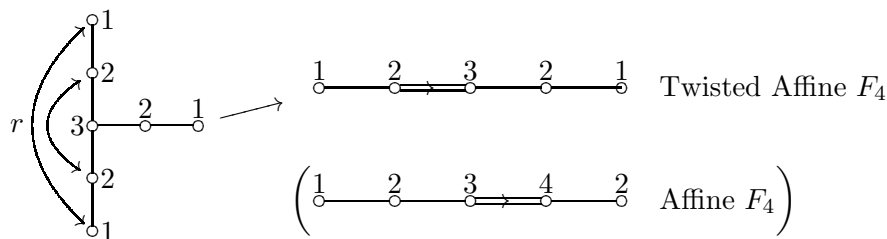
There are also some OUTER automorphisms of  $E_6$  coming from the diagram automorphism



The fixed point subalgebra has Dynkin diagram obtained by folding the  $E_6$  on itself. This is the  $F_4$  Dynkin diagram. The fixed points of  $E_6$  under the diagram automorphism is an  $F_4$  Lie algebra. So we get a real form of  $E_6$  with maximal compact subgroup  $F_4$ . This is probably the easiest way to construct  $F_4$ , by the way. Moreover, we can decompose  $E_6$  as a representation of  $F_4$ .  $\dim E_6 = 78$  and  $\dim F_4 = 52$ , so  $E_6 = F_4 \oplus 26$ , where 26 turns out to be irreducible (the smallest non-trivial representation of  $F_4$  ... the only one anybody actually works with). The roots of  $F_4$  look like  $(\dots, \pm 1, \pm 1 \dots)$  (24 of these) and  $(\pm \frac{1}{2} \dots \pm \frac{1}{2})$  (16 of these), and  $(\dots, \pm 1 \dots)$  (8 of them) ... the last two types are in the same orbit of the Weyl group.

The 26-dimensional representation has the following character: it has all norm 1 roots with multiplicity 1 and 0 with multiplicity 2 (note that this is not minuscule).

There is one other real form of  $E_6$ . To get at it, we have to talk about Kac's description of non-inner automorphisms of order  $N$ . The non-inner automorphisms all turn out to be related to diagram automorphisms. Choose a diagram automorphism of order  $r$ , which divides  $N$ . Let's take the standard thing on  $E_6$ . Fold the diagram (take the fixed points), and form a TWISTED affine Dynkin diagram (note that the arrow goes the wrong way from the affine  $F_4$ )



There are also numbers on the twisted diagram, but nevermind them. Find  $n_i$  so that  $r \sum n_i m_i = N$ . This is Kac's general rule. We'll only use the case  $N = 2$ .

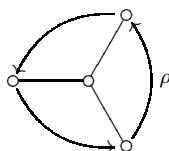
If  $r > 1$ , the only possibility is  $r = 2$  and one  $n_i$  is 1 and the corresponding  $m_i$  is 1. So we just have to find points of weight 1 in the twisted affine Dynkin diagram. There are just two ways of doing this in the case of  $E_6$



one of these gives us  $F_4$ , and the other has maximal compact subalgebra  $C_4$ , which is the split form since  $\dim C_4 = \#\text{roots of } F_4/2 = 24$ .

**Example 29.4.**  $F_4$ . The affine Dynkin is  $1 \text{---} 2 \text{---} 3 \text{---} 4 \text{---} 2$ . We can cross out one node of weight 1, giving the compact form (split form), or a node of weight 2 (in two ways), giving maximal compacts  $A_1 C_3$  or  $B_4$ . This gives us three real forms.

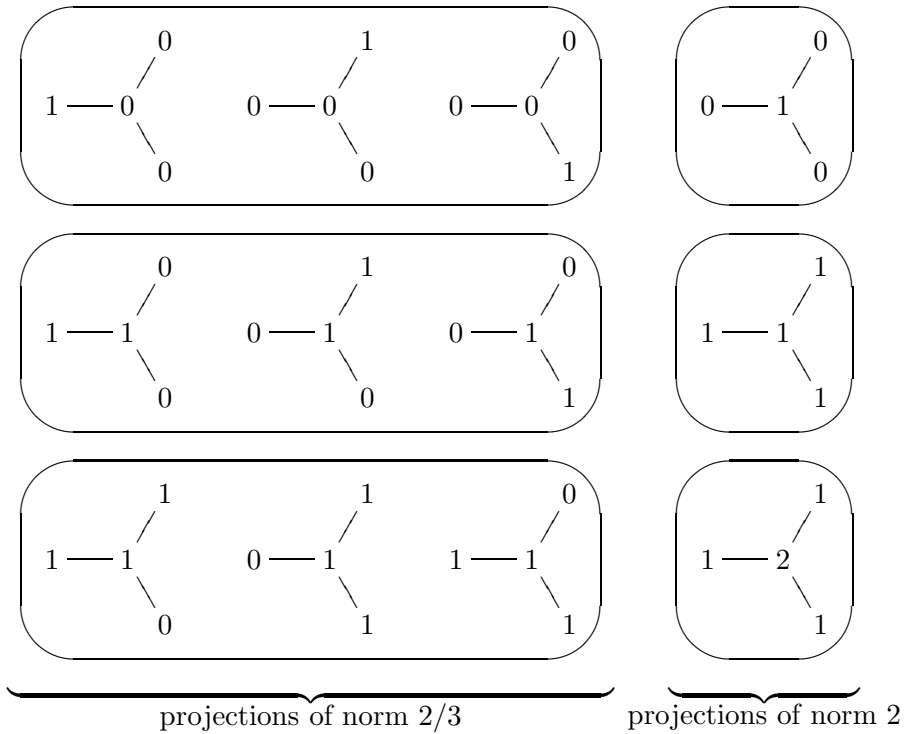
**Example 29.5.**  $G_2$ . We can actually draw this root system ... UCB won't supply me with a four-dimensional board. The construction is to take the  $D_4$  algebra and look at the fixed points of:



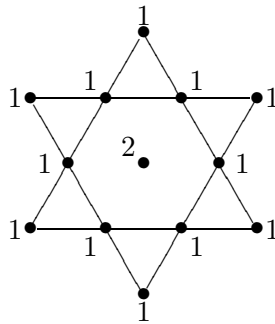
We want to find the fixed point subalgebra.

Fixed points on Cartan subalgebra:  $\rho$  fixes a two-dimensional space, and has 1-dimensional eigenspaces corresponding to  $\omega$  and  $\bar{\omega}$ , where  $\omega^3 = 1$ . The 2-dimensional space will be the Cartan of  $G_2$ .

Positive roots of  $D_4$  as linear combinations of simple roots (not fundamental weights):



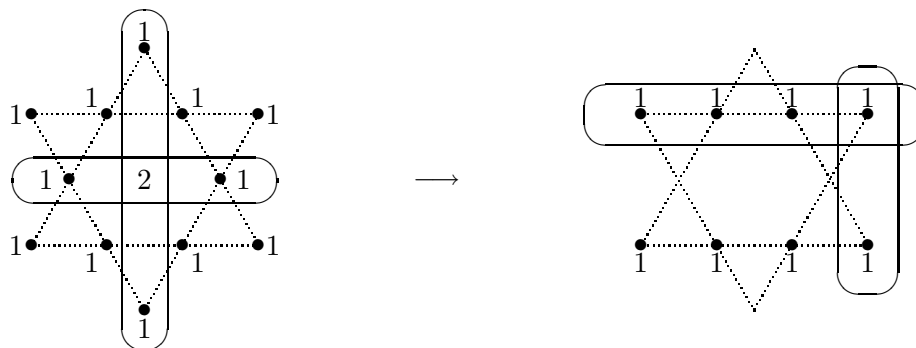
There are six orbits under  $\rho$ , grouped above. It obviously acts on the negative roots in exactly the same way. What we have is a root system with six roots of norm 2 and six roots of norm 2/3. Thus, the root system is  $G_2$ :



One of the only root systems to appear on a country's national flag. Now let's work out the real forms. Look at the affine:  $1 \text{---} 2 \text{=} 3$ . we can

delete the node of weight 1, giving the compact form:  $\times \text{---} \text{=} \text{=} \text{=} \text{---}$ . We can delete the node of weight 2, giving  $A_1 A_1$  as the compact subalgebra:  $\text{---} \text{---} \text{---} \times \text{---} \text{=} \text{=} \text{=} \text{---}$  ... this must be the split form because there is nothing else the split form can be.

