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Office hours Monday & Wednesday 2-3pm Goldsmith 208

Some terminology

What is a differential eqn?

It's an equation for an unknown function that involves its derivatives.

\[ y + 3 = 0 \implies y = -3 \] algebraic eqn

\[ \frac{d^2 y}{dt^2} + y = 0 \implies y(t) = ? \] differential eqn

As opposed to an algebraic eqn, we're solving for a function, not only one number!

The order of a diff. eqn is the
highest derivative that appears is $y^{(n)}$

\[ (1) \quad \frac{d^2 y}{dt^2} + y = 0 \quad 2nd\text{-order} \]

\[ \frac{d^2}{dt^2} y \sin(y) + y \frac{2d^4 y}{dt^4} + y = t^2 \quad 4th\text{-order} \]

We'll use notation:

\[ \frac{d^2 y}{dt^2} = y^{(2)} \]

E.g.  

|\[ y = \frac{dy}{dt} \]

To denote deriv $\frac{d}{dt}$

\[ \frac{d^2 y}{dt^2} + y = 0 \implies y^{(n)} + y = 0 \]

Finally, a diff. eq. $W(y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \frac{d^3 y}{dt^3}, \frac{d^4 y}{dt^4})$ is linear provided that $W$ is a linear fun of $y$ and its derivs, i.e. $W(...y^{(n-1)} + y^{(n-2)}...)$ can be written as $W(...y^{(n-1)},...)$ linear.
E.g. mass-spring \[ \frac{d^2 y}{dt^2} + y = 0 \] linear

\[ y + y \frac{dy}{dt} = 0 \] nonlinear

\[ \frac{d^4 y}{dt^4} + \frac{dy}{dt} + t^2 y = \sin(t) \] linear

\[ \frac{dy}{dt} + \sin(y) = 0 \] nonlinear

What do these equations represent?

Which ones are easy to solve, and which ones are harder?

Interactive exercise

Thomas Far

1) Name

Math & Physics

2) Year & major

3) Where from

4) What's your favorite thing about math?

5) Why did you sign up for course?
Find a partner and introduce yourselves.

Let's start by going more in-depth on a specific diff eq.

**Population growth** Let \( P(t) \) denote the size of population at time \( t \).

**Assumption:** population \( P(t) \) has a growth rate proportional to its size.

In mathematical terms,

\[
\frac{dP}{dt} = kP
\]

1st order, linear.

Even if we don't know the solution, we can learn a lot by studying the diff eqn. Qualitative/graphical analysis.

1) **Equilibria:** where does \( P \) stop changing?

i.e. \( \frac{dP}{dt} = 0 \Rightarrow P = 0 \) is the only eq pt.
2) Phase line
\[ P = 0 \] is an unstable Eq. If \[ P \] will keep growing and growing

3) Slope field
\[ \frac{dP}{dt} = kP \]

Understand qualitative behavior of solutions without solving anything!

This is an excellent place to start:

Analytic solution
\[ P(t) = Ce^{kt} \]
What is \( P(t) \)?

Guess: \( P(t) = C e^{kt} \)

\[
\frac{dP}{dt} = Cke^{kt} = kP
\]

And if we didn't know in advance, in this case we can use a technique for solving separable equations:

\[
\frac{dy}{dt} = g(y)h(t) \Rightarrow \int dy = \int g(y)h(t) dt
\]

In our case,

\[
\frac{dP}{dt} = kP \quad \Rightarrow \quad \int \frac{dP}{P} = \int k \, dt
\]

\[
\log P = kt + c \quad \Rightarrow \quad e^{\log P} = e^{kt+c} = e^c e^{kt}
\]

\[
P = Ce^{kt}
\]

Solution is exponential growth.

Does it a good model? What are its shortcomings?

Populations cannot grow forever. The environment has a carrying capacity.

\[
\dot{P} = kP(1 - P/Q)
\]

Logistic growth

Qualitative analysis
EQ^2. \ P = 0 \ \xi \ P = Q

Phase line \ \frac{dP}{dt}

\ P = 0 \ \text{is unstable, as before}
\ P = K \ \text{is a stable EQ point}

P will grow until it reaches K

Slope field

K
Analytic solution

\[ \frac{dP}{P(1-P/Q)} = \frac{A}{P} + \frac{B}{1-P/Q}, \]

where \( A = 1 \)
\( B = \frac{1}{6} \)

\[ \int \frac{1}{P} dP + \int \frac{1/6}{1-P/Q} dP = \int k dt \quad \Rightarrow \quad \log \frac{P}{1-P/Q} = kt + c \]

\[ \frac{P}{1-P/Q} = (c e^{kt}) \]

\[ P(t) = \frac{Ce^{kt}}{1 + \frac{Ce^{kt}}{Q}} \]

\[ \log \frac{P}{1-P/Q} = kt + c \quad \text{Check...} \quad \checkmark \]

Typically, diff eq involves

formulating ODE, i.e. modeling

What is the right eqn?
learning about solns

checking soln

You already can do a lot!

thinking about correct model, checking
analytic solns, typically assume small the e.g. and

In this course, we'll focus on

qualitative & analytic solution methods

We will learn several techniques

These work pretty well for linear


In general, not all nonlinear ODEs

not all (not many!) nonlinear ODEs

can be solved analytically

Need to rely on: qualitative analysis

approximation by e.g. (linear?) ODE's

- computation
Computers are very useful, we’ll discuss a bit (this happens to be my handwriting).

An example from economics

Solow-Swan model of capital growth

Let \( k(t) \) be the amount of capital at time \( t \).

According to this model,

\[
\text{rate of change of capital} = \{ \text{rate of investment} \} - \{ \text{depreciation rate} \}
\]

Assumptions:
- Investment rate = \( 5\sqrt{k} \)
- Saving rate = \( \frac{C}{K} \) (diminishing returns)
- Depreciation rate = \( 8k \)

\[
\frac{dk}{dt} = 5\sqrt{k} - 8k
\]

For exercises:
1) Compute EQ points & assess
2) Choose values of \( s \) and \( e \), draw a couple of slope fields, and plot a couple of solutions through it.