Integrating factors

Next we'll discuss an analytic technique for constructing particular solutions that is very effective for some types of equations.

Given \( \frac{dy}{dt} = b(t) \), we can just integrate.

E.g., \( \frac{dy}{dt} = \cos(t) \Rightarrow y(t) = \sin(t) + C \)

We'd like to use the same approach for the general linear eqn

\[ \frac{dy}{dt} = a(t) y + b(t) \]

Why can't we simply integrate?

Don't know \( y \), of course.

Don't change...
But, if we could rewrite as

\[
\frac{d}{dt} \left( y \right) = b
\]

we can use the same approach as before. Grouping unknown & known quantities as above,

\[
\frac{dy}{dt} - a(t) y = b(t)
\]

LHS is reminiscent of product rule

\[
\frac{d}{dt} (y \mu) = \left( \frac{dy}{dt} \right) \mu + y \frac{d\mu}{dt}
\]

with \( \mu(t) = 1 \), \( \frac{d\mu}{dt} = -a(t) \), quite work.

Idea: multiply both sides by a function that will allow us to use product rule

\[
(\text{ex}) \quad \frac{dy}{dt} = -2y + 3e
\]
\[\implies \quad \frac{dy}{dt} + 2y = 3e^{-2t}\]

\[\implies \quad e^{2t} \frac{dy}{dt} + 2e^{2t}y = 3e^{-2t}e^{2t}\]

\[\mu(t) = e^{2t} \text{ is the integrating factor}\]

\[\implies \quad \frac{d}{dt}(ye^{2t}) = 3\]

\[\implies \quad ye^{2t} = 3t + C\]

\[\implies \quad y(t) = 3te^{-2t} + Ce^{-2t} \text{ is the general solution}\]

\[\text{Good illustration of extended linearity principle}\]

\[\text{In this case, we could easily guess the correct factor.}\]

\[\text{In general, what to do in general?}\]

\[\mu(t) \frac{dy}{dt} - a(t) \mu(t)y = b(t) \mu(t)\]

\[\implies \quad A(t) = \mu(t), \quad \frac{dA}{dt} = -a(t) \mu(t)\]

\[\implies \quad \frac{d\mu}{dt} = -a(t) \mu(t)\]
\[ \frac{d\mu}{dt} = -\alpha(t)\mu \]

We have obtained a differential eqn for \( \mu(t) \)!

Solution is \( \mu(t) = e^{\int_{a(s)}^{t} \alpha(s) \, ds} \)

This is called the integrating factor

Unwinding,

\[ \frac{dy}{dt} - \alpha(t)y = b(t) \]

1) Multiply by int factor \( \mu(t) = e^{\int_{a(s)}^{t} \alpha(s) \, ds} \)

\[ \mu(t)\frac{dy}{dt} - \alpha(t)\mu(t)y = \mu(t)b(t) \]

2) Rewrite using product rule

\[ \frac{d}{dt}(\mu y) = \mu(t) b(t) \]

3) Integrate directly.
\[ m_y = \int \mu(s)b(s)ds \]

4) Divide by \( \mu(t) \)

\[ y(t) = \frac{1}{\mu(t)} \int_{s=0}^{t} \mu(s)b(s)ds \]

(ex)

\[ \frac{dy}{dt} + \frac{2}{t}y = t-1 \]

In this case, \( \mu(t) = e^{\int_{s=0}^{t} \frac{2}{s}ds} = 2 \ln t = t^2 \)

Can rewrite as

\[ \int \frac{d}{dt} (yt^2) = \int (t-1)t^2 \]

\[ yt^2 = \frac{t^4}{4} - \frac{t^3}{3} + C \]

\[ y(t) = \frac{t^2}{4} - \frac{t}{3} + C/t^2 \quad \text{is general solution} \]
Exercise

1) \( \frac{dy}{dt} = -2y + te^t \)

2) \( \frac{dy}{dt} + y = \cos(t) \)

Cannot always evaluate the integrals explicitly!

\[ \frac{dy}{dt} - t^2 y = t - 1 \quad \Rightarrow \quad \mu(t) = e^{-t^3/3} \]

\[ e^{-t^{3/3}} y = \int_0^t e^{-s^{3/3}} (s-1) \, ds \]
Not so bad. Identities, numerical meth