Let's say some more about the split form. What does the Lie algebra of  $G_2$  look like as a representation of the maximal compact subalgebra  $A_1 \times A_1$ ? In this case, it is small enough that we can just draw a picture:



We have two orthogonal  $A_1$ s, and we have leftover the stuff on the right. This thing on the right is a tensor product of the 4-dimensional irreducible representation of the horizontal and the 2-dimensional of the vertical. Thus,  $G_2 = 3 \times 1 + 1 \otimes 3 + 4 \otimes 2$  as irreducible representations of  $A_1^{(\text{horizontal})} \otimes A_1^{(\text{vertical})}$ .

Let's use this to determine exactly what the maximal compact subgroup is. It is a quotient of the simply connected compact group  $SU(2) \times SU(2)$ , with Lie algebra  $A_1 \times A_1$ . Just as for  $E_8$ , we need to identify which elements of the center act trivially on  $G_2$ . The center is  $\mathbb{Z}/2 \times \mathbb{Z}/2$ . Since we've decomposed  $G_2$ , we can compute this easily. A non-trivial element of the center of  $SU(2)$  acts as 1 (on odd dimensional representations) or  $-1$  (on even dimensional representations). So the element  $z \times z \in SU(2) \times SU(2)$  acts trivially on  $3 \otimes 1 + 1 \otimes 3 + 4 \otimes 2$ . Thus the maximal compact subgroup of the non-compact simple  $G_2$  is  $SU(2) \times SU(2)/(z \times z) \cong SO_4(\mathbb{R})$ , where  $z$  is the non-trivial element of  $\mathbb{Z}/2$ .

So we have constructed  $3 + 4 + 5 + 3 + 2$  (from  $E_8, E_7, E_6, F_4, G_2$ ) real forms of exceptional simple Lie groups.

There are another 5 exceptional real Lie groups: Take COMPLEX groups  $E_8(\mathbb{C}), E_7(\mathbb{C}), E_6(\mathbb{C}), F_4(\mathbb{C}),$  and  $G_2(\mathbb{C})$ , and consider them as REAL. These give simple real Lie groups of dimensions  $248 \times 2, 133 \times 2, 78 \times 2, 52 \times 2,$  and  $14 \times 2$ .

## Lecture 30 - Irreducible unitary representations of $SL_2(\mathbb{R})$

$SL_2(\mathbb{R})$  is non-compact. For compact Lie groups, all unitary representations are finite dimensional, and are all known well. For non-compact groups, the theory is much more complicated. Before doing the infinite-dimensional representations, we'll review finite-dimensional (usually not unitary) representations of  $SL_2(\mathbb{R})$ .

### Finite dimensional representations

Finite-dimensional complex representations of the following are much the same:  $SL_2(\mathbb{R})$ ,  $\mathfrak{sl}_2\mathbb{R}$ ,  $\mathfrak{sl}_2\mathbb{C}$  [branch  $SL_2(\mathbb{C})$  as a complex Lie group] (as a complex Lie algebra),  $\mathfrak{su}_2\mathbb{R}$  (as a real Lie algebra), and  $SU_2$  (as a real Lie group). This is because finite-dimensional representations of a simply connected Lie group are in bijection with representations of the Lie algebra. Complex representations of a REAL Lie algebra  $L$  correspond to complex representations of its complexification  $L \otimes \mathbb{C}$  considered as a COMPLEX Lie algebra.

Note: Representations of a COMPLEX Lie algebra  $L \otimes \mathbb{C}$  are not the same as representations of the REAL Lie algebra  $L \otimes \mathbb{C} \cong L + L$ . The representations of the real Lie algebra correspond roughly to (reps of  $L$ ) $\otimes$ (reps of  $L$ ).

Strictly speaking,  $SL_2(\mathbb{R})$  is not simply connected, which is not important for finite-dimensional representations.

Recall the main results for representations of  $SU_2$ :

1. For each positive integer  $n$ , there is one irreducible representation of dimension  $n$ .
2. The representations are completely reducible (every representation is a sum of irreducible ones). This is perhaps the most important fact.

The finite-dimensional representation theory of  $SU_2$  is EASIER than the representation theory of the ABELIAN Lie group  $\mathbb{R}^2$ , and that is because representations of  $SU_2$  are completely reducible.

For example, it is very difficult to classify pairs of commuting nilpotent matrices.

Completely reducible representations:

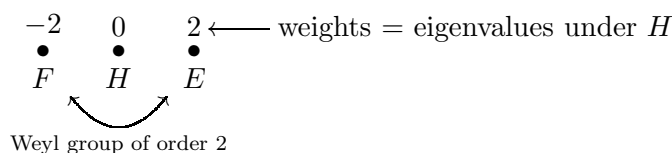
1. Complex representations of finite groups.
2. Representations of compact groups (Weyl character formula)

3. More generally, unitary representations of anything (you can take orthogonal complements of subrepresentations)
4. Finite-dimensional representations of semisimple Lie groups.

Representations which are not completely reducible:

1. Representations of a finite group  $G$  over fields of characteristic  $p \mid |G|$ .
2. Infinite-dimensional representations of non-compact Lie groups (even if they are semisimple).

We'll work with the Lie algebra  $\mathfrak{sl}_2\mathbb{R}$ , which has basis  $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ , and  $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ .  $H$  is a basis for the cartan  $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ .  $E$  spans the root space of the simple root.  $F$  spans the root space of the negative of the simple root. We find that  $[H, E] = 2E$ ,  $[H, F] = -2F$  (so  $E$  and  $F$  are eigenvectors of  $H$ ), and you can check that  $[E, F] = H$ .



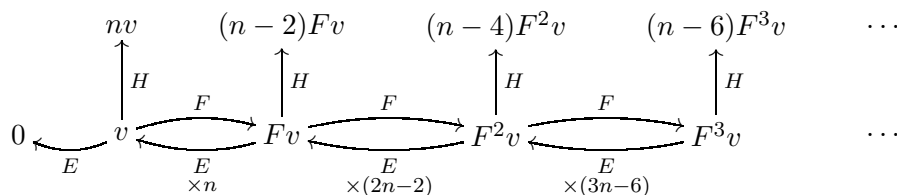
The Weyl group is generated by  $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and  $\omega^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ .

