The multiplicative-accelerator model

A macroeconomic model of business cycles

Let \( Y(t) = \text{size of economy} \)

auxiliary quantity

\( I(t) = \text{rate of investment} \)

\[
\frac{dY}{dt} = I(t) - sY
\]

\( s \) = saving factor

Key assumption: Investment intentions depends on market activity (Keynesian multiplier)

\[
\frac{dI}{dt} = v \frac{dY}{dt} - I
\]

\( v \) = velocity coefficient

\( \Rightarrow \frac{d^2Y}{dt^2} = \frac{dI}{dt} - \frac{dY}{dt} \)

...
\[ \frac{d^2 y}{dt^2} = (v - s) \frac{dy}{dt} - 1 \]
\[ = (1 - v \cdot s) \frac{dy}{dt} + s \cdot y \]

This eqn has the form
\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q \cdot y = 0 \]
for \[ p = s - v - 1 \]
\[ q = -1 \]

Exactly the same as damped harmonic oscillator.

Generality of mathematics

Now that we have some motivation, let's turn to the question of how to solve this equation.

First, let's try our favorite guess:
\[ y(t) = e^{rt} \]

\[ \frac{d^2 y}{dt^2} = r^2 y, \quad \frac{dy}{dt} = r y \]
\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0 \]

This is zero provided that

\[ r^2 + pr + q = 0 \]

characteristic equation
dt has roots

\[ r_\pm = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \]

Therefore \( A e^{r_+ t} + B e^{r_- t} \) are solns,

and \( y(t) = A e^{r_+ t} + B e^{r_- t} \)

is the general soln

(ex) \[ \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 0 \]

Trying \( e^{rt} \) results in
characteristic eqn

\[ r^2 + 3r + 2 = 0 \]
\[ (r+2)(r+1) = 0 \]
\[ r = -2 \quad \text{or} \quad r = -1 \]
\[ y(t) = Ae^{-2t} \quad y(t) = Be^{-t} \]

As for the bifurcations we observed earlier in the course, we should expect qualitatively different behavior depending on the discriminant in the quadratic formula.

Case 1: \( p^2 - 4q > 0 \) \( 2 \) real solutions \( \exp \text{ growth/decay} \)

Case 2: \( p^2 - 4q < 0 \) \( 2 \) imaginary roots

\[ r^2 - 4q = 0 \] \( \text{real soln w/multiplicity 2} \)
Case 3: \( p^2 - 4q = 0 \) real sin w/multiplicity 2

Come back later

Let's explore the case of imaginary roots in more detail.

Recall \( \frac{d^2y}{dt^2} + y = 0 \)

In this case \( p = 0 \), \( q = 1 \)

\( \Rightarrow p^2 - 4q < 0 \)

Roots are \( \pm \frac{\sqrt{-q}}{2} = \pm i \)

Sols \( e^{it} \), \( e^{-it} \)

What are these?

Hint: We already found general soln \( y(t) = A \sin(t) + B \cos(t) \)
Somehow, $e^{it}$ related to sines and cosines

In fact, $e^{ib} = \cos(b) + i\sin(b)$

Euler's formula

Moreover, for any complex number $a + ib$,

$e^{a+ib} = e^a e^{ib} = e^a \cos(b) + i e^a \sin(b)$

So, the real and imaginary parts of $e^{it}$ look good separately, but how to deal w/this complex-valued fun?

Theorem: If $y(t) = y_{re}(t) + iy_{im}(t)$ is a complex-valued solution to the drift eqn

\[
\frac{d^2y}{dt^2} + p \frac{dy}{dt} + q y = 0,
\]
where $p$ and $g$ are real numbers, then $y_{re}(t)$ and $y_{im}(t)$ are solutions as well.

**Proof**

By linearity,

\[
\frac{d^2(y_{re} + iy_{im})}{dt^2} + p \frac{d(y_{re} + iy_{im})}{dt} + g(y_{re} + iy_{im}) = 0
\]

\[
\begin{align*}
\Rightarrow & \quad \left(\frac{d^2 y_{re}}{dt^2} + p \frac{dy_{re}}{dt} + g y_{re}\right) + i \left(\ldots\right) = 0 \\
\Rightarrow & \quad \frac{d^2 y_{re}}{dt^2} + p \frac{dy_{re}}{dt} + g y_{re} = 0
\end{align*}
\]

**Upshot:** It's OK to work with complex-valued functions.
complex-valued fun like $e^{it}$, in the end, take real or imaginary parts

General soln:

$$y(t) = A \text{Re}(e^{it}) + B \text{Im}(e^{-it})$$

$$= A \cos(t) + B \sin(t)$$

Exercise:

Consider

$$\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 4y = 0$$

1) Write down char. poly.

\[ \text{solve for roots} \]

2) Give general soln

3) do something wrong?

Can you fix it?

$$y(t) = te^{-2t}$$

1. 

-2t

2.
\[ \frac{dy}{dt} = e^{-2t} - 2te^{-2t} \]

\[ \frac{d^2 y}{dt^2} = -2e^{-2t} - 2e^{-2t} + 4te^{-2t} \]

\[ = -4e^{-2t} + 4te^{-2t} \]