In our study of the mass connected to a spring, we obtained the second-order constant coefficient eqn

\[ m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0, \]

where \( m \) is mass, \( b \) is damping constant, and \( k \) is spring constant.

We showed it is a real p.t. if \( r \) is a root to the characteristic poly

\[ mr^2 + br + k = 0 \]

When \( b = 0 \), \( \frac{d^2 y}{dt^2} + \left( \frac{k}{m} \right)y = 0 \) and the char poly \( r^2 + \frac{k}{m} = 0 \) has roots \( \pm iw \) for \( w = \sqrt{k/m} \).

\[ e^{iwt} = \cos(\omega t) + i \sin(\omega t) \] by Euler's form, and we have a simple harmonic oscillator.

General soln: \( y(t) = C_1 \sin(\sqrt{\frac{k}{m}} t) + C_2 \cos(\sqrt{\frac{k}{m}} t) \)
For nonzero damping \((b \neq 0)\), we found that there are 3 possibilities depending on the parameters:

\[
\begin{align*}
(1) \quad & b^2 - 4km < 0, \quad 2 \\
& \text{complex-valued roots} \\
& r_1 = \alpha + i\omega \\
& r_2 = \alpha - i\omega \\
& \text{Underdamped} \\
& \text{Den soln: } y(t) = C_1 e^{r_1 t} \sin(\omega t) + C_2 e^{r_2 t} \cos(\omega t)
\end{align*}
\]

\[
\begin{align*}
(2) \quad & b^2 - 4km > 0, \quad 2 \\
& \text{distinct real roots} \\
& r_1, r_2 \\
& \text{Overdamped} \\
& \text{Den soln: } y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}
\end{align*}
\]

\[
\begin{align*}
(3) \quad & b^2 - 4km = 0, \quad \text{real root w/mult. 2} \\
& \text{Den soln: } y(t) = C_1 e^{rt}
\end{align*}
\]
Given solution: \( y(t) = C_1 e^{rt} + C_2 te^{rt} \) Critically damped

Next we study each case in detail

**Underdamped** \( b^2 - 4km < 0 \) - damping relatively small

(Ex) \( m = 1 \), \( b = 0.2 \), \( k = 1.01 \)

char. poly \( r^2 + 0.2r + 1.01 = 0 \)

two roots \( -0.1 \pm \sqrt{0.04 - 1.01} = \frac{-0.2 \pm \sqrt{-0.94}}{2} = -0.1 \pm i \)

Complex-valued solution is

\[ y(t) = e^{(-0.1\pm i)t} = e^{-0.1t} (e^{it}) \]

\[ = e^{-0.1t} (\cos(t) + i\sin(t)) \]

\[ = e^{-0.1t} (\cos(t) \pm i\sin(t)) \]

Gen. soln: \( y(t) = C_1 e^{-0.1t} \cos(t) + C_2 e^{-0.1t} \sin(t) \)

\( t \) Oscillations with Decay

Decay with oscillations
Overdamped

1. \[ b^2 - 4km > 0 \] - relatively large damping

(ex) \( m = 1, b = 3, k = 1 \)

no char poly \( r^2 + 3b + 1 = 0 \)

has roots \( x = \frac{-3 \pm 5}{2} \)

In this case, two real and negative roots,

faster decay, slower decay

then soln: \( y(t) = C_1 e^{\frac{-3 + 5}{2} t} + C_2 e^{\frac{-3 - 5}{2} t} \)

initial cond \((y_0, v_0) = (3, 0)\)

\((-0.25, 3)\)

Critically damped

\[ b^2 - 4km = 0 \] - damping balanced w/other params
(Ex) \( m=1, b=2, k=1 \)

\[ r^2 + 2r + 1 = 0 \]

which has a repeated root \( r = -1 \)

Gen soln: \( y(t) = C_1 e^{-t} + C_2 te^{-t} \)

\[ \text{Twice critical damping} \]

\[ \text{One-half critical damping} \]

\[ \text{No oscillations} \]

\[ \text{Most rapid relaxation to EQ} \]

In overdamped case, damp oscillations more quickly but slow down motion

Exercise

Consider

\[ \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + y = 0 \]
\[ m \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = 0 \]

What happens in the limit \( m \to 0 \)?

First, some algebra:

\[ b \cdot x + c = 0 \Rightarrow x = -\frac{c}{b} \]

\[ a \cdot x^2 + b \cdot x + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

What happens in limit \( a \to 0 \)? Do two roots converge to \( x = -c/b \)? Two roots become one?

We'll use:

For small, \( \sqrt{1 + x} \approx 1 + x/2 \) by first-order Taylor series.

\[ x = \frac{-b}{2a} + \frac{b}{2a} \sqrt{1 - \frac{4ac}{b^2}} \approx \frac{-b}{2a} + \frac{b}{2a} \left( 1 - \frac{2ac}{b^2} \right) \]

\[ x_+ = \frac{-b}{2a} + \frac{b}{2a} - \frac{c}{b} = -\frac{c}{b} \]

\[ x_- = \frac{-b}{2a} + \frac{c}{a} \to \infty \]

We recover limiting root. Second root runs off to \( \infty \).
1) Classify as over, under, or critically damped in terms of $m$.

2) Express the solution in terms of roots of $m$ (assume small $m$) over-damped.

$$\pm \frac{-3 \pm \sqrt{9-8m}}{2m}$$

Assume $m$ small (over-damped).

3) What are the limiting values of $r_e$ as $m \to 0$?

Hint: use $\sqrt{1+x} \approx 1 + \frac{x}{2}$ for $x$ small.

4) Compare to solution to limiting eqn

$$3 \frac{dy}{dt} + 2y = 0$$

How are your answers?
same/different?