Let  $V$  be a finite-dimensional irreducible complex representation of  $\mathfrak{sl}_2\mathbb{R}$ . First decompose  $V$  into eigenspaces of the Cartan subalgebra (weight spaces) (i.e. eigenspaces of the element  $H$ ). Note that eigenspaces of  $H$  exist because  $V$  is FINITE-DIMENSIONAL (remember this is a complex representation). Look at the LARGEST eigenvalue of  $H$  (exists since  $V$  is finite-dimensional), with eigenvector  $v$ . We have that  $Hv = nv$  for some  $n$ . Compute

$$\begin{aligned} H(Ev) &= [H, E]v + E(Hv) \\ &= 2Ev + Env = (n + 2)Ev \end{aligned}$$

So  $Ev = 0$  (lest it be an eigenvector of  $H$  with higher eigenvalue).  $[E, -]$  increases weights by 2 and  $[F, -]$  decreases weights by 2, and  $[H, -]$  fixes weights.

We have that  $E$  kills  $v$ , and  $H$  multiplies it by  $n$ . What does  $F$  do to  $v$ ?





What is  $E(Fv)$ ? Well,

$$\begin{aligned} EFv &= FEv + [E, F]v \\ &= 0 + Hv = nv \end{aligned}$$

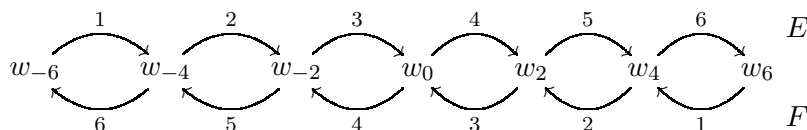
In general, we have

$$\begin{aligned} H(F^i v) &= (n - 2i)F^i v \\ E(F^i v) &= (ni - i(i - 1))F^{i-1} v \\ F(F^i v) &= F^{i+1} v \end{aligned}$$

So the vectors  $F^i v$  span  $V$  because they span an invariant subspace. This gives us an infinite number of vectors in distinct eigenspaces of  $H$ , and  $V$  is finite-dimensional. Thus,  $F^k v = 0$  for some  $k$ . Suppose  $k$  is the SMALLEST integer such that  $F^k v = 0$ . Then

$$0 = E(F^k v) = (nk - k(k - 1)) \underbrace{EF^{k-1} v}_{\neq 0}$$

So  $nk - k(k - 1) = 0$ , and  $k \neq 0$ , so  $n - (k - 1) = 0$ , so  $\boxed{k = n + 1}$ . So  $V$  has a basis consisting of  $v, Fv, \dots, F^n v$ . The formulas become a little better if we use the basis  $w_n = v, w_{n-2} = Fv, w_{n-4} = \frac{F^2 v}{2!}, \frac{F^3 v}{3!}, \dots, \frac{F^n v}{n!}$ .




This says that  $E(w_2) = 5w_4$  for example. So we've found a complete description of all finite-dimensional irreducible complex representations of  $\mathfrak{sl}_2\mathbb{R}$ . This is as explicit as you could possibly want.

These representations all lift to the group  $SL_2(\mathbb{R})$ :  $SL_2(\mathbb{R})$  acts on homogeneous polynomials of degree  $n$  by  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x, y) = f(ax + by, cx + dy)$ . This is an  $n + 1$ -dimensional space, and you can check that the eigenspaces are  $x^i y^{n-i}$ .

We have implicitly constructed VERMA MODULES. We have a basis  $w_n, w_{n-2}, \dots, w_{n-2i}, \dots$  with relations  $H(w_{n-2i}) = (n - 2i)w_{n-2i}$ ,  $Ew_{n-2i} = (n - i + 1)w_{n-2i+2}$ , and  $Fw_{n-2i} = (i + 1)w_{n-2i-2}$ . These are obtained by copying the formulas from the finite-dimensional case, but allow it to be infinite-dimensional. This is the universal representation generated by the highest weight vector  $w_n$  with eigenvalue  $n$  under  $H$  (highest weight just means  $E(w_n) = 0$ ).

Let's look at some things that go wrong in infinite dimensions.

 *Warning* 30.1. Representations corresponding to the Verma modules do NOT lift to representations of  $SL_2(\mathbb{R})$ , or even to its universal cover. The reason: look at the Weyl group (generated by  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ) of  $SL_2(\mathbb{R})$  acting on  $\langle H \rangle$ ; it changes  $H$  to  $-H$ . It maps eigenspaces with eigenvalue  $m$  to eigenvalue  $-m$ . But if you look at the Verma module, it has eigenspaces  $n, n-2, n-4, \dots$ , and this set is obviously not invariant under changing sign. The usual proof that representations of the Lie algebra lifts uses the exponential map of matrices, which doesn't converge in infinite dimensions.

*Remark* 30.2. The universal cover  $\widetilde{SL_2(\mathbb{R})}$  of  $SL_2(\mathbb{R})$ , or even the double cover  $Mp_2(\mathbb{R})$ , has NO faithful finite-dimensional representations.

*Proof.* Any finite dimensional representation comes from a finite dimensional representation of the Lie algebra  $\mathfrak{sl}_2\mathbb{R}$ . All such finite dimensional representations factor through  $SL_2(\mathbb{R})$ .  $\square$

All finite-dimensional representations of  $SL_2(\mathbb{R})$  are completely reducible. Weyl did this by Weyl's unitarian trick:

Notice that finite-dimensional representations of  $SL_2(\mathbb{R})$  are isomorphic (sort of) to finite-dimensional representations of the COMPACT group  $SU_2$  (because they have the same complexified Lie algebras. Thus, we just have to show it for  $SU_2$ . But representations of ANY compact group are completely reducible. Reason:

1. All unitary representations are completely reducible (if  $U \subseteq V$ , then  $V = U \oplus U^\perp$ ).
2. Any representation  $V$  of a COMPACT group  $G$  can be made unitary: take any unitary form on  $V$  (not necessarily invariant under  $G$ ), and average it over  $G$  to get an invariant unitary form. We can average because  $G$  is compact, so we can integrate any continuous function over  $G$ . This form is positive definite since it is the average of positive definite forms (if you try this with non-(positive definite) forms, you might get zero as a result).

## The Casimir operator

Set  $\Omega = 2EF + 2FE + H^2 \in U(\mathfrak{sl}_2\mathbb{R})$ . The main point is that  $\Omega$  commutes with  $\mathfrak{sl}_2\mathbb{R}$ . You can check this by brute force:

$$\begin{aligned}
 [H, \Omega] &= 2 \underbrace{([H, E]F + E[H, F])}_0 + \dots \\
 [E, \Omega] &= 2[E, E]F + 2E[F, E] + 2[E, F]E \\
 &\quad + 2F[E, E] + [E, H]H + H[E, H] = 0 \\
 [F, \Omega] &= \text{Similar}
 \end{aligned}$$

Thus,  $\Omega$  is in the center of  $U(\mathfrak{sl}_2\mathbb{R})$ . In fact, it generates the center. This doesn't really explain where  $\Omega$  comes from.

*Remark 30.3.* Why does  $\Omega$  exist? The answer is that it comes from a symmetric invariant bilinear form on the Lie algebra  $\mathfrak{sl}_2\mathbb{R}$  given by  $(E, F) = 1$ ,  $(E, E) = (F, F) = (F, H) = (E, H) = 0$ ,  $(H, H) = 2$ . This bilinear form is an invariant map  $L \otimes L \rightarrow \mathbb{C}$ , where  $L = \mathfrak{sl}_2\mathbb{R}$ , which by duality gives an invariant element in  $L \otimes L$ , which turns out to be  $2E \otimes F + 2F \otimes E + H \otimes H$ . The invariance of this element corresponds to  $\Omega$  being in the center of  $U(\mathfrak{sl}_2\mathbb{R})$ .

