Differential equations may include higher-order derivatives such as \[ \frac{d^2y}{dt^2}, \frac{d^3y}{dt^3} \text{ etc.} \]

Second-order diff eqs arise frequently in physics.

Consider a block having mass \( m \) attached to a spring w/spring constant \( k \).

Let \( x \) denote displacement from
Newton's law says

\[ F = ma = m \frac{d^2x}{dt^2} \]

According to Hooke's law, the force exerted by a spring is

\[ F = -kx \]

"linear restoring force"

That is, the mass-spring system obeys

\[ m \frac{d^2x}{dt^2} = -kx \]

\[ \therefore \frac{d^2x}{dt^2} = -\frac{k}{m} x \]
Let's consider the case $k = 1$, so that
\[
\frac{d^2 x}{dt^2} = -x
\]

What few do you know of that satisfy this diff eq?

\[x(t) = \cos(t) \text{ and } x(t) = \sin(t)\]

As was the case for linear 1st-order eqns, there is a linearity principle for linear 2nd-order eqns such as this one. 2) Constant multiples of solns are solns, and sums of solns are solns. Therefore, any function of the form
$X(t) = A \sin t + B \cos t$

is a general solution.

In fact, all solutions have this form.

**Question:** What is a suitable IVP for this eqn?

\[
\begin{align*}
\frac{d^2x}{dt^2} &= -x \\
x(0) &= 0
\end{align*}
\]

OK? No! There are two undetermined constants to set, so we need two pieces of information.

This makes sense:

2\textsuperscript{nd} order eqn e.g. \[\frac{d^2x}{dt^2} = f(t)\]

\[\Rightarrow \text{integrate } \int_0^t f(s)ds + AtB\]
\[ x(t) = \int \int f(s) ds + A + B \]

get two constants of integration.

Physicist's argument: of course not!

Need to know not only initial position but also initial velocity. If the mass is flicked at \( t = 0 \), behaves very differently (oscillates back and forth).

This generalizes to the case of \( n \)-th-order equations. Need initial conditions for \( n \) quantities.

Examples of suitable IVPs.
Examples of second-order equations

(Ex 1) \[
\begin{aligned}
d^{2}x &= -x \\
x(0) &= 0 \\
\frac{dx}{dt}(0) &= 0
\end{aligned}
\]

(Ex 2) \[
\begin{aligned}
\frac{d^{2}x}{dt^{2}} &= -x \\
x(0) &= 0 \\
\frac{dx}{dt}(0) &= 1
\end{aligned}
\]

\[x(t) = 0 \quad \text{and} \quad x(t) = \sin(t)\]

Going back to the original problem,

\[
\frac{d^{2}x}{dt^{2}} = -\frac{k}{m} x,
\]

the general solution is

\[x(t) = A \sin\left(\sqrt{\frac{k}{m}} t\right) + B \cos\left(\sqrt{\frac{k}{m}} t\right)\]

so IVP \[
\begin{aligned}
x(0) &= 0 \\
\frac{dx}{dt}(0) &= 0
\end{aligned}
\]

\[\Rightarrow \quad x(t) = \sin\left(\sqrt{\frac{k}{m}} t\right)\]

Plot the solution:

\[\begin{aligned}
\text{spring} \\
\text{oscillates}
\end{aligned}\]
period \( \frac{2\pi \sqrt{\frac{m}{k}}}{k} \), frequency \( \frac{1}{\text{period}} = \frac{1}{2\pi \sqrt{\frac{k}{m}}} \)

Observations: increasing mass \( \Rightarrow \) longer period
increasing spring stiffness \( \Rightarrow \) shorter period

does this a good model? What's missing?

Mass won't oscillate forever, there is damping because of air resistance

Model can be improved by allowing a damping force

Friction force \( \tau = -b \frac{dx}{dt} \)
\[ m \frac{d^2 x}{dt^2} = -kx - b \frac{dx}{dt} \]

Rearrange and divide by \( m \):

\[ \frac{d^2 x}{dt^2} + \frac{p}{m} \frac{dx}{dt} + q x = 0 \]

This is the eqn for a damped harmonic oscillator, or any constant-coefficient 2nd-order diff eqn.