Since  $\Omega$  is in the center of  $U(\mathfrak{sl}_2\mathbb{R})$ , it acts on each irreducible representation as multiplication by a constant. We can work out what this constant is for the finite-dimensional representations. Apply  $\Omega$  to the highest vector  $w_n$ :

$$\begin{aligned} (2EF + 2FE + HH)w_n &= (2n + 0 + n^2)w_n \\ &= (2n + n^2)w_n \end{aligned}$$

So  $\Omega$  has eigenvalue  $2n + n^2$  on the irreducible representation of dimension  $n + 1$ . Thus,  $\Omega$  has DISTINCT eigenvalues on different irreducible representations, so it can be used to separate different irreducible representations. The main use of  $\Omega$  will be in the next lecture, where we'll use it to deal with infinite-dimensional representation.

To finish today's lecture, let's look at an application of  $\Omega$ . We'll sketch an algebraic argument that the representations of  $\mathfrak{sl}_2\mathbb{R}$  are completely reducible. Given an exact sequence of representations

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

we want to find a splitting  $W \rightarrow V$ , so that  $V = U \oplus W$ .

Step 1: Reduce to the case where  $W = \mathbb{C}$ . The idea is to look at

$$0 \rightarrow \text{Hom}_{\mathbb{C}}(W, U) \rightarrow \text{Hom}_{\mathbb{C}}(W, V) \rightarrow \text{Hom}_{\mathbb{C}}(W, W) \rightarrow 0$$

and  $\text{Hom}_{\mathbb{C}}(W, W)$  has an obvious one-dimensional subspace, so we can get a smaller exact sequence

$$0 \rightarrow \text{Hom}_{\mathbb{C}}(W, U) \rightarrow \text{subspace of } \text{Hom}_{\mathbb{C}}(W, V) \rightarrow \mathbb{C} \rightarrow 0$$

and if we can split this, the original sequence splits.

Step 2: Reduce to the case where  $U$  is irreducible. This is an easy induction on the number of irreducible components of  $U$ .

► **Exercise 30.1.** Do this.

Step 3: This is the key step. We have

$$0 \rightarrow U \rightarrow V \rightarrow \mathbb{C} \rightarrow 0$$

with  $U$  irreducible. Now apply the Casimir operator  $\Omega$ .  $V$  splits as eigenvalues of  $\Omega$ , so is  $U \oplus \mathbb{C}$  UNLESS  $U$  has the same eigenvalue as  $\mathbb{C}$  (i.e. unless  $U = \mathbb{C}$ ).

Step 4: We have reduced to

$$0 \rightarrow \mathbb{C} \rightarrow V \rightarrow \mathbb{C} \rightarrow 0$$

which splits because  $\mathfrak{sl}_2(\mathbb{R})$  is perfect<sup>1</sup> (no homomorphisms to the abelian algebra  $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ ).

Next time, in the final lecture, we'll talk about infinite-dimensional unitary representations.

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<sup>1</sup> $L$  is *perfect* if  $[L, L] = L$

## Lecture 31 - Unitary representations of $SL_2(\mathbb{R})$

Last lecture, we found the finite-dimensional (non-unitary) representations of  $SL_2(\mathbb{R})$ .

### Background about infinite-dimensional representations

(of a Lie group  $G$ ) What is a finite-dimensional representation?

1st guess Banach space acted on by  $G$ ?

This is not good for some reasons: Look at the action of  $G$  on the functions on  $G$  (by left translation). We could use  $L^2$  functions, or  $L^1$  or  $L^p$ . These are completely different Banach spaces, but they are essentially the same representation.

2nd guess Hilbert space acted on by  $G$ ? This is sort of okay.

The problem is that finite dimensional representations of  $SL_2(\mathbb{R})$  are NOT Hilbert space representations, so we are throwing away some interesting representations.

Solution (Harish-Chandra) Take  $\mathfrak{g}$  to be the Lie algebra of  $G$ , and let  $K$  be the maximal compact subgroup. If  $V$  is an infinite-dimensional representation of  $G$ , there is no reason why  $\mathfrak{g}$  should act on  $V$ .

The simplest example fails. Let  $\mathbb{R}$  act on  $L^2(\mathbb{R})$  by left translation. Then the Lie algebra is generated by  $\frac{d}{dx}$  (or  $i\frac{d}{dx}$ ) acting on  $L^2(\mathbb{R})$ , but  $\frac{d}{dx}$  of an  $L^2$  function is not in  $L^2$  in general.

Let  $V$  be a Hilbert space. Set  $V_\omega$  to be the  $K$ -finite vectors of  $V$ , which are the vectors contained in a finite-dimensional representation of  $K$ . The point is that  $K$  is compact, so  $V$  splits into a Hilbert space direct sum finite-dimensional representations of  $K$ , at least if  $V$  is a Hilbert space. Then  $V_\omega$  is a representation of the Lie algebra  $\mathfrak{g}$ , not a representation of  $G$ .  $V_\omega$  is a representation of the group  $K$ . It is a  $(\mathfrak{g}, K)$ -module, which means that it is acted on by  $\mathfrak{g}$  and  $K$  in a “compatible” way, where compatible means that

1. they give the same representations of the Lie algebra of  $K$ .
2.  $k(u)v = k(u(k^{-1}v))$  for  $k \in K$ ,  $u \in \mathfrak{g}$ , and  $v \in V$ .

The  $K$ -finite vectors of an irreducible unitary representation of  $G$  is ADMISSIBLE, which means that every representation of  $K$  only occurs a *finite* number of times. The GOOD category of representations is the representations of admissible  $(\mathfrak{g}, K)$ -modules. It turns out that this is a really well behaved category.

We want to find the unitary irreducible representations of  $G$ . We will do this in several steps:

1. Classify all irreducible admissible representations of  $G$ . This was solved by Langlands, Harish-Chandra et. al.
2. Find which have hermitian inner products  $(\cdot, \cdot)$ . This is easy.
3. Find which ones are positive definite. This is VERY HARD. We'll only do this for the simplest case:  $SL_2(\mathbb{R})$ .

### The group $SL_2(\mathbb{R})$

We found some generators (in  $Lie(SL_2(\mathbb{R})) \otimes \mathbb{C}$  last time:  $E, F, H$ , with  $[H, E] = 2E$ ,  $[H, F] = -2F$ , and  $[E, F] = H$ . We have that  $H = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $E = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ , and  $F = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$ . Why not use the old  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ , and  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ?

Because  $SL_2(\mathbb{R})$  has two different classes of Cartan subgroup:  $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ , spanned by  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , and  $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ , spanned by  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , and the second one is COMPACT. The point is that non-compact (abelian) groups need not have eigenvectors on infinite-dimensional spaces. An eigenvector is the same as a weight space. The first thing you do is split it into weight spaces, and if your Cartan subgroup is not compact, you can't get started. We work with the compact subalgebra so that the weight spaces exist.

Given the representation  $V$ , we can write it as some direct sum of eigenspaces of  $H$ , as the Lie group  $H$  generates is compact (isomorphic to  $S^1$ ). In the finite-dimensional case, we found a HIGHEST weight, which gave us complete control over the representation. The trouble is that in infinite dimensions, there is no reason for the highest weight to exist, and in general they don't. The highest weight requires a finite number of eigenvalues.

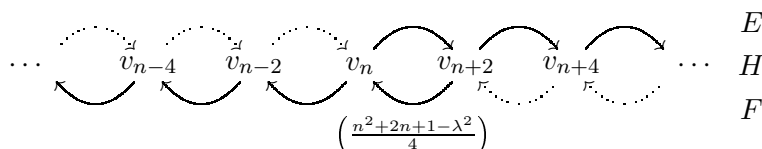
A good substitute for the highest weight vector: Look at the Casimir operator  $\Omega = 2EF + 2FE + H^2 + 1$ . The key point is that  $\Omega$  is in the center of the universal enveloping algebra. As  $V$  is assumed admissible, we can conclude that  $\Omega$  has eigenvectors (because we can find a finite-dimensional space acted on by  $\Omega$ ). As  $V$  is irreducible and  $\Omega$  commutes with  $G$ , all of  $V$  is an eigenspace of  $\Omega$ . We'll see that this gives us about as much information as a highest weight vector.

Let the eigenvalue of  $\Omega$  on  $V$  be  $\lambda^2$  (the square will make the interesting representations have integral  $\lambda$ ; the  $+1$  in  $\Omega$  is for the same reason).

Suppose  $v \in V_n$ , where  $V_n$  is the space of vectors where  $H$  has eigenvalue  $n$ . In the finite-dimensional case, we looked at  $Ev$ , and saw that  $HEv = (n+2)Ev$ . What is  $FEv$ ? If  $v$  was a highest weight vector, we could control this. Notice that  $\Omega = 4FE + H^2 + 2H + 1$  (using  $[E, F] = H$ ), and  $\Omega v = \lambda^2 v$ .

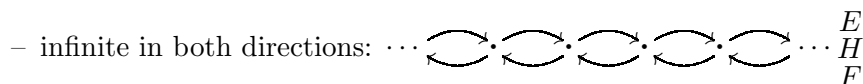
This says that  $4FEv + n^2v + 2nv + v = \lambda^2v$ . This shows that  $FEv$  is a multiple of  $v$ .

Now we can draw a picture of what the representation looks like:

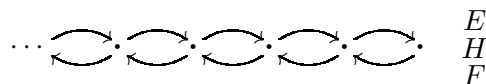


Thus,  $V_\omega$  is spanned by  $V_{n+2k}$ , where  $k$  is an integer. The non-zero elements among the  $V_{n+2k}$  are linearly independent as they have different eigenvalues. The only question remaining is whether any of the  $V_{n+2k}$  vanish.

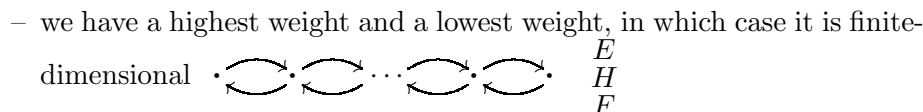
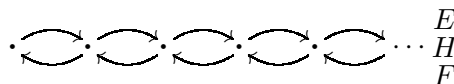
There are four possible shapes for an irreducible representation



– a lowest weight, and infinite in the other direction:

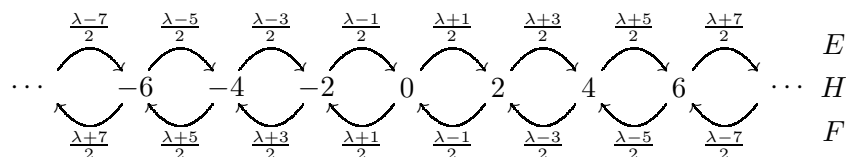


– a highest weight, and infinite in the other direction:



We'll see that all these show up. We also see that an irreducible representation is completely determined once we know  $\lambda$  and some  $n$  for which  $V_n \neq 0$ . The remaining question is to construct representations with all possible values of  $\lambda \in \mathbb{C}$  and  $n \in \mathbb{Z}$ .  $n$  is an integer because it must be a representations of the circle.

If  $n$  is even, we have



It is easy to check that these maps satisfy  $[E, F] = H$ ,  $[H, E] = 2E$ , and  $[H, F] = -2F$

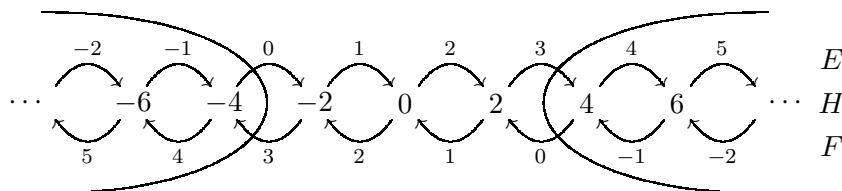
► **Exercise 31.1.** Do the case of  $n$  odd.

Problem: These may not be irreducible, and we want to decompose them into irreducible representations. The only way they can fail to be irreducible is if  $Ev_n = 0$  or  $Fv_n = 0$  for some  $n$  (otherwise, from any vector, you can generate the whole space). The only ways that can happen is if

$$\begin{aligned} n \text{ even: } & \lambda \text{ an odd integer} \\ n \text{ odd: } & \lambda \text{ an even integer.} \end{aligned}$$

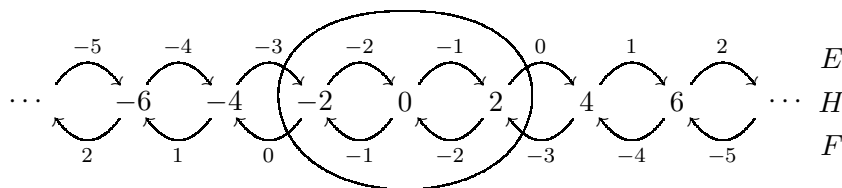
What happens in these cases? The easiest thing is probably just to write out an example.

**Example 31.1.** Take  $n$  even, and  $\lambda = 3$ , so we have



You can just see what the irreducible subrepresentations are ... they are shown in the picture. So  $V$  has two irreducible subrepresentations  $V_-$  and  $V_+$ , and  $V/(V_- \oplus V_+)$  is an irreducible 3-dimensional representation.

**Example 31.2.** If  $n$  is even, but  $\lambda$  is negative, say  $\lambda = -3$ , we get



Here we have an irreducible finite-dimensional representation. If you quotient out by that subrepresentation, you get  $V_+ \oplus V_-$ .

► **Exercise 31.2.** Show that for  $n$  odd, and  $\lambda = 0$ ,  $V = V_+ \oplus V_-$ .

So we have a complete list of all irreducible admissible representations:

1. if  $\lambda \notin \mathbb{Z}$ , you get one representation (remember  $\lambda \equiv -\lambda$ ). This is the bi-infinite case.
2. Finite-dimensional representation for each  $n \geq 1$  ( $\lambda = \pm n$ )
3. Discrete series for each  $\lambda \in \mathbb{Z} \setminus \{0\}$ , which is the half infinite case: you get a lowest weight when  $\lambda < 0$  and a highest weight when  $\lambda > 0$ .



4. two “limits of discrete series” where  $n$  is odd and  $\lambda = 0$ .

Which of these can be made into *unitary* representations?  $H^\dagger = -H$ ,  $E^\dagger = F$ , and  $F^\dagger = E$ . If we have a hermitian inner product  $(\cdot, \cdot)$ , we see that

$$\begin{aligned} (v_{j+2}, v_{j+2}) &= \frac{2}{\lambda + j + 1} (Ev_j, v_{j+2}) \\ &= \frac{2}{\lambda + j + 1} (v_j, -Fv_{j+2}) \\ &= -\frac{2}{\lambda + j + 1} \frac{\overline{\lambda - j - 1}}{2} (v_j, v_j) > 0 \end{aligned}$$

where we fix the sign errors. So we want  $-\frac{\overline{\lambda - j - 1}}{\lambda + j + 1}$  to be real and positive whenever  $j, j + 2$  are non-zero eigenvectors. So

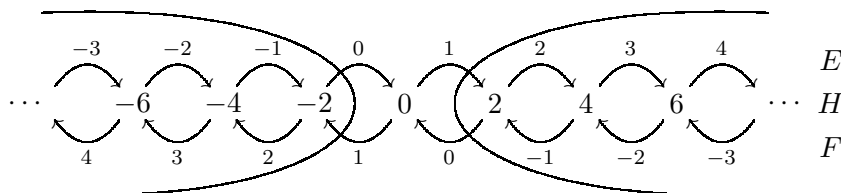
$$-(\lambda - 1 - j)(\lambda + 1 + j) = -\lambda^2 + (j + 1)^2$$

should be positive for all  $j$ . Conversely, when you have this, blah.

This condition is satisfied in the following cases:

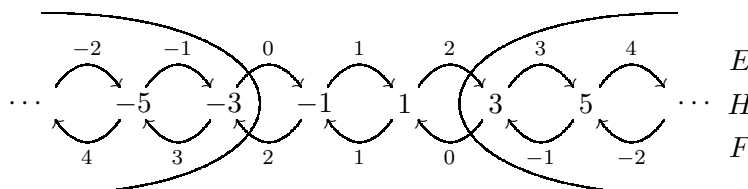
1.  $\lambda^2 \leq 0$ . These representations are called **PRINCIPAL SERIES** representations. These are all irreducible *except* when  $\lambda = 0$  and  $n$  is odd, in which case it is the sum of two limits of discrete series representations
2.  $0 < \lambda < 1$  and  $j$  even. These are called **COMPLEMENTARY SERIES**. They are annoying, and you spend a lot of time trying to show that they don't occur.
3.  $\lambda^2 = n^2$  for  $n \geq 1$  (for some of the irreducible pieces).

If  $\lambda = 1$ , we get



We see that we get two discrete series and a 1-dimensional representation, all of which are unitary

For  $\lambda = 2$  (this is the more generic one), we have



The middle representation (where  $(j+1)^2 < \lambda^2 = 4$  is NOT unitary, which we already knew. So the DISCRETE SERIES representations ARE unitary, and the FINITE dimensional representations of dimension greater than or equal to 2 are NOT.

Summary: the irreducible unitary representations of  $SL_2(\mathbb{R})$  are given by

1. the 1-dimensional representation
2. Discrete series representations for any  $\lambda \in \mathbb{Z} \setminus \{0\}$
3. Two limit of discrete series representations for  $\lambda = 0$
4. Two series of principal series representations:

$$j \text{ even: } \lambda \in i\mathbb{R}, \lambda \geq 0$$

$$j \text{ odd: } \lambda \in i\mathbb{R}, \lambda > 0$$

5. Complementary series: parameterized by  $\lambda$ , with  $0 < \lambda < 1$ .

The nice stuff that happened for  $SL_2(\mathbb{R})$  breaks down for more complicated Lie groups.

Representations of finite covers of  $SL_2(\mathbb{R})$  are similar, except  $j$  need not be integral. For example, for the double cover  $\widehat{SL_2(\mathbb{R})} = Mp_2(\mathbb{R})$ ,  $2j \in \mathbb{Z}$ .

► **Exercise 31.3.** Find the irreducible unitary representations of  $Mp_2(\mathbb{R})$ .

## Solutions to (some) Exercises

**Solution 1.1.** Yes. Consider  $\mu^{-1}(e) \subseteq G \times G$ . We would like to use the implicit function theorem to show that there is a function  $f$  (which is as smooth as  $\mu$ ) such that  $(h, g) \in \mu^{-1}(e)$  if and only if  $g = f(h)$ . This function will be  $\iota$ . You need to check that for every  $g$ , the derivative of left multiplication by  $g$  at  $g^{-1}$  is non-singular (i.e. that  $dl_g(g^{-1})$  is a non-singular matrix). This is obvious because we have an inverse, namely  $dl_{g^{-1}}(e)$ .

**Solution 1.2.** Just do it.

**Solution 4.1.** We calculate:

$$\begin{aligned} \frac{d}{dt} \|g(t)\|^2 &= 2 \left\langle \frac{d}{dt} g, g \right\rangle \\ &\leq 2 \left\| \frac{d}{dt} g \right\| \|g\| \\ &\leq 2 \|\xi\| \|g\|^2. \end{aligned}$$

That is,  $\eta(t) := \|g(t)\|^2$  satisfies the differential inequality:

$$\frac{d}{dt} \eta(t) \leq \|\xi\| \eta(t),$$

which in turn implies (Gronwall's inequality) that

$$\eta(t) \leq e^{2 \int_{t_0}^t \|\xi(s)\| ds}$$

so that

$$\begin{aligned} \|g\| &\leq e^{\int_{t_0}^t \|\xi(s)\| ds} \\ &\leq C' |t - t_0| \end{aligned}$$

since for  $|t - t_0|$  sufficiently small, exponentiation is Lipschitz.

**Solution 8.2.** We would like to compute the coefficients of the product  $(X^a H^b Y^r)(X^s H^c Y^d)$  once it is rewritten in the PBW basis by repeatedly applying the relations  $XY - YX = \varepsilon H$ ,  $HX = XH$ , and  $HY = YH$ . Check by induction that

$$Y^r X^s = \sum_{n=0}^{\infty} \varepsilon^n (-1)^n n! \binom{r}{n} \binom{s}{n} X^{r-n} H^n Y^{s-n}.$$

It follows that  $p_n^{I,J}$  is zero unless  $I = (Y, \dots, Y)$  and  $J = (X, \dots, X)$ , in which case  $p_n^{I,J} = \frac{(-1)^n}{n!} H^n$ .

**Solution 9.3.** We have  $[a, b]_h := \sum_{n=0}^{\infty} h^n m_n(a, b)$ , where  $m_0(a, b) = [a, b]$ . Now we compute

$$\begin{aligned} [a, [b, c]_h]_h &= [a, \sum_{l \geq 0} h^l m_l(b, c)]_h \\ &= \sum_{l \geq 0} h^l \sum_{k \geq 0} h^k m_k(a, m_l(b, c)) \\ &= \sum_{N \geq 0} h^N m_k(a, m_{N-k}(b, c)) \quad (N = k + l) \end{aligned}$$

Adding the cyclic permutations and looking at the coefficient of  $h^N$ , we get the desired result.

**Solution 11.1.** Obvious.

**Solution 11.2.**  $[G, G]$  is normal because  $r[g, h]r^{-1} = [rgr^{-1}, rhr^{-1}]$ . To see that  $[G, G]$  is connected, let  $\gamma_{gh} : [0, 1] \rightarrow G$  be a path from  $g$  to  $h$  and define the path  $t \mapsto g\gamma(t)g^{-1}\gamma(t)^{-1}$  in  $[G, G]$  from the identity to  $[g, h]$ . Since all the generators of  $[G, G]$  are connected to  $e \in G$  by paths, all of  $[G, G]$  is connected to  $e$ .

Now we show that the Lie algebra of  $[G, G]$  is  $\mathcal{D}\mathfrak{g}$ . Consider the Lie algebra homomorphism  $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathcal{D}\mathfrak{g}$ . Since  $G$  is simply connected, this lifts to some Lie group homomorphism  $p : G \rightarrow H$ :

$$\begin{array}{ccccc} \mathcal{D}\mathfrak{g} & \longrightarrow & \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{g}/\mathcal{D}\mathfrak{g} \\ & & \exp \downarrow & & \downarrow \exp \\ [G, G] & \longrightarrow & G & \xrightarrow{p} & H \cong \mathbb{R}^n \end{array}$$

where  $H$  is the simply connected Lie group with Lie algebra  $\mathfrak{g}/\mathcal{D}\mathfrak{g}$ . Note that the Lie algebra of the kernel of  $p$  must be contained in  $\ker \pi = \mathcal{D}\mathfrak{g}$ . Also,  $\mathfrak{g}/\mathcal{D}\mathfrak{g}$  is abelian, so  $H$  is abelian, so  $[G, G]$  is in the kernel of  $p$ . This shows that  $\text{Lie}([G, G]) \subseteq \mathcal{D}\mathfrak{g}$ .

To see that  $\mathcal{D}\mathfrak{g} \subseteq \text{Lie}([G, G])$ , assume that  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ . Then for  $X, Y \in \mathfrak{g}$  consider the path  $\gamma(t) = \exp(X\sqrt{t}) \exp(Y\sqrt{t}) \exp(-X\sqrt{t}) \exp(-Y\sqrt{t})$  in  $[G, G]$ :

$$\begin{aligned} \gamma(t) &= \left(1 + X\sqrt{t} + \frac{1}{2}X^2t + \dots\right) \left(1 + Y\sqrt{t} + \frac{1}{2}Y^2t + \dots\right) \times \\ &\quad \left(1 - X\sqrt{t} + \frac{1}{2}X^2t + \dots\right) \left(1 - Y\sqrt{t} + \frac{1}{2}Y^2t + \dots\right) \\ &= 1 + \sqrt{t}(X + Y - X - Y) + \\ &\quad t(XY - X^2 - XY - YX - Y^2 + X^2 + Y^2) + \dots \\ &= 1 + t[X, Y] + O(t^{3/2}) \end{aligned}$$

So  $\gamma'(0) = [X, Y]$ . This shows that  $[G, G]$  is a connected component of the kernel of  $p$ .

Since  $[G, G]$  is a connected component of  $p^{-1}(0)$ , it is closed in  $G$ .

**Solution 11.3.** Let  $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\text{rad } \mathfrak{g}$  be the canonical projection, and assume  $\mathfrak{a} \in \mathfrak{g}/\text{rad } \mathfrak{g}$  is solvable. Then  $\mathcal{D}^k \mathfrak{a} = 0$  for some  $k$ , so  $\mathcal{D}^k \pi^{-1}(\mathfrak{a}) \subseteq \text{rad } \mathfrak{g}$ . Since  $\text{rad } \mathfrak{g}$  is solvable, we have that  $\mathcal{D}^N \pi^{-1}(\mathfrak{a}) = 0$  for some  $N$ . By definition of  $\text{rad } \mathfrak{g}$ , we get that  $\pi^{-1}(\mathfrak{a}) \subseteq \text{rad } \mathfrak{g}$ , so  $\mathfrak{a} = 0 \subseteq \mathfrak{g}/\text{rad } \mathfrak{g}$ . Thus,  $\mathfrak{g}/\text{rad } \mathfrak{g}$  is semi-simple.

**Solution 12.1.** An invariant form  $B$  induces a homomorphism  $\mathfrak{g} \rightarrow \mathfrak{g}^*$ . Invariance says that this homomorphism is an intertwiner of representations of  $\mathfrak{g}$  (with the adjoint action on  $\mathfrak{g}$  and the coadjoint action on  $\mathfrak{g}^*$ ). Since  $\mathfrak{g}$  is simple, these are both irreducible representations. By Schur's Lemma, any two such homomorphisms must be proportional, so any two invariant forms must be proportional.

**Solution 12.2.** Done in class.

**Solution 12.3.** yuck.

**Solution 12.4.** The complex for computing cohomology is

$$\begin{aligned}
 0 \longrightarrow k \xrightarrow{d_0} \text{Hom}(\mathfrak{sl}_2, k) &\xrightarrow{d_1} \text{Hom}(\Lambda^2 \mathfrak{sl}_2, k) \xrightarrow{d_2} \text{Hom}(\Lambda^3 \mathfrak{sl}_2, k) \longrightarrow 0 \\
 c \longmapsto dc(x) = -x \cdot c = 0 & \\
 f \longmapsto df(x, y) = f([x, y]) & \\
 \alpha \longmapsto d\alpha(x, y, z) = \alpha([x, y], z) - \alpha([x, z], y) & \\
 & \quad + \alpha([y, z], x)
 \end{aligned}$$

We have that  $\ker d_1 = k$ , so  $H^0(\mathfrak{sl}_2, k) = k$ . The kernel of  $d_1$  is zero, as we computed in Remark 12.11. Since  $\text{Hom}(\mathfrak{sl}_2, k)$  and  $\text{Hom}(\Lambda^2 \mathfrak{sl}_2, k)$  are both three-dimensional, it follows that  $d_1$  is surjective, and since the kernel of  $d_2$  must contain the image of  $d_1$ , we know that  $d_2$  is the zero map. This tells us that  $H^1(\mathfrak{sl}_2, k) = 0$ ,  $H^2(\mathfrak{sl}_2, k) = 0$ , and  $H^3(\mathfrak{sl}_2, k) = \text{Hom}(\Lambda^3 \mathfrak{sl}_2, k) \cong k$ .

**Solution 12.5.** Let  $D \in \text{Der}(\mathfrak{g})$  and let  $X, Y \in \mathfrak{g}$ . Then

$$\begin{aligned}
 [D, ad_X]_{\text{Der}(\mathfrak{g})}(Y) &= D([X, Y]) - [X, D(Y)] \\
 &= [D(X), Y] + [X, D(Y)] - [X, D(Y)] \\
 &= ad_{D(X)}(Y).
 \end{aligned}$$

**Solution 13.1.** Let  $x \in \mathfrak{h}$ , so  $[x, h] = 0$ . Since  $ad_{x_n}$  is a polynomial in  $ad_x$ , we get that  $[x_n, h] = 0$ , so  $x_n \in \mathfrak{h}$ . Thus, it is enough to show that any nilpotent element in  $\mathfrak{h}$  is zero. We do this using property 4, that the Killing

form is non-degenerate on  $\mathfrak{h}$ . If  $y \in \mathfrak{h}$ , then  $B(x_n, y) = \text{tr}(ad_{x_n} \circ ad_y)$ .  $\mathfrak{h}$  is abelian by the proposition, so  $[x_n, y] = 0$ , so  $ad_{x_n}$  commutes with  $ad_y$ . Thus, we can simultaneously upper triangularize  $ad_{x_n}$  and  $ad_y$ . Since  $ad_{x_n}$  is nilpotent, it follows that  $\text{tr}(ad_{x_n} \circ ad_y) = 0$ . So  $x_n = 0$  by non-degeneracy of  $B$ .

**Solution 13.2.** Since  $\Delta$  is a finite set in  $\mathfrak{h}^*$ , we can find some  $h \in \mathfrak{h}$  so that  $\alpha(h) \neq \beta(h)$  for distinct roots  $\alpha$  and  $\beta$ . Then this  $h$  is a regular element which gives the right Cartan subalgebra, and the desired properties follow from the properties on page 60.

**Solution 13.3.** If  $\Delta$  does not span  $\mathfrak{h}^*$ , then there is some non-zero  $h \in \mathfrak{h}$  such that  $\alpha(h) = 0$  for all  $\alpha \in \Delta$ . This means that all of the eigenvalues of  $ad_h$  are zero. Since  $h$  is semisimple,  $ad_h = 0$ . And since  $ad$  is faithful, we get  $h = 0$ , proving property 1.

To prove 2, consider the  $\alpha$ -string through  $\beta$ . It must be of the form  $\mathfrak{g}_{\beta+n\alpha} \oplus \mathfrak{g}_{\beta+(n-1)\alpha} \oplus \cdots \oplus \mathfrak{g}_{\beta+m\alpha}$  for some integers  $n \geq 0 \geq m$ . From the characterization of irreducible finite dimensional representations of  $\mathfrak{sl}_2$ , we know that each eigenvalue of  $H_\alpha$  is an integer, so  $\beta(H_\alpha) = r \in \mathbb{Z}$  (since  $[H_\alpha, X_\beta] = \beta(H_\alpha)X_\beta$ ). We also know that the eigenvalues of  $H_\alpha$  are symmetric around zero, so we must have  $-r = (\beta + s\alpha)(H_\alpha)$  for some  $s$  for which  $\mathfrak{g}_{\beta+s\alpha}$  is in the  $\alpha$ -string through  $\beta$ . Then we get  $\beta(H_\alpha) + s\alpha(H_\alpha) = \beta(\alpha) + 2s = -r = -\beta(\alpha)$ , from which we know that  $s = -\beta(H_\alpha)$ . Thus,  $\mathfrak{g}_{\beta-\beta(H_\alpha)\alpha} \neq 0$ , so  $\beta - (\beta(H_\alpha))\alpha$  is a root.

Finally, we prove 3. If  $\alpha$  and  $\beta = c\alpha$  are roots, then by property 2, we know that  $\alpha(H_\beta) = 2/c$  and  $\beta(H_\alpha) = 2c$  are integers (note that  $H_\beta = H_\alpha/c$ ). It follows that  $c = \pm\frac{1}{2}, \pm 1$ , or  $\pm 2$ . Therefore, it is enough to show that  $\alpha$  and  $2\alpha$  cannot both be roots. To see this, consider the  $\alpha$ -string through  $2\alpha$ . We have that  $[H_\alpha, X_{2\alpha}] = 2\alpha(H_\alpha)X_{2\alpha} = 4X_{2\alpha}$ , so the  $\alpha$ -string must have a non-zero element  $[Y_\alpha, X_{2\alpha}] \in \mathfrak{g}_\alpha$ , which is spanned by  $X_\alpha$ . But then we would have that  $X_{2\alpha}$  is a multiple of  $[X_\alpha, X_\alpha] = 0$ , which is a contradiction.

**Solution 14.1.** If  $\Delta$  is reducible, with  $\Delta = \Delta_1 \cup \Delta_2$ , then set  $\mathfrak{h}_i$  to be the span of  $\Delta_i$ , and set  $\mathfrak{g}_i = \mathfrak{h}_i \oplus \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha$  (for  $i = 1, 2$ ). Then we have that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  as a vector space. We must check that  $\mathfrak{g}_1$  is an ideal (the by symmetry,  $\mathfrak{g}_2$  will also be an ideal). From the relation  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ , we know that it is enough to check that for  $\alpha \in \Delta_1$ ,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}_1, \text{ and} \tag{1}$$

$$[\mathfrak{g}_\alpha, \mathfrak{h}_2] = 0. \tag{2}$$

Letting  $\beta \in \Delta_2$ , we have that  $\beta([X_\alpha, Y_\alpha]) = \beta(H_\alpha) = \frac{2(\alpha, \beta)}{(\beta, \beta)} = 0$  because  $\Delta_1$  and  $\Delta_2$  are orthogonal; 1 follows because  $\Delta_2$  spans the orthogonal complement of  $\mathfrak{h}_1$  in  $\mathfrak{h}$ . Similarly, we have  $[X_\alpha, H_\beta] = \alpha(H_\beta)X_\alpha = 0$ ; 2 follows because the  $H_\beta$  span  $\mathfrak{h}_2$ .

If, conversely,  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , then since  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are ideals, we get that the roots coming from  $\mathfrak{g}_1$  are orthogonal to the roots from  $\mathfrak{g}_2$ , showing that the root system of  $\mathfrak{g}$  is reducible.

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## Index

- adjoint representation, [27](#)
- Ado's Theorem, [5](#)
- $\alpha$ -string, [64](#)
- antipode, [21](#)
- $\mathfrak{b}$ , [47](#)
- Baker-Campbell-Hausdorff, [11](#)
- bialgebra, [22](#)
- Borcherds, Richard E., [3](#), [105–170](#)
- Cartan decomposition, [61](#)
- Cartan subalgebra, [61](#)
- Cartan's Criterion, [54](#)
- Casimir operator, [55](#)
- central extension, [43](#)
- cohomology
  - of Lie algebras, [37](#)
- Complete reducibility, *see* Weyl's Theorem
- comultiplication, [21](#)
- coroot, [66](#)
- counit, [21](#)
- covering map, [16](#)
- Coxeter diagram, [71](#)
- Coxeter group, [66](#)
- $\mathfrak{d}$ , [47](#)
- Danish, [21](#)
- deformation
  - of a Lie algebra, [42](#)
- deformations
  - of associative algebras, [32](#)
- derived series, [47](#)
- dual pairing, [23](#)
- Dynkin diagram, [71](#)
- Engel's Theorem, [48](#)
- exponential map, [10](#)
- filtered space, [28](#)
- $\mathfrak{gl}_\infty$ , [44](#)
- $\mathfrak{gl}_n$ , [4](#), [47](#)
- Heisenberg algebra, [33](#), [63](#)
- Hochschild cohomology, [5](#)
- Hopf Algebras, [21](#)
- Hopf ideal, [25](#)
- invariant form, [52](#)
- Jacobi identity, [4](#)
- joke, [178](#)
- Jordan decomposition, [53](#)
  - absolute, [59](#), [60](#)
- Kac-Moody algebra, [44](#)
- Killing form, [52](#)
- Knutson, Alan, [3](#)
- Kontsevitch, Maxim, [35](#)
- L<sup>A</sup>T<sub>E</sub>X, [3](#)
- Lie algebra, [4](#)
  - of a Lie group, [6](#)
- Lie algebra cohomology, [37](#)
- Lie derivative, [5](#)
- Lie group, [4](#)
- Lie ideal, [6](#)
- Lie's Theorem, [49](#)
- loop algebra, [44](#)
- loop space, [44](#)
- lower central series, [47](#)
- nilpotent, [47](#)
  - element, [60](#)
- one-parameter subgroup, [10](#)
- PBW, *see* Poincaré-Birkhoff-Witt
- Poincaré-Birkhoff-Witt, [31](#)
- rank, [62](#), [67](#)
- real form, [4](#)
- regular element, [60](#)

- representations, [26](#)
- Reshetikhin, Nicolai, [3–45](#)
- root, [61](#)
  - positive, [69](#)
  - simple, [69](#)
- root space, [61](#)
- root decomposition, [61](#)
- root system
  - abstract, [67](#)
  - irreducible, [67](#)
  
- semisimple, [51](#)
  - element, [60](#)
- Serganova, Vera, [3, 46–104](#)
- $\mathfrak{sl}_2$ , [61, 63](#)
- $\mathfrak{sl}_3$ , [62](#)
- $\mathfrak{sl}_n$ , [60](#)
- solvable, [47](#)
- star product, [34](#)
  
- unitary trick, [58](#)
- universal cover, [17](#)
- universal enveloping algebra, [24–32](#)
- upper triangular, *see*  $\mathfrak{b}$
- useful facts, [47–48](#)
  
- variety of Lie algebras, [5](#)
- Verma module, [90–101](#)
  
- Weyl group, [68](#)
- Weyl's Theorem, [58](#)
- Whitehead's Theorem, [55](#)
  
- Zariski open set, [65](#)